CHAPTER 22

THE ELECTRIC FIELD
II CONTINUOUS CHARGE DISTRIBUTIONS

• Calculating $\vec{E}$ from Coulomb’s Law
  - Finite line of charge
  - Infinite line of charge
  - Ring of charge
  - Uniformly charged disk
  - Uniformly charged infinite plate

• Gauss’s Law
  - Definition of Electric Flux

• Calculating $\vec{E}$ from Gauss’s Law
  - Outside/inside a spherical conductor
  - Infinite line of charge
  - Uniformly charged infinite plate
  - Uniformly charged insulating sphere

• $\vec{E}$ inside hollow conductors

Electric fields are important because the force on a charge $q$ is $\vec{F} = q\vec{E}$ and so we can determine how a charge moves in the field, using Newton’s 2nd Law: viz:

$$\ddot{a} = \frac{\vec{F}}{m} = \frac{q\vec{E}}{m}.$$ 

Basically, there are two approaches to calculate $\vec{E}$:

- using Coulomb’s Law
- using Gauss’s Law

1: Using Coulomb’s Law:

Consider the force on a test charge $q_o$ due to an element of charge $dQ$ within some charge distribution:

$$\text{The force on } q_o \text{ due to } dQ \text{ is } d\vec{F} = q_o d\vec{E}. \quad \text{d}\vec{F} = q_o \frac{d\vec{E}}{d\vec{r}} \hat{r}.$$ 

$$\therefore d\vec{E}(\text{at } q_o) = \frac{d\vec{F}}{q_o} = \frac{1}{q_o} \frac{d\vec{F}}{d\vec{r}} \hat{r}$$ 

i.e., $$\vec{E}(\text{at } q_o) = k \int \frac{dQ}{r^2} \hat{r}.$$
Line of charge uniformly distributed
• length 2a, total charge Q

\[ dQ = \lambda \, dy \]

The y-component of the contribution from the lower segment cancels the y-component from the upper segment. Thus, the resultant field is confined to the x-direction. So, the magnitude of the electric field along \( \hat{i} \) due to the segment \( dy \) (at \( y \)) is

\[ dE_x = dE \cdot \cos \theta = k \frac{dQ}{r^2} \cdot \frac{x}{r} = k \frac{x \lambda dy}{(x^2 + y^2)^{3/2}}. \]

\[ \therefore E_x = kx \lambda \int_{-a}^{a} \frac{dy}{(x^2 + y^2)^{3/2}} \]

since \( k, x \) and \( \lambda \) are constants. However, the integral is a standard integral, viz:
\[ \frac{a}{2} \int_{-a}^{a} \frac{dy}{\left( x^2 + y^2 \right)^{3/2}} = [\frac{y}{x^2 \sqrt{x^2 + y^2}}]_a^{-a} \]
\[ \therefore E_x = kx \lambda \left[ \frac{y}{x^2 \sqrt{x^2 + y^2}} \right]_a^{-a} \]
\[ = kx \lambda \left[ \frac{a}{x^2 \sqrt{x^2 + a^2}} - \frac{(-a)}{x^2 \sqrt{x^2 + (-a)^2}} \right] = k \left[ \frac{2a \lambda}{x \sqrt{x^2 + a^2}} \right], \]
\[ \bar{E} = E_x \hat{i} = k \frac{Q}{x \sqrt{x^2 + a^2}} \hat{i}. \]

What difference would it make, if any, if the charge was not distributed uniformly nor symmetrically along the wire?

- Note: when \( x \gg a \), then \( \bar{E} = k \frac{Q}{x^2} \hat{i} \).
- If \( a \to \infty \), i.e., an infinite line of charge, then
  \[ E = \lim_{a \to \infty} k \frac{Q}{x \sqrt{x^2 + a^2}} = k \frac{Q}{xa} = \frac{2k \lambda}{x}, \]
  i.e., \[ \bar{E} = \frac{2k \lambda}{x} \hat{i} = \frac{\lambda}{2 \pi \epsilon_0 x} \hat{i}. \]
On the axis of a line of charge:
- length 2a, total charge Q

The electric field is in the $\hat{j}$ direction. The magnitude of the electric field a distance $d$ from the center of the line of charge due to the segment $dy$ is

$$dE_y = k \frac{dQ}{(d-y)^2} = k \frac{\lambda dy}{(d-y)^2}.$$  

$$\therefore E_y = k\lambda \int_{-a}^{a} \frac{dy}{(d-y)^2}.$$  

Change the variable and put $u = (d-y)$, then $dy = -du$; also, when $y = \pm a$, $u = d \mp a$.  

