CHAPTER 2

MOTION IN ONE DIMENSION

- Why objects move

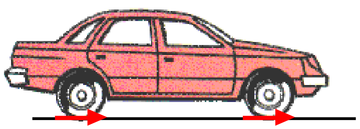
- Displacement and velocity
    - Average velocity
    - Average speed
    - Relative velocity
    - Instantaneous velocity
    - Reading displacement-time graphs

- Acceleration
    - Average acceleration
    - Instantaneous acceleration
    - Reading velocity-time graphs
    - Free fall

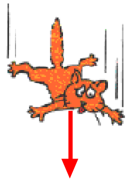
- *Equations of motion using calculus (You study)*

This chapter is about *MOTION* ... so, *why* do objects move?

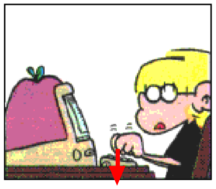
It is *FORCE* that causes a change in motion (starting, stopping, going round corners, etc).



It is the force of the road on the tires that cause a car to move!



It is the force of gravity downward that makes the cat fall! (Its weight.)



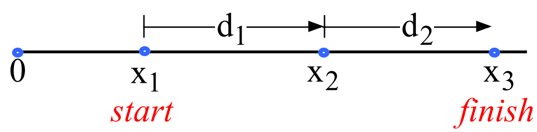
It is the force exerted by the typists fingers that makes the keys move!

For the moment we won't worry about what *causes* motion but the rules that tell us *how* objects move.

Displacement and distance traveled

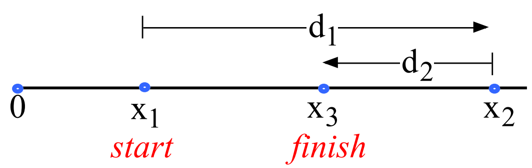
The *displacement* is $(x_2 - x_1)$ $(= \Delta x)$.

The *distance traveled* is $(x_2 - x_1)$ also.



The *displacement* is $(x_3 - x_1)$.

The *distance traveled* is $(x_3 - x_1) = (d_1 + d_2)$.

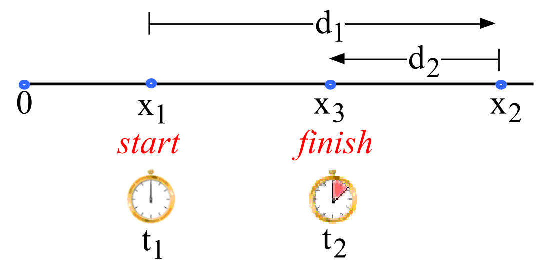


The *displacement* is $(x_3 - x_1)$.

The *distance traveled* is $(d_1 + d_2)$.

- distance traveled $\geq$ displacement
- displacement can be positive *or* negative.

Velocity and average speed



$$\Delta t = (t_2 - t_1)$$

- Average velocity $v_{av} = \dfrac{\text{displacement}}{\text{time}} = \dfrac{\Delta x}{\Delta t}$.
- Average speed $= \dfrac{\text{distance}}{\text{time}} = \dfrac{(d_1 + d_2)}{\Delta t}$.

But, in this example, $(d_1 + d_2) > (x_3 - x_1)$, so

average speed > average velocity

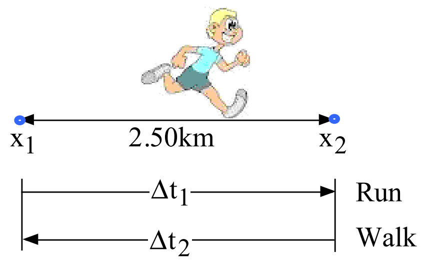
Dimensions: velocity and speed $\Rightarrow \dfrac{[L]}{[T]}$

Units: m/s, km/h, ft/s, mi/h.

DISCUSSION PROBLEM [2.1]:

Estimate the average velocity of the winning car at the Indianapolis 500 ... (total distance 500 miles in about $2\frac{1}{2}$ hours).

