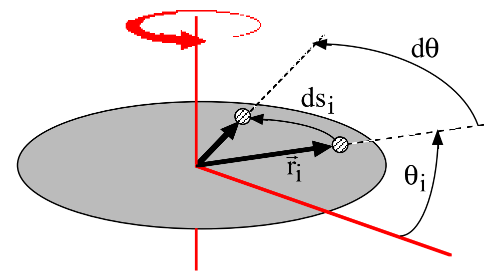
CHAPTER 9

ROTATION

- Angular velocity and angular acceleration
  - equations of rotational motion
- Torque and Moment of Inertia
  - Newton's 2nd Law for rotation
- Determination of the Moment of Inertia
  - Parallel axis theorem
  - Perpendicular axis theorem
- Rotational kinetic energy
  - power
- Rolling objects (with no slip)

Angular velocity and angular acceleration



The arc length moved by the ith element in a rotating rigid, non-deformable disk is:

$$ds_i = r_i|d\theta|$$

where dθ is in radians. The *angular velocity* of the rotating disk is defined as:

$$\omega = \frac{d\theta}{dt},$$

and so the *linear velocity* of the ith element (in the direction of the tangent) is:

$$v_i = \frac{ds_i}{dt} = r_i\omega,$$

where r_i is the distance from the rotation axis.

DISCUSSION PROBLEM [9.1]:

You have a friend who lives in Minnesota, and you live in Florida. As the Earth rotates, your *linear velocity* is ____________ hers, and your *angular velocity* is _____________ hers.

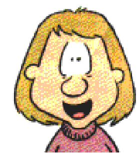
A: less than; equal to

B: equal to; greater than

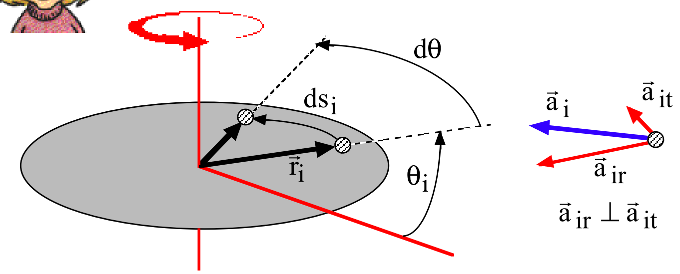
C: greater than; less than

D: less than; greater than

E: greater than; equal to



If the angular velocity changes there is angular acceleration ...



The *angular acceleration* of the disk is:

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}.\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2},$$

and the *tangential acceleration* of the i th element is:

$$a_{it} = \frac{dv_i}{dt} = r_i \frac{d\omega}{dt} = r_i\alpha.$$

But, because the ith element is traveling in a circle, it experiences a *radial (centripetal) acceleration*:

$$a_{ir}\ (= a_{ic}) = \frac{v_i^2}{r_i} = r_i\omega^2.$$

The resultant *linear* acceleration is $|\vec{a}_i| = \sqrt{a_{ir}^2 + a_{it}^2}$.

Angular velocity (ω) (*vector*)

Dimension: $\omega \Rightarrow \frac{1}{[T]}$

(Check: $v_i = r_i\omega \Rightarrow [L]\frac{1}{[T]} = \frac{[L]}{[T]}$).

Units: rad/s

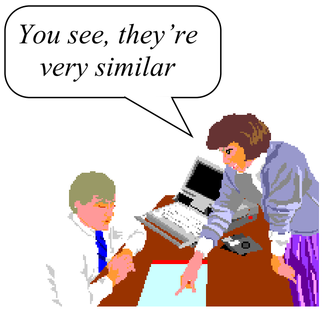
Angular acceleration (α) (*vector*)

Dimension: $\alpha \Rightarrow \frac{1}{[T]^2}$

Units: rad/s^2

CONNECTION BETWEEN LINEAR AND ROTATIONAL MOTION

Linear motion	*Rotational motion*
$a \Rightarrow$ constant	$\alpha \Rightarrow$ constant
$v = v_\circ + at$	$\omega = \omega_\circ + \alpha t$
$(x - x_\circ) = v_\circ t + \frac{1}{2}at^2$	$(\theta - \theta_\circ) = \omega_\circ t + \frac{1}{2}\alpha t^2$
$v^2 = v_\circ^2 + 2a(x - x_\circ)$	$\omega^2 = \omega_\circ^2 + 2\alpha(\theta - \theta_\circ)$

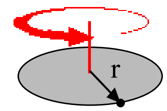


Question 9.1: A disk of radius 12 cm, initially at rest, begins rotating about its axis with a constant angular acceleration of 3.00 rad/s^2. After 5 s, what are

(a) the angular velocity of the disk, and

(b) the tangential and centripetal accelerations of a point on the perimeter of the disk?

(c) How many revolutions were made by the disk in those 5 s?



$r = 0.12\text{ m}: \omega_\circ = 0: \theta_\circ = 0: t = 5\text{ s}$

$\alpha = 3.00\text{ rad/s}^2.$

$\omega = ?: \theta = ?: a_t = ?: a_c = ?$

(a) $\omega = \omega_\circ + \alpha t = (3.00\text{ rad/s}^2)(5\text{ s}) = 15.0\text{ rad/s}.$

(b) $v_i = r_i\omega = (0.12\text{ m})(15.0\text{ rad/s}) = 1.80\text{ m/s}$ (*linear*).

- *tangential acceleration*: $a_t = r_i\alpha$

$$= (0.12\text{ m})(3.00\text{ rad/s}^2) = 0.36\text{ m/s}^2.$$

- *centripetal acceleration*: $a_c = r_i\omega^2$

$$= (0.12\text{ m})(15.0\text{ rad/s})^2 = 27.0\text{ m/s}^2.$$

(Check ... $a_c = \dfrac{v^2}{r} = \dfrac{(1.80\text{ m/s})^2}{0.12\text{ m}} = 27.0\text{ m/s}^2$.)

(c) $(\theta - \theta_\circ) = \omega_\circ t + \dfrac{1}{2}\alpha t^2$

$$= \frac{1}{2}(3.00\text{ rad/s}^2)(5\text{ s})^2 = 37.5\text{ rad},$$

$$\Rightarrow n = \frac{37.5\text{ rad}}{2\pi} = 5.97\text{ rev}.$$

In many applications a belt or chain is pulled from or wound onto a pulley or gear wheel ...