If $y \gg a$ then:

$$E_y \hat{j} = k \frac{Q}{y^2-a^2} \hat{j}$$

i.e., wire looks like a point charge.
Therefore, we get

\[ E = -k\lambda \int_{d+a}^{d-a} \frac{du}{u^2} = k\lambda \int_{d+a}^{d-a} \frac{1}{u} \left( \frac{1}{d-a} - \frac{1}{d+a} \right) \]

\[ = k\lambda \left( \frac{(d+a) - (d-a)}{(d-a)(d+a)} \right) = \frac{2k\lambda a}{(d^2 - a^2)} = k \frac{Q}{(d^2 - a^2)}. \]

\[ \therefore \hat{E} = E_y \hat{j} = k \frac{Q}{(d^2 - a^2)} \hat{j}. \]

- If \( d \gg a \), then \( \hat{E} \rightarrow k \frac{Q}{d^2} \hat{j} \), i.e., the line of charge looks like a point charge.

**Question 22.1**: If the charge distribution along the wire is not uniform nor symmetrical, what would be the effect on the electric field at \( y \)?

A: No affect at all.
B: There would be a component of the field in the \( x \)-direction.
C: The field at \( y \) is along the \( y \)-direction but could be larger or smaller depending on the exact distribution.
D: Any of the above ... impossible to say without knowing the precise charge distribution.
Imagine the most extreme cases, i.e., when all the charge is either at one end or the other (above); the electric fields are both in the y-direction. However, the strengths of the fields are different:

\[ \vec{E}_1 = k \frac{Q}{(y-a)^2} \hat{j} \quad \text{and} \quad \vec{E}_2 = k \frac{Q}{(y+a)^2} \hat{j} \]

i.e., \( E_1 > E_2 \). So, the magnitude of the electric field does depend on how the charge is distributed.

Therefore, the answer is C.

**Question 22.2**: An infinitely long line of charge that has a uniform linear charge density equal to \(-1.50 \mu C/m\) lies parallel to the y-axis at \( x = -2.00 \) m. A positive point charge of magnitude 1.30 \( \mu C \) is located at \( (x,y) = (1.00 \, \text{m}, 2.00 \, \text{m}) \).

Find the total electric field at \( (x,y) = (2.00 \, \text{m}, 1.50 \, \text{m}) \).
The total electric field at $P$ is the vector sum of two fields, i.e., $\vec{E}_R = \vec{E}_1 + \vec{E}_2$.

$$\vec{E}_1 = \frac{\lambda}{2\pi\varepsilon_0 x} \hat{i} = -1.5 \times 10^{-6} \hat{i} = -6.75 \times 10^3 \hat{i} \text{ N/C.}$$

$$|\vec{E}_2| = k\frac{q_2}{r^2} = 9 \times 10^9 \times \frac{1.3 \times 10^{-6}}{0.5^2 + 1.0^2} = 9.36 \times 10^3 \text{ N/C.}$$

$$\therefore \vec{E}_2 = |\vec{E}_2| \cos(26.6^\circ) \hat{i} = 8.37 \times 10^3 \hat{i} \text{ N/C}$$

and

$$\vec{E}_2 = -|\vec{E}_2| \sin(26.6^\circ) \hat{j} = -4.19 \times 10^3 \hat{j} \text{ N/C.}$$

$$\therefore \vec{E}_R = (-6.75 \times 10^3 \hat{i} + 8.37 \times 10^3 \hat{i} - 4.19 \times 10^3 \hat{j}) \text{ N/C}$$

$$= (1.62 \times 10^3 \hat{i} - 4.19 \times 10^3 \hat{j}) \text{ N/C.}$$

On the axis of a ring of uniformly distributed charge ($Q$):

By symmetry, the components parallel to $\hat{j}$ and $\hat{k}$ cancel so the electric field is confined to the x-direction ($\hat{i}$).

The magnitude of the electric field at $P$ due to the element $dQ$ is

$$dE_x = \cos \theta dE = \left(\frac{x}{r}\right) k \frac{dQ}{r^2} = k \frac{xdQ}{r^3}.$$

$$\therefore E_x = k \int \frac{xdQ}{Q \left(x^2 + a^2\right)^{3/2}} = \frac{kx}{\left(x^2 + a^2\right)^{3/2}} \int dQ$$

$$= k \frac{Qx}{\left(x^2 + a^2\right)^{3/2}},$$

since $x$ and $a$ are constant.
∴ $E = E_x \hat{i} = k \frac{Qx}{(x^2 + a^2)^{3/2}} \hat{i}$.

• When $x = 0$: $E = 0$.

• When $x \gg a$: $E = k \frac{Q}{x^2} \hat{i}$

i.e., the ring looks like a point charge.