Question 2.1: A runner runs for 2.5 km in a straight line in 9.0 min and takes 30.0 mins to walk back to the starting point. (a) What is the runner's velocity for the first 9.0 min? (b) What is the average velocity for the time spent walking? (c) What is the average velocity for the whole trip? (d) What is the average speed for the whole trip? (e) Plot a position versus time graph for this problem.



(a) From $x_1 \to x_2$ (run): $v_{av} = \dfrac{(x_2 - x_1)}{\Delta t_1}$

$$= \frac{2.5 \text{ km}}{9 \text{ min}} = \frac{2500 \text{ m}}{9 \times 60 \text{ s}} = 4.63 \text{ m/s}.$$

(b) From $x_2 \to x_1$ (walk): $v_{av} = \dfrac{(x_1 - x_2)}{\Delta t_2}$

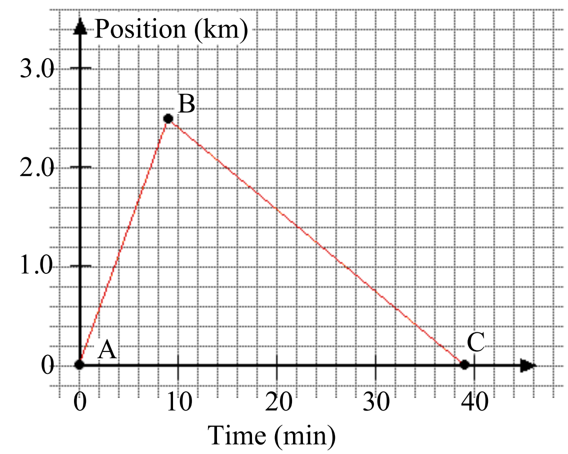
$$= \frac{-2.5 \text{ km}}{30 \text{ min}} = \frac{-2500 \text{ m}}{30 \times 60 \text{ s}} = -1.39 \text{ m/s}.$$

(c) $v_{av} = \dfrac{\text{displacement}}{\text{total time}} = 0.$

(d) av. speed $= \dfrac{\text{distance}}{\text{total time}} = \dfrac{5000 \text{ m}}{39 \text{ min}} = \dfrac{5000 \text{ m}}{2340 \text{ s}}$

$$= 2.14 \text{ m/s}.$$

(e) Position versus time graph for this problem; note that the displacement is the runner's position relative to the starting point.



• The *slope* of AB is $\dfrac{2.5 \text{ km}}{9 \text{ min}} \Rightarrow 4.63$ m/s, i.e., the *average velocity* for the first 9 min.

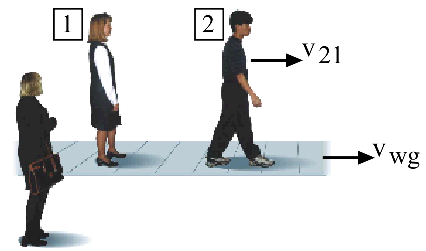
• The *slope* of BC is $\dfrac{-2.5 \text{ km}}{30 \text{ min}} \Rightarrow -1.39$ m/s, i.e., the *average velocity* while walking.

Thus, velocity is obtained from the slope of a displacement *vs* time plot.

DISCUSSION PROBLEM [2.2]:

You lean out of a window and throw a ball straight down. It bounces off the sidewalk and returns to your hand 2.0 s later. (a) What is the average velocity of the ball? (b) What is its average speed?

Relative velocity



If the velocity of the walkway relative to the ground (i.e., the stationary observer) is v_{wg}, and the velocity of $\boxed{2}$ with respect to $\boxed{1}$ is v_{21}, then the velocity of $\boxed{2}$ with respect to a stationary observer (the *relative velocity*) is:

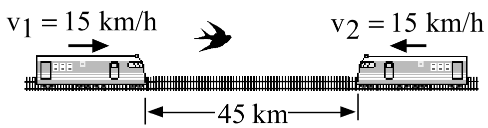
$$(v_{wg} + v_{21})$$

Note, if $\boxed{2}$ is walking in the *opposite* direction, his velocity relative to the observer is

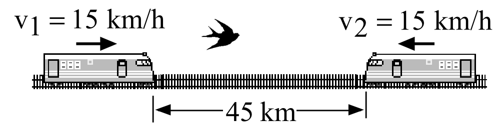
$$(v_{wg} - v_{21})$$

In that case, what "happens" if $|v_{wg}| = |v_{21}|$?