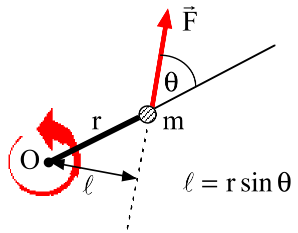
As the string (chain or belt) is removed (or added), its *instantaneous velocity* is the same as the *tangential velocity* at the rim of the wheel, providing there is no slip:

$$\text{i.e.,}\quad v_t = R\omega.$$

Also, under the same conditions, the *instantaneous acceleration* of the string is the same as the *tangential acceleration* at the rim of the wheel:

$$\text{i.e.,}\quad a_t = \frac{dv_t}{dt} = R\frac{d\omega}{dt} = R\alpha.$$

As we saw in chapter 4 that *force produces change in motion*. However, force does not always produce a change in *rotational motion*. It is *torque* that produces a change in rotational motion. Consider a mass m attached to a massless rigid rod that rotates around an axis O. The force F shown will cause the mass to rotate.



The *magnitude* of the torque due to a force F on m is:

$$\boxed{\tau = \ell F = (r\sin\theta)F,}$$

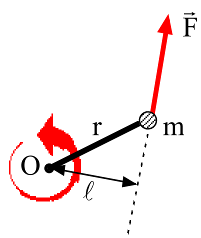
where ℓ is called the *lever arm*. The lever arm is the *perpendicular distance* from the axis of rotation (O) to *the line of action of the force*.

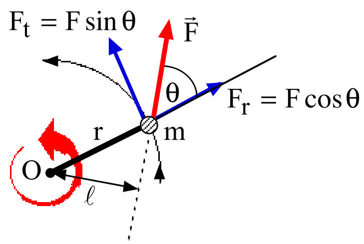
NOTE: if $\vec{F}$ passes through O, i.e., $\ell = 0$, then $\tau = 0$ and there will be no change in rotational motion.

$\underline{\underline{\text{Dimension}}}$: $\tau \Rightarrow [L]\frac{[M][L]}{[T]^2} = \frac{[M][L]^2}{[T]^2}$ (*vector*)

$\underline{\underline{\text{Unit}}}$: $N \cdot m$

Form the radial and tangential components of the force:

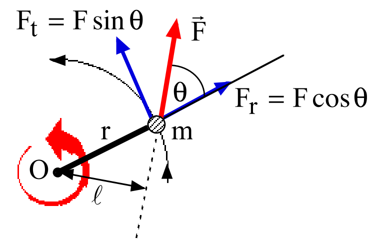




$$\text{Then } \tau = \ell F = (r\sin\theta)F = r(F\sin\theta)$$

$$\text{i.e., } \tau = rF_t.$$

Note: the <u>radial component</u> F_r*, which passes through the axis of rotation, does not produce a torque and therefore it does not produce rotation; only the <u>tangential component</u>* F_t *produces a torque that results in rotation.*



Newton's 2nd Law tells us that the tangential component of the force F_t produces a tangential acceleration a_t,

$$\text{i.e., } F_t = ma_t.$$

Therefore, the torque is

$$\tau = rF_t = mra_t,$$

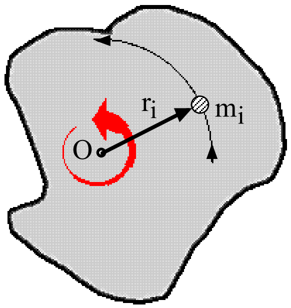
but, from earlier, the tangential (linear) acceleration is related to the angular acceleration α, viz: $a_t = r\alpha$.

$$\therefore \tau = mra_t = mr^2\alpha.$$

So torque (τ) produces angular acceleration (α).

A rigid object that rotates about a fixed axis can be thought of as a collection of small, individual elements of mass m_i that each move in a circular path of radius r_i, where r_i is measured from the axis of rotation.

From above, for each separate mass element, we have



$$\tau_i = m_i r_i^2 \alpha,$$

where τ_i is the net torque on the ith element. Summing over *all* elements, the *total* net torque on the object is:

$$\tau_{net} = \Sigma_i \tau_i = \Sigma_i m_i r_i^2 \alpha = \left(\Sigma_i m_i r_i^2\right)\alpha$$

$$= I\alpha,$$

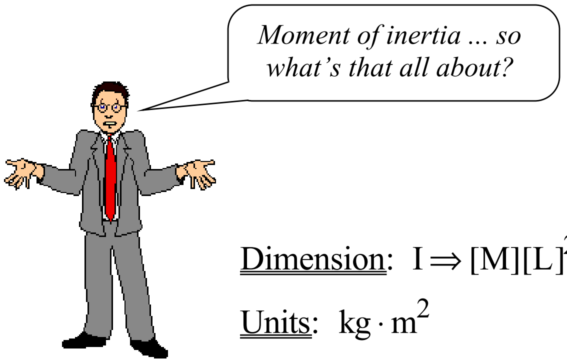
where $I = \Sigma_i m_i r_i^2$ is called the *MOMENT OF INERTIA*.

This is Newton's 2nd Law for rotation, i.e.,

$$\tau_{net} = I\alpha.$$

"*A net external torque acting on a body produces an angular acceleration,* α*, of that body given by* $I\alpha$*, where* I *is the moment of inertia.*"

(viz: $F_{net} = \Sigma_i F_i = ma$.)



Dimension: $I \Rightarrow [M][L]^2$ (*scalar*)

Units: $kg \cdot m^2$

Every object has a moment of inertia about an axis of rotation. Its value depends not simply on mass but on *how* the mass is distributed around that axis. For a *discrete collection* of i objects, the moment of inertia about the rotation axis is:

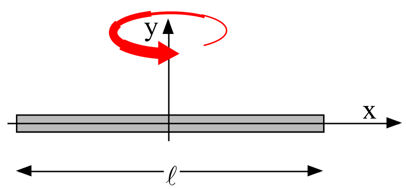
$$I = \Sigma_i I_i = \Sigma_i m_i r_i^2.$$

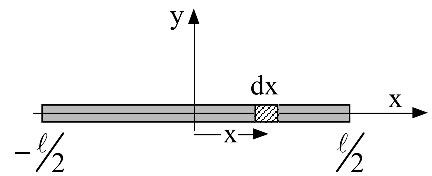
For a 'continuous' object:

$$I = \lim_{m_i \to 0} \Sigma_i m_i r_i^2 = \int r^2 dm,$$

where m is a function of r.

Question 9.2: Find the moment of inertia of a uniform thin rod of length ℓ and mass M rotating about an axis perpendicular to the rod and through its center.