On the axis of a ring of uniformly distributed charge ($Q$):

$E_x \hat{i} = k \frac{Q}{x^2} \hat{i}$

$max at$

$x = \frac{a}{\sqrt{2}}$

$E_x < 0$

$(x < 0)$

$E_x > 0$

$(x > 0)$
**Question 22.3**: If the charge Q is not uniform nor symmetrical around the wire, what difference, if any, would it make to the electric field at x?

A: No effect at all.
B: There would be components of the field in the y- and/or z-directions
C: The field at x is along the x-direction but could be larger or smaller depending on the distribution
D: Any of the above ... not possible to say without knowing the precise charge distribution

Look at an extreme case, when the charge is concentrated in one region, at the bottom, say. Clearly, there is a y-component.

In general, with a non-uniform charge distribution, the y- and z-components of the electric fields from diametrically opposite sides will not cancel. Therefore, the resultant electric field will not be confined to the x-axis; there will be components in the y- and z-directions, depending on exactly how the charge is distributed. (Non-trivial calculation!)

Therefore, the answer is B.
**Question 22.4:** Two uniformly charged rings, one with charge $Q_1$ and the other with charge $Q_2$, are placed 1.00 m apart with their central axes aligned, as shown above. The smaller ring, with charge $Q_1$, has radius $a_1 = 0.20$ m; the larger ring has radius $a_2 = 0.30$ m. If $Q_1 = 1.50 \times 10^{-6}$ C, what is $Q_2$ if the electric field midway between the rings is zero?

The magnitude of the electric field at a distance $x$ along the axis of a uniformly charged ring of radius $a$ is

$$E = k \frac{Qx}{(x^2 + a^2)^{3/2}},$$

where $Q$ is the charge on the ring. The electric fields at $x = 0.50$ m are directed in opposite directions and so for $E_{net} = 0$, we require

$$k \frac{Q_1 x}{(x^2 + a_1^2)^{3/2}} = k \frac{Q_2 x}{(x^2 + a_2^2)^{3/2}},$$

i.e.,

$$Q_2 = \left( \frac{x^2 + a_2^2}{x^2 + a_1^2} \right)^{3/2} Q_1.$$
Substituting the given values we obtain

$$Q_2 = \left( \frac{0.50m^2 + 0.30m^2}{0.50m^2 + 0.20m^2} \right)^{3/2} (1.50 \times 10^{-6} C)$$

$$= \left( \frac{0.34}{0.29} \right)^{3/2} (1.50 \times 10^{-6} C) = (1.269)(1.50 \times 10^{-6} C)$$

$$= 1.90 \times 10^{-6} C.$$

Uniformly charged disk of radius R

(Charge per unit area ⇒ \( \sigma = \frac{Q}{\pi R^2} \))

If \( R >> x \), i.e., a very large plate then:

$$\vec{E} = E_x \hat{i} = \frac{\sigma}{2\varepsilon_0} \left[ 1 - \frac{1}{\sqrt{\left( \frac{R^2}{x^2} \right) + 1}} \right] \hat{i}$$

NOTE: when \( R >> x \) the field is perpendicular to the plate and independent of \( x \).
The ring element has charge \( dQ = 2\pi r \sigma \, dr \). From before, the magnitude of the electric field due to the ring element is
\[
dE_x = k \frac{dQ \cdot x}{\left(x^2 + r^2\right)^{3/2}} = k \frac{(2\pi r \sigma \, dr) \cdot x}{\left(x^2 + r^2\right)^{3/2}}.
\]
\[
\therefore \ E_x = 2\pi k x \sigma \frac{R \, dr}{\sqrt{(x^2 + r^2)^3}}.
\]
We change the variable and put \( u = x^2 + r^2 \). Then\( \, du = 2r \, dr \), i.e., \( r \, dr = \frac{du}{2} \). Also, when \( r = 0 \), \( u = x^2 \) and when \( r = R \), \( u = x^2 + R^2 \).

Then
\[
E_x = \pi k x \sigma \frac{x^2 + R^2}{\sqrt{u}} = \pi k x \sigma \left[ \frac{-x^2 + R^2}{\sqrt{u}} \right]
\]
\[
= 2\pi k x \sigma \left[ \frac{1}{x} - \frac{1}{\sqrt{x^2 + R^2}} \right] = 2\pi k x \sigma \left[ 1 - \frac{x}{\sqrt{x^2 + R^2}} \right]
\]
\[
= \frac{\sigma}{2\varepsilon_o} \left[ 1 - \frac{1}{\sqrt{1 + \left(\frac{R^2}{x^2}\right)}} \right] \hat{i}.
\]
i.e., \( \hat{E} = E_x \hat{i} = \frac{\sigma}{2\varepsilon_o} \left[ 1 - \frac{1}{\sqrt{1 + \left(\frac{R^2}{x^2}\right)}} \right] \hat{i} \).

• When \( R \gg x \), i.e., the plate is very large
\[
\hat{E} \rightarrow \frac{\sigma}{2\varepsilon_o} \hat{i}
\]
i.e., the electric field is perpendicular to the plate and independent of \( x \).
• When $x \gg R$, i.e., $R^2/x^2 \ll 1$, we expect the disc to look like a point charge. When $R^2/x^2 \ll 1$, then
\[
1 - \frac{1}{\sqrt{1 + \left( \frac{R^2}{x^2} \right)^2}} = 1 - \left( 1 + \left( \frac{R^2}{x^2} \right)^2 \right)^{-1/2} \\
\approx 1 - \left( 1 + \left( -\frac{1}{2} \right) \left( \frac{R^2}{x^2} \right)^2 \right) = \frac{R^2}{2x^2}.
\]
\[\therefore E_x = \frac{\sigma}{2\varepsilon_o} \left( \frac{R^2}{2x^2} \right).\]

But $\sigma = \frac{Q}{4\pi R^2}$ and $\varepsilon_o = \frac{1}{4\pi k}$, so
\[E_x = k \left( \frac{Q}{x^2} \right),\]
i.e., the disc looks like a point charge ... good!

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Electric field due to two large uniformly charged parallel plates

(Charge per unit area $\Rightarrow \sigma$)

Consider each plate separately ...

\[\tilde{E}_1 = -\frac{\sigma}{2\varepsilon_o} \hat{i} \quad \tilde{E}_1 = \frac{\sigma}{2\varepsilon_o} \hat{i}\]

\[\tilde{E}_2 = \frac{\sigma}{2\varepsilon_o} \hat{i} \quad \tilde{E}_2 = -\frac{\sigma}{2\varepsilon_o} \hat{i}\]
Two large uniformly charged parallel plates ...

In region A: \( \vec{E}_A = \vec{E}_1 + \vec{E}_2 = \left( -\frac{\sigma}{2\varepsilon_0} \right) \hat{i} + \frac{\sigma}{2\varepsilon_0} \hat{i} = 0 \)

In region B: \( \vec{E}_B = \vec{E}_1 + \vec{E}_2 = \frac{\sigma}{2\varepsilon_0} \hat{i} + \frac{\sigma}{\varepsilon_0} \hat{i} = \frac{\sigma}{\varepsilon_0} \hat{i} \)

In region C: \( \vec{E}_C = \vec{E}_1 + \vec{E}_2 = \frac{\sigma}{2\varepsilon_0} \hat{i} + \left( -\frac{\sigma}{2\varepsilon_0} \right) \hat{i} = 0 \)

NOTE: the field is parallel and uniform between the plates and zero outside ... see photo in chapter 21.

**Question 22.5:** A uniformly charged ring, with a charge of 2.50\( \mu \)C and radius 0.20 m, and a very large, uniformly charged sheet, with charge density 1.00\( \mu \)C/m\(^2\), are placed 0.50 m apart such that the axis of the ring is perpendicular to the sheet, as shown above. What is the electric field at the point \( \times \) midway between the ring and the sheet?
Since the charged sheet is very large and positively charged, the electric field at $\times$ due to the sheet is

$$E_1 = \frac{\sigma}{2\varepsilon_o} \hat{i}.$$

The electric field at $\times$ due to the positively charged ring is

$$E_2 = k \frac{Qx}{(x^2 + a^2)^{3/2}} (-\hat{i}),$$

where $x$ is the distance from the center of the ring to $\times$ and $a$ is the radius of the ring.

$$\therefore E_{net} = E_1 + E_2 = \left[ \frac{\sigma}{2\varepsilon_o} - k \frac{Qx}{(x^2 + a^2)^{3/2}} \right] \hat{i}.$$

Since $k = \frac{1}{4\pi\varepsilon_o}$, $E_{net} = \frac{1}{2\varepsilon_o} \left[ \sigma - \frac{Qx}{2\pi(2x^2 + a^2)^{3/2}} \right] \hat{i}.$

Substituting the given values:

$$E_{net} = \frac{1}{2 \times 8.854 \times 10^{-12}} \left[ 1.00 \times 10^{-6} - \frac{2.50 \times 10^{-6} \times 0.25}{2\pi(0.25^2 + 0.20^2)^{3/2}} \right] \hat{i}$$

$$= (-1.15 \times 10^5 \text{ N/C}) \hat{i}.$$

Hence, a positive test charge placed at $\times$ will experience a force directed in the $-\hat{i}$ direction.