A rod is a 'continuous' object, so the moment of inertia is

$$I = \int r^2 dm = \int_{x=-\ell/2}^{x=\ell/2} x^2 dm.$$

The mass per unit length of the rod is M/ℓ, so the mass of the small element of length dx is

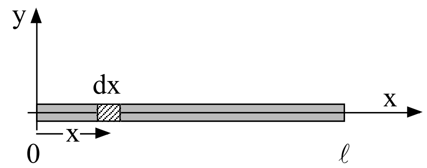
$$dm = (M/\ell)dx.$$

Substituting for dm, the integral becomes

$$I = \int_{x=-\ell/2}^{x=\ell/2} x^2 (M/\ell) dx = (M/\ell)\left[\frac{x^3}{3}\right]_{-\ell/2}^{\ell/2}$$

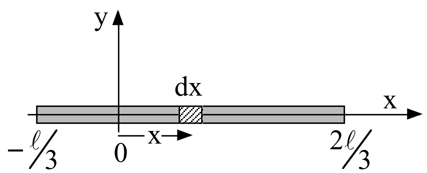
$$= (M/\ell)\left[\frac{\ell^3}{24} - \frac{(-\ell^3)}{24}\right] = \frac{1}{12}M\ell^2.$$

[1] Show for yourselves that the moment of inertia of a rod of mass M and length ℓ about one end is



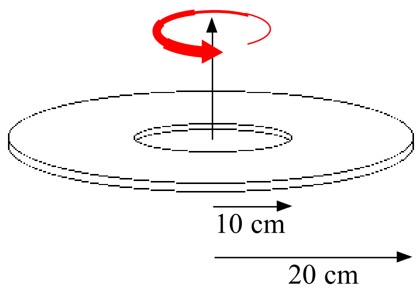
$$I = \frac{1}{3}M\ell^2.$$

[2] Show for yourselves that the moment of inertia of a rod of mass M and length ℓ about an axis one-third the distance from one end is

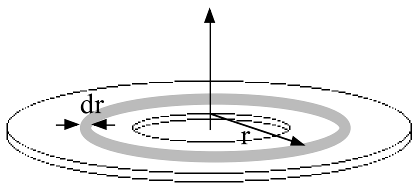


$$I = \frac{1}{9}M\ell^2.$$

<u>Question 9.3</u>: Find the moment of inertia of the circular disk shown below, rotating about an axis perpendicular to the plane and through its center. The mass of the disk is 1.50 kg.



The moment of inertia is given by $I = \int_{r_1}^{r_2} r^2 dm$. Consider a ring of radius r and width dr. If the mass of the object is M, the mass of the ring is



$$dm = M\frac{2\pi r dr}{\pi\left(r_2^2 - r_1^2\right)},$$

where r_1 and r_2 are the inner and outer radii of the object. Then, substituting for dm,

$$I = \int_{r_1}^{r_2} r^2 M\frac{2\pi r dr}{\pi\left(r_2^2 - r_1^2\right)} = \frac{2M}{\left(r_2^2 - r_1^2\right)}\int_{r_1}^{r_2} r^3 dr$$

$$= \frac{2M}{\left(r_2^2 - r_1^2\right)}\left[\frac{r^4}{4}\right]_{r_1}^{r_2} = \frac{M}{2\left(r_2^2 - r_1^2\right)}\left(r_2^4 - r_1^4\right)$$

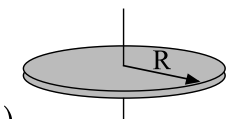
$$= \frac{M}{2\left(r_2^2 - r_1^2\right)}\left(r_2^2 - r_1^2\right)\left(r_2^2 + r_1^2\right) = \frac{1}{2}M\left(r_2^2 + r_1^2\right).$$

$$\therefore I = \frac{1}{2}M\left(r_2^2 + r_1^2\right)$$

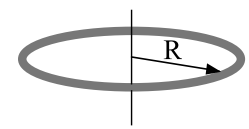
$$= \frac{1}{2}(1.50 \text{ kg})\left((0.20 \text{ m})^2 + (0.10 \text{ m})^2\right)$$

$$= 3.75 \times 10^{-2} \text{ kg} \cdot \text{m}^2.$$

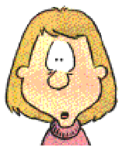
[1] If $r_1 = 0$, then $I_{disk} = \frac{1}{2}MR^2$, where R is the radius of the disk $(= r_2)$.



[2] For a thin hoop, $r_1 \approx r_2 = R$, then $I_{hoop} = \frac{1}{2}M\left(2R^2\right) = MR^2$.



Providing the thicknesses of the disks are uniform, the moments of inertia do not depend on thickness. So, these expressions also apply to cylinders and tubes.



Sure, but, what's the significance of I?

Remember, from chapter 4 ...

Mass ⇒ a measure of resistance to a change in *linear* motion, e.g., how difficult it is to start or stop *linear* motion.

Moment of Inertia ⇒ a measure of resistance to a change in *rotational* motion, i.e., how difficult it is to start or stop *rotational* motion.

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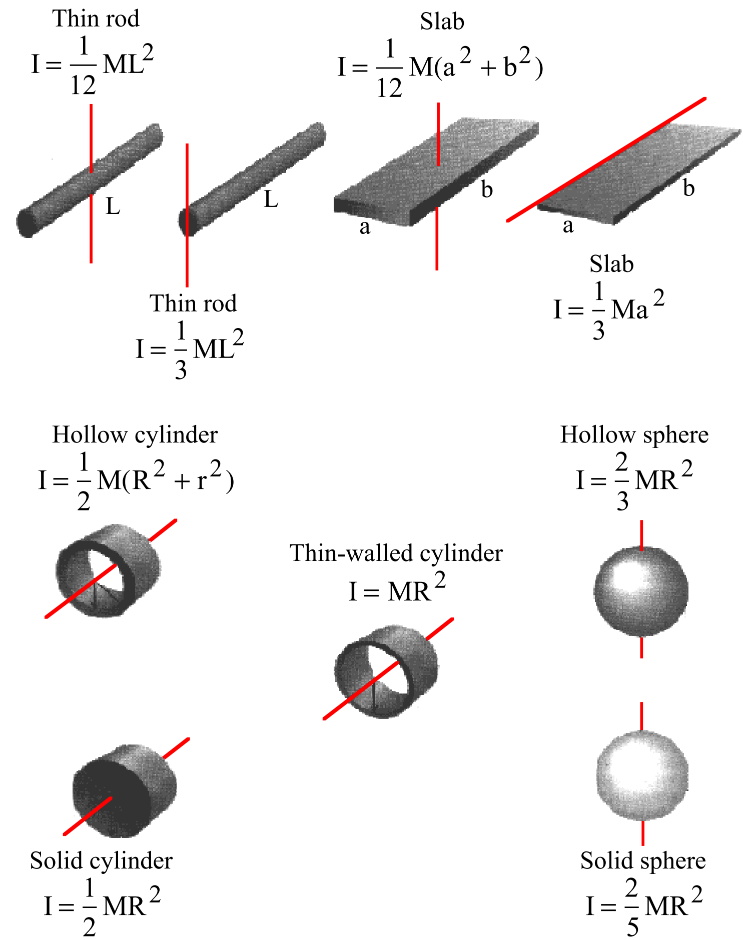
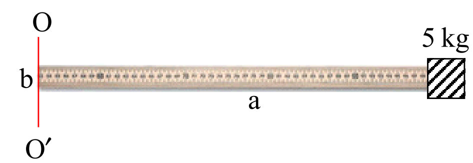
Same torques
Different moments of inertia