Show for yourself that if $Q = 8.24 \times 10^{-7} \text{ C (0.824\mu C)}$, then the net electric field at $\times$ is zero.
Imagine an electric field $\mathbf{E}$ passing perpendicularly through an area $A$. We define the electric flux:

$$\Phi = EA$$

If the electric field $\mathbf{E}$ passes through at an angle $\theta$ to the surface normal. The electric flux (or simply the flux) is:

$$\Phi = \mathbf{E} \cdot \mathbf{A} = EA \cos \theta.$$ 

Note that area $\mathbf{A}$ is actually a vector (directed along the surface normal) so the flux can be $\pm$ depending on $\theta$, i.e., the relative directions of $\mathbf{E}$ and $\mathbf{A}$.

If the electric field varies over the surface then the flux is:

$$\Phi = \int \mathbf{E} \cdot d\mathbf{A}.$$

**Question 22.6**: A uniform electric field $\mathbf{E} = a\mathbf{i} + b\mathbf{j}$ intersects a surface of area $\mathbf{A}$. Find the flux through this area if the surface lies in the
(a) yz plane,
(b) xz plane,
(c) xy plane.
2: Using Gauss’s Law:

Gauss’s Law: The net flux through any closed 3-d surface is:

\[
\Phi = \frac{Q_{\text{en}}}{\varepsilon_0} = \oint E \cdot d\mathbf{A},
\]

where \(Q_{\text{en}}\) is the net charge enclosed by the surface.

It is valid for all closed surfaces and all charge distributions. Flux can flow outward and inward through a surface; thus, the net flux is the difference between the outward flux and the inward flux.

Check out a proof of Gauss’s law for an isolated point charge on the web-page (under the useful notes link).
Take any Gaussian surface enclosing a charge Q. Since we can take any closed surface, we’ll take a sphere.

![Gaussian Surface Diagram]

The total flux $\Phi$ through the surface depends on the number of electric field lines $N$ leaving (or entering) the charge,

i.e., $\Phi \propto N$.

But $\Phi = \frac{Q}{\varepsilon_0}$.

∴ $N \propto Q$

i.e., the number of lines leaving or entering a charge $Q$ is proportional to $Q$.

**Question 22.7:** A spherical Gaussian surface is drawn around a point charge positioned at its center. If the radius of the sphere is decreased, does the flux increase, decrease or stay the same?
Assume the initial radius is \( r_1 \) and the final radius is \( r_2 \) (\( r_2 < r_1 \)). The flux is through an element of the surface is defined as \( d\Phi = \mathbf{E} \cdot d\mathbf{A} \). But, at every point on a spherical Gaussian surface \( \mathbf{E} \parallel d\mathbf{A} \) and so 
\[
\mathbf{E} \cdot d\mathbf{A} = E \cdot dA.
\]

Thus, the total flux through the surface is 
\[
\Phi = \int E \cdot dA = E \int dA = EA,
\]
where \( A \) is the area of the surface and \( E \) is the value of the electric field on the surface. If the radius is \( r \), then 
\[
\Phi = k \frac{q}{r^2} 4\pi r^2 = 4\pi k q = \frac{q}{\varepsilon_0},
\]
(since \( k = \frac{1}{4\pi\varepsilon_0} \), from the previous chapter). So, \( \Phi \) is independent of \( r \), the radius, i.e., it remains constant. In fact, we can generalize this result; the flux is independent of the size of any Gaussian surface.

Therefore, the answer is C.

Note: if two charges of equal magnitude but of opposite sign, i.e., \( Q_2 = -Q_1 \), are enclosed within a Gaussian surface, then the net charge enclosed by the Gaussian surface is zero. Therefore, the net flux through the surface is zero. Although lines of electric field are passing through the surface, there are equal numbers flowing INWARDS and OUTWARDS so the net flux is zero!
Gauss’s Law can be used to obtain the electric field from a given charge distribution. It is normally useful only when there is a good deal of symmetry.

Three helpful pieces of additional information are:

[1] Any *excess* charge on a conductor resides entirely on the surface. (Why?)

*By EXCESS charge we mean the net charge!*

[2] The electric field at the surface of a conductor is always perpendicular to the surface. (Next slide.)

[3] Choose a Gaussian surface that match the symmetry of the electric field, so that

(a) the surface is either perpendicular or parallel to the electric field lines,

(b) the electric field has the same magnitude at each point on the surface. (See the following examples.)
[2] The electric field is always perpendicular to the surface of a conductor ...

If the electric field was at some angle to the surface, there would be parallel, \( \vec{E}_{//} \), and perpendicular, \( \vec{E}_\perp \), components of the electric field. The parallel component would produce a force on the surface charges and so they would move! Static equilibrium, which is established very rapidly, requires that there is no parallel component \( \vec{E}_{//} \), which means \( \vec{E} \) is perpendicular to the surface at all points.

Electric field due to a charged conductor:

Since the excess charge resides on the surface, it doesn’t matter whether the sphere is solid or hollow!

OUTSIDE THE SPHERE \((r > R)\) ...

Use a spherical Gaussian surface centered on the charged sphere, with surface area \(4\pi r^2\). Gauss’s Law tells us:

\[
\Phi = \frac{Q_{en}}{\varepsilon_0} = \int \vec{E} \cdot d\vec{A} = \int E dA = EA, 
\]

since the electric field is constant over the surface.

\[
\therefore E(r) = \frac{\Phi}{4\pi r^2} = \frac{Q}{4\pi \varepsilon_0 r^2} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} = k \frac{Q}{r^2} \quad (r > R) 
\]

i.e., it looks just like a point charge!
INSIDE THE SPHERE...

Using Gauss’s Law:

\[ E(r) = \frac{\Phi}{A} = \frac{Q_{en}}{A\varepsilon_0} = 0 \quad r < R \]

At the surface \( E_s = kQ/R^2 = \frac{\sigma}{\varepsilon_0} \), but \( \sigma = \frac{Q}{4\pi R^2} \).

\[ \therefore E_s = \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\pi R^2\sigma}{R^2} = \frac{\sigma}{\varepsilon_0}. \]

**Question 22.8**: The electric field at the Earth’s surface is about 200 N/C and is directed towards the center of the Earth. What is the charge on the Earth? (Radius of the Earth = 6400 km.)
For any Gaussian surface around the surface of the Earth we have: \[ \Phi = E \cdot A = \frac{Q_{en}}{\varepsilon_0}. \]

Take a spherical Gaussian surface centered on the Earth, of radius equal to the radius of the Earth (actually just slightly larger!). Then we have

\[ \Phi = \vec{E} \cdot \vec{A} = -E \times 4\pi R^2, \]
as \( \vec{E} \) (inwards) and \( \vec{A} \) (outwards) are anti-parallel.

\[ \therefore \Phi = -200 \times 4\pi \times (6400 \times 10^3)^2 = \frac{Q_{en}}{8.85 \times 10^{-12}}, \]
i.e., \[ Q_{en} = -200 \times 4\pi \times (6400 \times 10^3)^2 \times 8.85 \times 10^{-12} \]
\[ = -9.11 \times 10^5 \text{C}. \]

Which means that the charge on the Earth is \textit{negative}.

Also, since \( Q_{en} = ne \), that charge corresponds to

\[ n = \frac{Q_{en}}{e} = \frac{-9.11 \times 10^5}{-1.60 \times 10^{-19}} = 5.69 \times 10^{24} \]
excess electrons!

\[ \text{Electric field due to an infinitely long charged wire:} \]

\[ \text{Charge per unit length = } \lambda \]

Take a cylindrical Gaussian surface of length \( \ell \) and radius \( r \), centred on the wire. The electric field is \textit{radial} and |\( \vec{E} \)| is the same over the curved surface so |\( \vec{E} \)| through the ends is zero. \textit{Why??}

Using Gauss’s Law:

\[ \Phi = \frac{Q_{en}}{\varepsilon_0} = \int \vec{E} \cdot d\vec{A} = \int E dA = E \int dA = EA \]

\[ \therefore E(r) = \frac{Q_{en}}{A\varepsilon_0} = \frac{\ell \lambda}{2\pi r \varepsilon_0} = \frac{\lambda}{2\pi \varepsilon_0 r}. \]

This is the same result we got using Coulomb’s Law.
Choose a cylindrical Gaussian surface of radius $r$ and length $\ell$. Note that $\mathbf{E}$ is perpendicular to the sheet and the ends of cylinder, i.e., there is no $\mathbf{E}$ through the curved surface (and so $\ell$ is not important). From Gauss’s Law:

$$\Phi = \frac{Q_{en}}{\varepsilon_0} = \int \mathbf{E} \cdot d\mathbf{A} = \int E \, dA = EA$$

$$\therefore E = \frac{Q_{en}}{A \varepsilon_0} = \frac{\pi r^2 \sigma}{2(\pi r^2) \varepsilon_0} = \frac{\sigma}{2 \varepsilon_0}.$$

Same as the result using Coulomb’s Law.

Choose a rectangular box as a Gaussian surface. From Gauss’s Law the flux through each surface is:

$$\Phi = \frac{Q_{en}}{\varepsilon_0} = \mathbf{E} \cdot \mathbf{A} = EA$$

since $\mathbf{E}$ is perpendicular to the ends of box and there is no $\mathbf{E}$ through the sides.

Total flux: $\Phi_{\text{total}} \Rightarrow E(2A)$.

$$\therefore E = \frac{Q_{en}}{2A \varepsilon_0} = \frac{\sigma A}{2A \varepsilon_0} = \frac{\sigma}{2 \varepsilon_0}.$$

Same result as before!
Electric field due to a uniformly charged insulating sphere:
- **radius** \( R \), **Total charge** \( Q \).