Values of the moment of inertia for "simple" shapes ...



Question 9.4: A 1 m ruler has a mass of 0.25 kg. A 5 kg mass is attached to the 100 cm end of the rule. What is its moment of inertia about the 0 cm end?



The combined moment of inertia is $I = I_{ruler} + I_{mass}$, where I_{ruler} and I_{mass} are calculated about $O - O'$.

Since a >> b we assume the meter rule is a rod (viz: Question 9.2) with $I = \frac{1}{3}Ma^2$ about $O - O'$.

$$\therefore I_{ruler} = \frac{1}{3}Ma^2 = \frac{1}{3}(0.25 \text{ kg})(1 \text{ m})^2$$

$$= 0.083 \text{ kg} \cdot \text{m}^2.$$

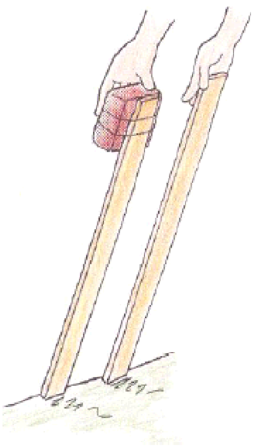
The moment of inertia of the 5 kg mass about $O - O'$ is

$$I_{mass} = ma^2 = (5.0 \text{ kg})(1 \text{ m})^2 = 5.0 \text{ kg} \cdot \text{m}^2.$$

Thus, the total moment of inertia of the ruler and mass is

$$I_{ruler} + I_{mass} = 5.083 \text{ kg} \cdot \text{m}^2.$$

DISCUSSION PROBLEM [9.2]:



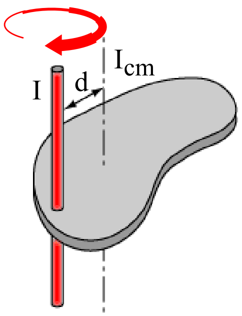
A pair of meter rulers are placed so that their lower ends are against a wall. One of the rulers has a large mass attached to its upper end. If the meter rulers are released at the same time and allowed to fall, which one hits the floor first?

A: The meter ruler with the mass.

B: The meter ruler without the mass.

C: They hit the floor at the same time.

Parallel axis theorem:



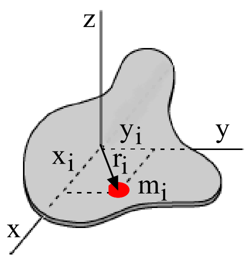
Usually, the moment of inertia is given for an axis that passes through the center of mass (cm) of the object. What if the object rotates about an axis parallel to the axis through the center of mass for which we don't know the moment of inertia? The moment of inertia about a *general (parallel) axis* is given by:

$$I = I_{cm} + Md^2$$

where I_{cm} is the moment of inertia about the center of mass, M is the mass of the object and d is the distance between the parallel axes.

Note: the two rotation axes must be parallel

Perpendicular axis theorem:



Consider a planar object (e.g., a thin disk or sheet) in the x,y plane. By definition, the moment of inertia about the z-axis (perpendicular to the plane of the object) is

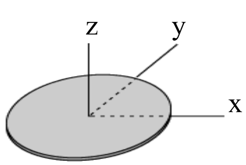
$$I_z = \Sigma_i m_i r_i^2 = \Sigma_i m_i (x_i^2 + y_i^2) = \Sigma_i m_i x_i^2 + \Sigma_i m_i y_i^2$$

But $\Sigma_i m_i x_i^2 = I_x$, i.e., the moment of inertia about x, and $\Sigma_i m_i y_i^2 = I_y$, i.e., the moment of inertia about y.

$$\therefore I_z = I_x + I_y.$$

Note: the object must be planar

Example of a disk:

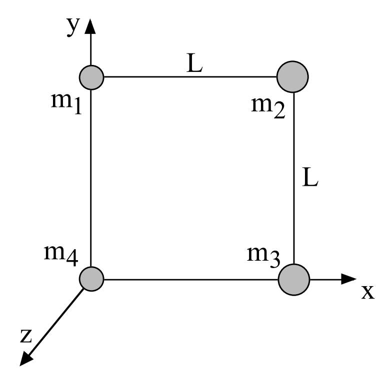


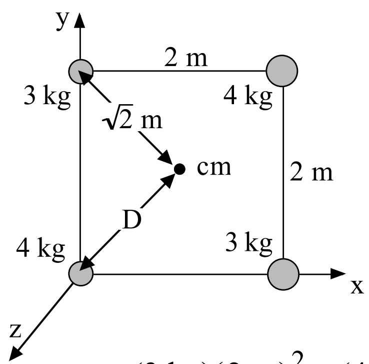
$$I_z = \frac{1}{2}MR^2.$$

But, by symmetry, $I_x = I_y$.

$$\therefore I_x = I_y = \frac{1}{4}MR^2.$$

<u>Question 9.5</u>: Four masses at the corners of a square with side length $L = 2$ m are conncted by massless rods. The masses are $m_1 = m_3 = 3$ kg and $m_2 = m_4 = 4$ kg. Find (a) the moment of inertia about the z-axis, (b) the moment of inertia about an axis that is perpendicular to the plane of the ensemble and passes through the center of mass of the system, (c) the moment of inertia about the x-axis, which passes through m_3 and m_4.