**Uniform volume charge density:**
\[
\rho = \frac{Q}{V_{\text{total}}} = \frac{Q}{\frac{4}{3} \pi R^3}
\]

**INSIDE THE SPHERE** (\( r < R \)) ... At radius \( r \), the enclosed charge \( = \rho V \), where \( V \) is the enclosed volume.

\[ \therefore \text{Enclosed charge } \Rightarrow Q_{\text{en}} = \rho \frac{4}{3} \pi r^3 = \frac{Q r^3}{R^3}. \]

Using Gauss’s Law:
\[
\Phi = \frac{Q_{\text{en}}}{\varepsilon_0} = \int \vec{E} \cdot d\vec{A} = \int E dA = E \int dA = EA.
\]

\[ \therefore E(r) = \frac{Qr^3}{4\pi \varepsilon_0 R^3}, \]

i.e., \[ E(r) = \frac{1}{4\pi \varepsilon_0} \frac{Qr}{R^3} = k \frac{Qr}{R^3} \quad (r < R) \]

It can be shown very easily (prove it for yourselves) that for \( r > R \)
\[ E(r) = k \frac{Q}{r^2} \quad (r > R), \]

i.e., **outside** the sphere it looks like a point charge.

**NOTE:** The electric field at the surface of an insulating sphere is the same as a conducting sphere, \( E_s = k \frac{Q}{R^2} \).
**Question 22.9:** A nonconducting sphere of radius 6.00 cm has a uniform volume charge density of 450 nC/m³.

(a) What is the total charge on the sphere?

Find the electric field at the following distances from the center of the sphere:

(b) 2.00 cm,

(c) 5.90 cm,

(d) 6.10 cm, and

(e) 10.0 cm.

(a) Total charge \[ \Rightarrow V_{\text{tot}} = \frac{4}{3} \pi R^3 \rho \]
\[ = \frac{4}{3} \pi (0.06)^3 \times 450 \times 10^{-9} = 4.07 \times 10^{-10} \text{C}. \]

(b) \[ \Phi(r) = E(r) \cdot A \cdot \frac{Q_{\text{en}}}{\varepsilon_0}. \]
\[ \therefore E(r) = \frac{Q_{\text{en}}}{A \varepsilon_0} = \frac{(4/3) \pi r^3 \rho}{4 \pi r^2 \varepsilon_0} = \frac{rp}{3\varepsilon_0}. \]

With \( r = 2.00 \text{ cm} \Rightarrow E(r) = 339 \text{ N/C}. \)

(c) With \( r = 5.90 \text{ cm} \Rightarrow E(r) = 999 \text{ N/C} \)

(d) When \( r > R, \) \( Q_{\text{en}} = 4.07 \times 10^{-10} \text{C}. \)

Therefore, with \( r = 6.10 \text{ cm}, \)
\[ E(r) = \frac{Q_{\text{en}}}{A \varepsilon_0} = \frac{4.07 \times 10^{-10}}{4 \pi r^2 \varepsilon_0} = 983 \text{ N/C}. \]

(e) With \( r = 10.0 \text{ cm} \Rightarrow E(r) = 366 \text{ N/C}. \)

Note: (d) and (e) are the same for a conducting sphere (solid or hollow) with a total charge of \( 4.07 \times 10^{-10} \text{C}. \)
The charge on a conductor resides entirely on the surface but what if the conductor has a hollow cavity?

Let’s start with a solid conductor with positive charges distributed over its surface.

If there is a cavity within the conductor that contains no charge, draw a Gaussian surface within the conductor but outside the cavity.

From earlier we found that the electric field strength \( E = 0 \) everywhere within a conductor, i.e., there is no net flux within a conductor. Therefore, Gauss’s Law tells us that there is no charge within the Gaussian surface, i.e., there is no charge on the surface of the cavity.
But, what if the cavity contains a charge +q?

Everywhere within a conductor $E = 0$, so Gauss’s Law tells us that there must be no net charge inside the Gaussian surface. So, there must be a charge of $-q$ over the inner surface of the cavity. If the conductor were uncharged initially, then the $-q$ charge had to come from somewhere ... by redistributing charge in the conductor so the outer surface has a charge $+q$.

If the conductor had an existing charge $+Q$ initially, then inserting a charge of $+q$ in the cavity will produce a charge of $-q$ over the inside of the cavity and $(q + Q)$ on the outside of the conductor.