(a) Since we are dealing with discrete masses

$$I = \Sigma_i I_i = \Sigma_i m_i r_i^2.$$

Moment of inertia about the z-axis:

$$I_z = \Sigma_i m_i r_i^2$$

$$= (3\text{ kg})(2\text{ m})^2 + (4\text{ kg})(2\sqrt{2}\text{ m})^2$$

$$+(3\text{ kg})(2\text{ m})^2 + (4\text{ kg})(0) = 56\text{ kg}\cdot\text{m}^2.$$

(b) By symmetry, the center of mass is at the center of the square.

$$\therefore I_{cm} = \Sigma_i m_i r_i^2 = (3\text{ kg})(\sqrt{2}\text{ m})^2 + (4\text{ kg})(\sqrt{2}\text{ m})^2$$

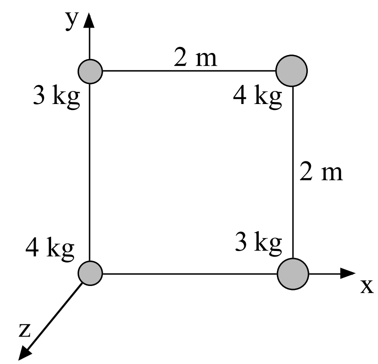
$$+(3\text{ kg})(\sqrt{2}\text{ m})^2 + (4\text{ kg})(\sqrt{2}\text{ m})^2 = 28\text{ kg}\cdot\text{m}^2.$$

- Check, using the parallel axis-theorem

$$I_z = I_{cm} + MD^2$$

$$\therefore I_{cm} = I_z - MD^2 = (56\text{ kg}\cdot\text{m}^2) - (14\text{ kg})(\sqrt{2}\text{ m})^2$$

$$= 28\text{ kg}\cdot\text{m}^2.$$



(c) Since the ensemble is planar and confined to the x,y plane, we can use the perpendicular axis theorem, i.e.,

$$I_z = I_x + I_y.$$

But, by symmetry, $I_x = I_y$.

$$\therefore I_x = \frac{1}{2} I_z = \frac{1}{2}\left(56\text{ kg}\cdot\text{m}^2\right)$$

$$= 28\text{ kg}\cdot\text{m}^2.$$

Check:

$$I_x = \Sigma_i m_i r_i^2 = (3\text{ kg})(2\text{ m})^2 + (4\text{ kg})(2\text{ m})^2$$

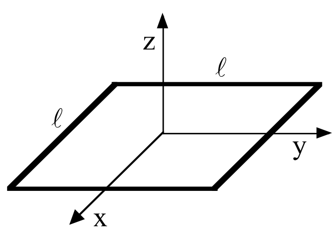
$$= 28\text{ kg}\cdot\text{m}^2.$$

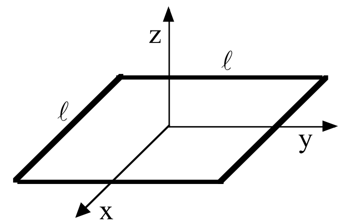
Question 9.6: Four thin rods, each of length ℓ and mass M, are arranged to form a square, in the x,y plane, as shown. If the origin of the axes is at the center of the square,

(a) using the parallel axis theorem, show that

$$I_z = \frac{4}{3}M\ell^2.$$

(b) Hence find I_x and I_y.





(a) Using the parallel axis theorem, we have for <u>each</u> rod

$$I_z = I_{cm} + Md^2,$$

where I_{cm} is the moment of inertia through the center of mass of each rod and $d = \ell/2$.

$$\therefore I_z(\text{total}) = 4\left(\frac{1}{12}m\ell^2 + m\frac{\ell^2}{4}\right) = \frac{4}{3}m\ell^2.$$

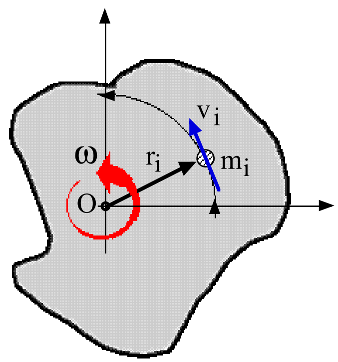
(b) Since the object is planar we can use the perpendicular axis theorem, i.e.,

$$I_z = I_x + I_y.$$

But because of symmetry $I_x = I_y$.

$$\therefore I_x = I_y = \frac{1}{2}I_z = \frac{2}{3}m\ell^2.$$

We saw in chapter 6 that a linearly moving object has *translational* kinetic energy ... an object rotating about an axis has *rotational* kinetic energy ...



If a rigid object is rotating with angular velocity ω, the kinetic energy of the i[th] element is:

$$K_i = \frac{1}{2} m_i v_i^2$$

$$= \frac{1}{2} m_i r_i^2 \omega^2, \text{ since } v_i = r_i \omega.$$

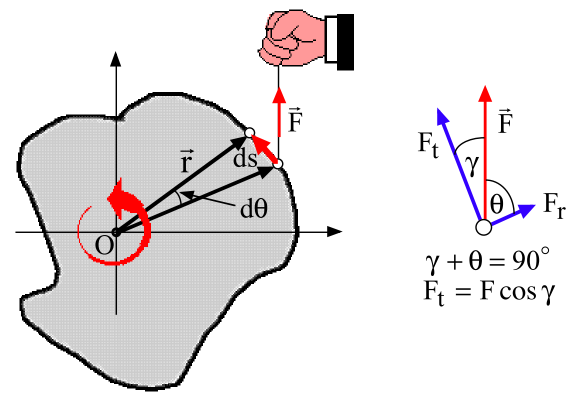
So, the *total* *rotational* kinetic energy is:

$$K = \Sigma_i K_i = \frac{1}{2} \Sigma_i m_i r_i^2 \omega^2$$

$$= \frac{1}{2} I \omega^2.$$

* $K_{rot} = \frac{1}{2} I\omega^2$ is the analog of $K_{trans} = \frac{1}{2} mv^2$.

A *torque* is required to rotate (or slow down) an object ... but torque involves *force* ...



When force is applied over a distance, work is done, given by:

$$dW = \vec{F} \bullet d\vec{s} = F.ds\cos\gamma = F\cos\gamma.ds$$

$$= F_t.ds = F_t.rd\theta = \tau.d\theta \qquad \text{(J or N.m)}.$$

Power is the rate at which the torque does work,

$$\text{i.e.,} \quad P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega \qquad \text{(watts)}.$$

Note: P is the *instantaneous* power.

- $dW = \tau.d\theta$ is the analog of $dW = F.ds$.
- $P = \tau\omega$ is the analog of $P = Fv$.

Question 9.7: The 3.9 liter V-8 engine fitted to a 488GTB Ferrari develops 560 ft · lb of torque at 3000 rev/min. What is the power developed by the engine with these parameters? (1 ft · lb = 1.36 N · m.)

From earlier, power $P = \tau\omega$. Convert the torque to $N \cdot m$ and angular velocity to rad/s,

$$\text{i.e.,} \quad \tau = 1.36 \times 560 = 762\ N \cdot m,$$

and

$$\omega = \frac{2\pi(3700\ \text{rev/min})}{60\ \text{s/min}} = 387.5\ \text{rad/s}.$$

$$\therefore P = (762\ N \cdot m)(387.5\ \text{rad/s}) = 2.95 \times 10^5\ \text{watts}.$$

But 746 watts = 1 HP

$$\therefore P = \frac{2.95 \times 10^5\ \text{watts}}{746\ \text{watts/HP}} = 396\ \text{HP}.$$

Question 9.8: An electric motor exerts a constant torque of $10.0\ \mathrm{N \cdot m}$ to the shaft of a grindstone with mass $16.0\ \mathrm{kg}$ and radius $0.50\ \mathrm{m}$. If the system starts from rest, find

(a) the rotational kinetic energy of the grindstone after 8.0 s,

(b) the work done by the motor during this time, and

(c) the average power delivered by the motor.

(a) To find the rotational kinetic energy we need to know the angular velocity (ω). Treating the grindstone as a solid disk

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(16.0\ \mathrm{kg})(0.50\ \mathrm{m})^2 = 2.0\ \mathrm{kg \cdot m^2}.$$

Since $\tau = I\alpha$, $\alpha = \frac{\tau}{I} = \frac{10\ \mathrm{N \cdot m}}{2\ \mathrm{kg \cdot m^2}} = 5.0\ \mathrm{rad/s^2}$.

The grindstone starts from rest ($\omega_\circ = 0$) so

$$\omega = \alpha t = (5.0\ \mathrm{rad/s^2})(8.0\ \mathrm{s}) = 40\ \mathrm{rad/s}.$$

$$\therefore K = \frac{1}{2}I\omega^2 = \frac{1}{2}(2\ \mathrm{kg \cdot m^2})(40\ \mathrm{rad/s})^2 = 1600\ \mathrm{J}.$$

(b) There are two ways to determine the work done by the motor.

(i) By the work-kinetic energy theorem we would expect the motor to have done 1600 J of work.

(ii) We can use the expression $W = \tau\theta$, but we need to find θ, i.e., the angle through which the grindstone has turned in 8.0 s.

$$\theta = \omega_\circ t + \frac{1}{2}\alpha t^2 = \frac{1}{2}(5.0\ \mathrm{rad/s^2})(8.0\ \mathrm{s})^2 = 160\ \mathrm{rad}$$

$$\therefore W = \tau\theta = (10.0 \text{ N}\cdot\text{m})(160 \text{ rad}) = 1600 \text{ J}.$$

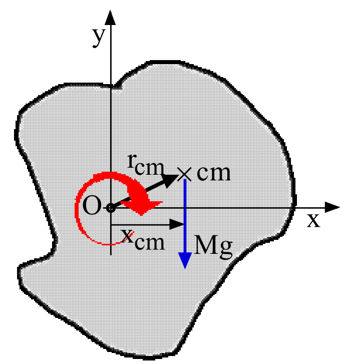
(c) The *average* power is the total work done divided by the total time interval, i.e.,

$$P_{av} = \frac{\Delta W}{\Delta t} = \frac{1600 \text{ J}}{8.0 \text{ s}} = 200 \text{ W}.$$

Note: the expression $P = \tau\omega$ we derived earlier is actually the *instantaneous* power. We cannot use that expression here as ω is not constant.

The instantaneous power actually increases linearly from zero at $t = 0$ to $(10.0 \text{ N}\cdot\text{m}) \times (40 \text{ rad/s}) = 400 \text{ W}$ at $t = 8.0 \text{ s}$.

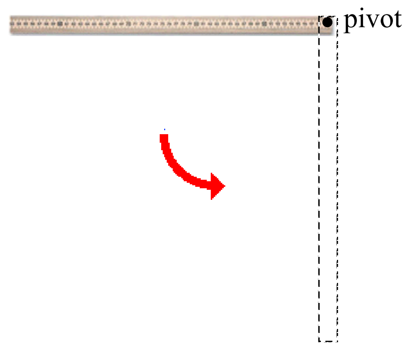
If a rigid object is suspended from an arbitrary point O and is free to rotate about that point, it will turn until the center of mass is vertically beneath the suspension point.



If the y-direction is vertical and the suspension point is not at the center of mass, the object will experience a net torque given by

$$\tau = Mgx_{cm},$$

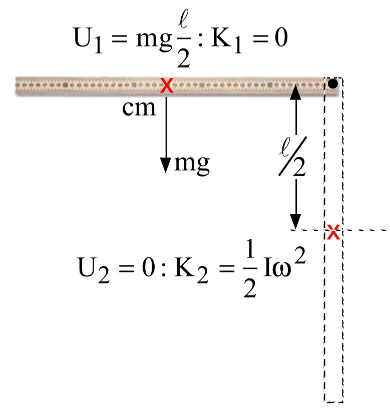
where x_{cm} is the x component of the center of mass. Therefore, the object will rotate *until* $x_{cm} = 0$, i.e., the suspension point is direction above the center of mass.



Question 9.9: The "zero" end of a 1 m ruler of mass 0.25 kg is attached to a frictionless pivot, the other end is free to rotate in the vertical plane. If the ruler is released from rest in the horizontal position, what is

(a) the initial acceleration of the 100 cm end of the ruler, and

(b) the linear speed of the 100 cm end as the ruler passes through the vertical?



(a) Use Newton's 2nd Law for rotation, i.e., $\tau = I\alpha$, where $\tau = mg\,\ell/2$ and $I = (1/3)m\ell^2$. Then

$$\alpha = \tau/I = 3g/2\ell,$$

which is the <u>initial</u> angular acceleration of the center of mass. The linear acceleration of the 100 cm end is then

$$a = \ell\alpha = 3g/2 = 14.7\ \text{m/s}^2,$$

which is greater than g! Also note, the torque τ varies as the ruler swings down.