**Question 22.10:**

The inner conducting sphere, of radius 1.00 m, has a surface charge density of 1.00 $\mu$C/m$^2$. What is the charge density on the inner surface of the outer sphere (inner radius = 2.00 m)?
If the total charge on the inner sphere is \(+Q\), then from our notes, the total charge on the inner surface of the outer sphere is \(-Q\). So, if \(A_1\) is the surface area of the inner sphere and \(A_2\) is the inner surface area of the outer sphere:

\[ A_2\sigma_2 = -A_1\sigma_1. \]

\[ \therefore \sigma_2 = -\frac{A_1}{A_2}\sigma_1 = -\frac{4\pi R_1^2}{4\pi R_2^2}\sigma_1 \]

\[ = -\frac{(1.00 \text{ m})^2}{(2.00 \text{ m})^2}\sigma_1 = -0.25 \mu\text{C/m}^2. \]

Since the charge on the outer surface is \(+Q\), the charge density on the outer surface is:

\[ \sigma_3 = -\frac{A_2}{A_3}\sigma_2 = \frac{A_1}{A_3}\sigma_1 = \frac{R_1^2}{R_3^2}\sigma_1, \]

where \(R_3\) is the outer radius of the outer sphere.

How come the electric field is zero inside a conductor even if there’s an external field? ... because the charges inside are re-distributed to produce a “compensating” field, \(E_{\text{comp}}\), ...

Initially uncharged

\[ \therefore \sigma_R \Rightarrow +\sigma \]

and

\[ \sigma_L \Rightarrow -\sigma \]

\[ E_{\text{inside}} = E_{\text{comp}} - E = 0. \]

If \(E_{\text{inside}} = 0\) then \(E_{\text{comp}} = \frac{\sigma}{\varepsilon_0}\).

\[ \therefore E = \frac{\sigma}{\varepsilon_0}, \text{ so the charge on} \]

the surfaces of the conductor is \(\pm\sigma = \pm\varepsilon_0 E\), where \(E\) is the external field.
What about a hollow conducting box in an electric field? ... why is the field inside zero?

If an uncharged conducting hollow box is placed in an electric field ($\vec{E}$) the charges within the conductor are redistributed. But, the electric field created by the redistributed charge ($\vec{E}_{\text{comp}}$) is such that the net field inside ($E_{\text{in}}$) is zero. This configuration is known as a **FARADAY CAGE** and is a way of “protecting” sensitive equipment.

\[
\sigma_L + \sigma_R = 0
\]

\[
E_{\text{comp}} = \frac{\sigma}{\varepsilon_0} = E \quad -\sigma \quad +\sigma
\]

\[
\therefore \sigma = \varepsilon_0 E
\]

**Examples of a Faraday cage ...**

... coaxial (antenna) wire.

The copper braid acts as a Faraday cage. Since the electric field inside the braid is zero the center conducting wire is “protected” from extraneous, external signals.
A large spark jumps to the car’s body and then jumps across the car’s front tire. The person inside is unharmed because there is no electric field inside the vehicle.

More examples of Faraday cages!

Would you volunteer??
Earlier we saw that $\vec{E}$ is always perpendicular to the surface of a charged conductor. What happens if a *neutral* conductor is placed in an electric field?

The charges within the conductor are redistributed by the electric field. However, the *resultant electric field* - due to the field created by the conductor and the original field - is still perpendicular to the surface of the conductor. So, no matter whether it is charged or not, the electric field at the surface of a conductor is always perpendicular to the surface. Of course, the electric field *inside* the conductor is zero.

Since *we* are conductors, we “distort” the electric field of the Earth ...

Note that the electric field lines are perpendicular to the body at all points!
**Question 22.11:** A cylindrical Gaussian surface encloses a region perpendicular to two, very large parallel plates with different charge densities, \( \sigma_L = 2.50 \text{ nC/m}^2 \) and \( \sigma_R = 7.50 \text{ nC/m}^2 \). If \( \Phi_L \) and \( \Phi_R \) are the fluxes through the left and right ends of the cylinder, what are \( \Phi_L \) and \( \Phi_R \)?

The electric field due to the \( \sigma_L \) plate is \( E = \sigma_L/2\varepsilon_0 \) and the electric field due to the \( \sigma_R \) plate is \( E = \sigma_R/2\varepsilon_0 \). Thus, the total electric field towards the left is:

\[
E_L = \frac{\sigma_L}{2\varepsilon_0} + \frac{\sigma_R}{2\varepsilon_0} = \left(10.0 \text{ nC/m}^2\right)/\varepsilon_0,
\]

and the total electric field towards the right is:

\[
E_R = \frac{\sigma_R}{2\varepsilon_0} + \frac{\sigma_L}{2\varepsilon_0} = \left(10.0 \text{ nC/m}^2\right)/\varepsilon_0.
\]

The areas (A) of the ends of the cylinder are equal, and the electric fields are constant over the ends, so

\[\Phi_L = E_L A \quad \text{and} \quad \Phi_R = E_R A.\]

But \( E_L = E_R \),

\[
\therefore \Phi_L = \Phi_R.
\]