(b) To find the speed of the 100 cm end as it passes the vertical we use the conservation of mechanical energy,

$$\text{i.e., } U_1 + K_1 = U_2 + K_2.$$

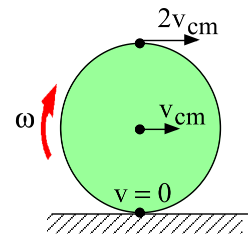
Taking the zero of the gravitational potential energy at the point where the center of mass is at its lowest point, then

$$mg\frac{\ell}{2} + 0 = 0 + \frac{1}{2}I\omega^2.$$

$$\therefore \omega = \sqrt{\frac{mg\ell}{I}} = \sqrt{\frac{3g}{\ell}}.$$

The linear velocity of the 100 cm end as it passes the vertical is then

$$v = \ell\omega = \sqrt{3g\ell} = \sqrt{3(9.81 \text{ m/s}^2)(1.0 \text{ m})}$$
$$= 5.42 \text{ m/s}.$$



Consider a ball, cylinder, wheel or disc) rolling on a surface without slipping.

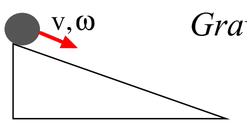
- The point in contact with the surface has *zero* instantaneous velocity relative to the surface.
- The velocity of the cm is v_{cm} $(= r\omega = v)$,
- The velocity of a point at the top is $2v_{cm}$ $(= 2v)$.

Since the object has the same linear velocity as the cm, i.e., v, the *total kinetic energy* is:

$$K_{tot} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

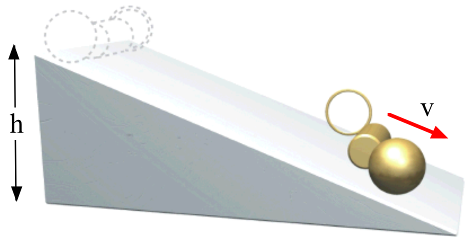
translational + *rotational*

where $\omega = v/r$. So, as a ball rolls down a hill ...



Gravitational potential energy ⇒ *translational energy* + *rotational energy*

Consider rolling a sphere, cylinder and a hoop down an incline. Do they have the same velocity at the bottom?



For each object, conservation of energy gives:

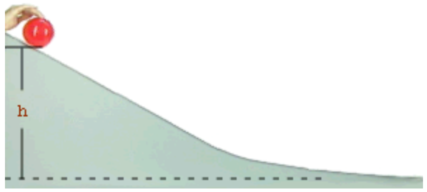
$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2,$$

and with *no slip* $v = R\omega$, where R is the radius of the object. After manipulation we find:

$$v^2 = \frac{2gh}{1+\left(I/mR^2\right)}.$$

∴ For *maximum velocity*: $I \Rightarrow$ *smallest value*.

So, in a race between objects rolling down a slope, the order would be (1) *sphere*, (2) *cylinder*, (3) *hoop*, and is <u>*completely independent*</u> of m and R!



RMA07VD2.MOV

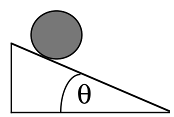
$$v^2 = \frac{2gh}{1+\left(I/mR^2\right)}$$

For a solid sphere $I = \frac{2}{5}mR^2$.

$$\therefore v^2 = \frac{2gh}{1+2/5}, \text{ i.e., } v = \sqrt{\frac{10gh}{7}}.$$

Since this is independent of m and R, a bowling ball and a pool ball would have the same speeds at the bottom of an incline, despite their different masses and radii!

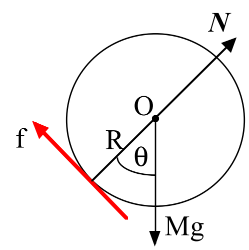
Question 9.10: A uniform solid ball of mass M and radius R rolls without slipping down an incline at an angle θ to the horizontal. Find (a) the frictional force acting at the point of contact with the surface, and (b) the acceleration of the center of mass of the ball, in terms of M, R, g and θ.



NOTE: there must be friction at the point of contact otherwise the ball would slide down the incline!

(a) Draw the free body diagram of all the forces acting on the ball. Use Newton's 2nd Law down the incline:

$$Mg\sin\theta - f = Ma_{cm} \quad \ldots \quad (i)$$



where a_{cm} is the acceleration of the center of mass. To get an expression for a_{cm}, we take torques about O, the center of the ball. (**Note**: the normal force $\boldsymbol{N}$ and the weight force Mg do not contribute to the torque as their lines of action pass through O.)

$$\therefore \tau = fR = I\alpha, \quad \text{i.e.,} \quad \alpha = \frac{fR}{I},$$

where I is the moment of inertia of the ball and α its angular acceleration. With no slip

$$a_{cm} = R\alpha = \frac{fR^2}{I} \quad \ldots \quad \ldots \quad (ii)$$

Substituting for a_{cm} in equation (i), we get

$$Mg\sin\theta - f = M\frac{fR^2}{I} = \frac{MfR^2}{\left(\frac{2}{5}MR^2\right)} = \frac{5}{2}f.$$

$$\therefore Mg\sin\theta = \frac{5}{2}f + f = \frac{7}{2}f,$$

$$\text{i.e., } f = \frac{2}{7}Mg\sin\theta.$$

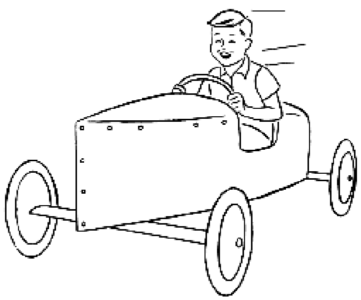
Note that this is a *static* frictional force (because there is no slip).

(b) Substituting for f in equation (ii), we get

$$a_{cm} = \frac{fR^2}{I} = \left(\frac{2}{7}Mg\sin\theta\right)\frac{R^2}{\left(\frac{2}{5}MR^2\right)} = \frac{5}{7}g\sin\theta.$$

You can show yourselves that if we put $I = \beta MR^2$, where $\beta = 1$ for a hoop, $\beta = 1/2$ for a disk, etc., then a general expression for the static frictional force acting on an object rolling down an incline, with no slip, and the acceleration of the center of mass are:

$$f = \frac{Mg\sin\theta}{1+\beta^{-1}} \quad \text{and} \quad a_{cm} = \frac{g\sin\theta}{1+\beta}.$$



<u>Question 9.11</u>: Suppose you can choose wheels of *any* design for a soapbox derby race car. If the total weight of the vehicle is fixed, which type of wheel design should you choose if you want to have the best chance to win the race?

The total mechanical energy of the car is:

translational kinetic energy + *rotational kinetic energy.*

Conservation of energy gives:

$$Mgh = K_{trans} + K_{rot} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2,$$

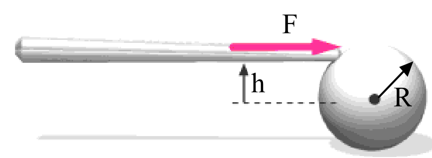
where M is the total mass of the car and I is the moment of inertia of the wheels. With four wheels of radius r, for example, we have, using the no-slip condition,

$$K_{rot} = 4 \times \frac{1}{2}I\omega^2 = 2(\beta mr^2)\left(v/r\right)^2 = 2\beta mv^2,$$

$$\therefore Mgh = \frac{1}{2}Mv^2 + 2\beta mv^2,$$

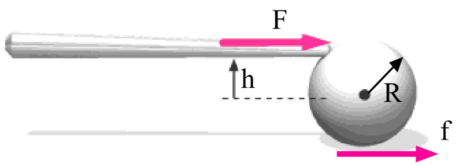
$$\text{i.e.,} \qquad v^2 = \frac{2Mgh}{(M + 4\beta m)}.$$

So, for *maximum* speed (v) at the bottom of the hill (and greater average speed) with M fixed, we want m and β *to be as small as possible*. **Note**: that the result does not depend on the radius of the wheels (r). So, solid wheels are a good choice with the mass concentrated as close to the axle as possible.



Question 9.12: A cue ball is struck by a horizontal cue a distance h above the center of the ball. If the cue ball is to roll without slipping, what is h? Express your answer in terms of the radius R of the ball.

You can assume that the frictional force of the table on the ball is negligible compared with the applied force F.



The net torque about the center is $\tau = Fh - fR$.

From earlier, if there is no-slip then $v_{cm} = R\omega$,

$$\text{i.e., } a_{cm} = \frac{dv_{cm}}{dt} = R\frac{d\omega}{dt} = R\alpha.$$

Using Newton's 2nd Law $F + f = ma_{cm}$.

If $F >> f$, then $F \approx ma_{cm}$ and $\tau \approx Fh = I\alpha$.

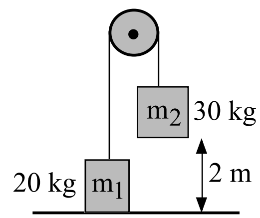
$$\therefore \frac{F}{m} = a_{cm} = R\alpha = R\frac{Fh}{I}, \text{ i.e., } h = \frac{I}{mR}.$$

For a solid sphere $I = \frac{2}{5}mR^2$.

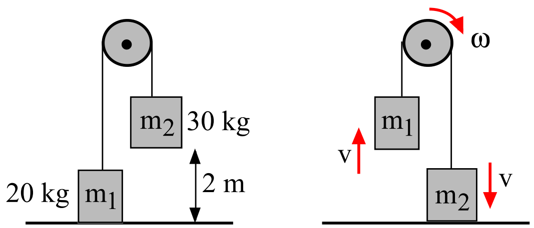
$$\therefore h = \frac{2}{5}R.$$

- $h > \frac{2}{5}R \Rightarrow$ *top spin*
- $h < \frac{2}{5}R \Rightarrow$ *back spin*

"SLIP"



<u>Question 9.13</u>: The 30 kg mass, shown above, is released from rest, from a distance of 2 m above the ground. Modeling the pulley as a uniform disk with a radius of 10 cm and mass 5 kg, find (a) the speed of the 30 kg mass <u>just before</u> it strikes the ground, (b) the angular velocity of the pulley at that instant, (c) the tensions in the two strings, and (d) the time it takes for the 30 kg block to reach the ground. Assume the bearings in the pulley are frictionless and there is no slip between the string and the pulley.



We have both translational and rotational motion. The moment of inertia of the pulley is:

$$I = \frac{1}{2}mR^2 = \frac{1}{2}(5\text{ kg})(0.10\text{ m})^2 = 0.025\text{ kg}\cdot\text{m}^2.$$

(a) Using conservation of energy, as m_2 hits the ground

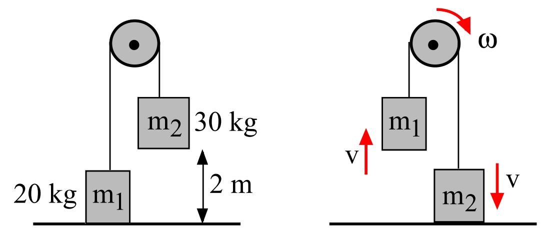
$$m_2gh = \frac{1}{2}m_1v^2 + m_1gh + \frac{1}{2}m_2v^2 + \frac{1}{2}I\omega^2.$$

The angular velocity of the pulley $\omega = \frac{v}{R}$, so

$$(30\text{ kg})(9.81\text{ m/s}^2)(2\text{ m}) = \frac{1}{2}(20\text{ kg})v^2$$

$$+(20\text{ kg})(9.81\text{ m/s}^2)(2\text{ m}) + \frac{1}{2}(30\text{ kg})v^2$$

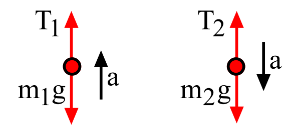
$$+\frac{1}{2}(0.025\text{ kg}\cdot\text{m}^2)\left(\frac{v}{(0.10\text{ m})}\right)^2,$$



i.e., $589 = 10v^2 + 392 + 15v^2 + 1.3v^2$

$$\therefore v = \sqrt{\frac{197}{26.3}} = 2.74\text{ m/s}.$$

(b) $\omega = \frac{v}{R} = \frac{2.74}{0.10} = 27.4\text{ rad/s}.$

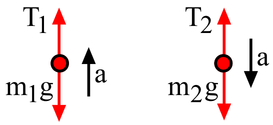
(c) 

Draw the free-body diagrams for the masses.

What is the upward acceleration of m_1?

$$v^2 = \cancel{v_\circ^2} + 2ah.$$

$$\therefore a = \frac{v^2}{2h} = \frac{(2.74\text{ m/s})^2}{2(2\text{ m})} = 1.88\text{ m/s}^2.$$



Then $T_1 - m_1 g = m_1 a$,

i.e., $T_1 = m_1(g + a) = 234\ \text{N}$.

Also $T_2 - m_2 g = m_2(-a)$,

i.e., $T_2 = m_2(g - a) = 238\ \text{N}$.

(d) $(y - y_\circ) = h = \frac{1}{2}at^2$.

$$\therefore t = \sqrt{\frac{2h}{a}} = \sqrt{\frac{2(2\ \text{m})}{1.88\ \text{m/s}^2}} = 1.46\ \text{s}.$$