This is called a *Galilean transformation*.



Question 2.2: Two trains, 45 km apart and traveling at 15 km/h, are approaching each other on parallel tracks. A bird flies back and forth at 20 km/h between the trains until the trains pass each other. (a) For how long does the bird fly? (b) How far does the bird fly?



Av. speed of bird $\Rightarrow \dfrac{\text{distance flown}}{\text{time}} = \dfrac{s}{t} = 20\ \text{km/h}.$

$$\therefore s = 20 \times t.$$

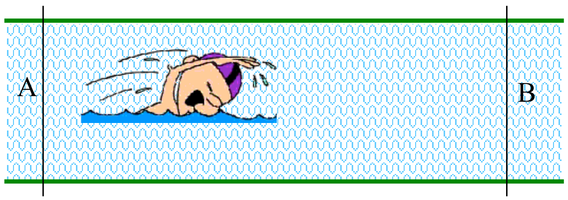
But what is t? It's the time before the trains meet! Imagine you are sitting on the front of the left hand train, then the relative speed of the right hand train, i.e., relative to you, is

$$15\ \text{km/h} + 15\ \text{km/h} = 30\ \text{km/h}.$$

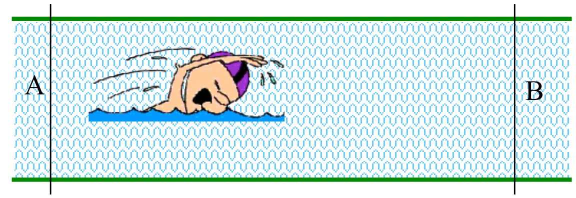
So, the time taken until the trains pass is

$$\frac{\text{distance they travel}}{\text{relative speed}} = \frac{45\ \text{km}}{30\ \text{km/h}} = 1.50\ \text{h}.$$

$$\therefore s = 20 \times 1.5 = 30\ \text{km}.$$



Question 2.3: A river flows from left to right with a velocity of 0.20 m/s with respect to the bank. If you swim with a velocity of 1.00 m/s relative to the water, does it take you *more time*, *less time*, or the *same time* to swim from $A \rightarrow B \rightarrow A$ than if the river was not flowing at all?



When you swim from $A \rightarrow B$ your velocity relative to the bank is $v_{AB} = 1.00 + 0.20 = 1.20$ m/s. When you swim from $B \rightarrow A$, your velocity relative to the bank is $v_{BA} = 1.00 - 0.20 = 0.80$ m/s. If the distance AB is d, then the times taken (in seconds) are:

$$t_{AB} = \frac{d}{1.20} \quad \text{and} \quad t_{BA} = \frac{d}{0.80},$$

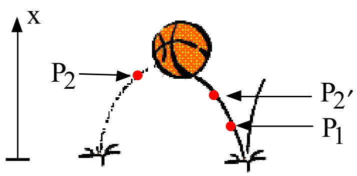
so the total time is

$$t = \frac{d}{0.80} + \frac{d}{1.20} = \frac{(1.20 + 0.80)d}{0.80 \times 1.20} = \frac{2.00d}{0.96} = 2.08d.$$

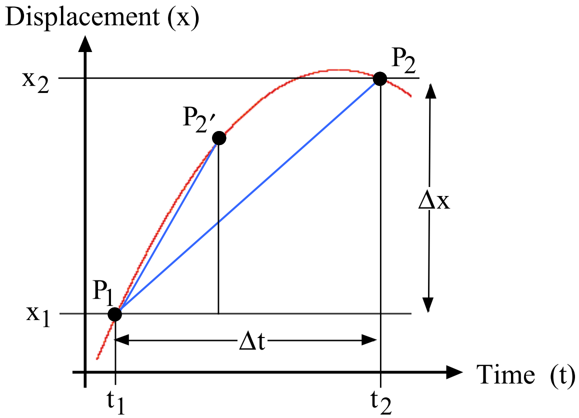
But, if the river was *not* flowing, then

$$t = \frac{d}{1.00} + \frac{d}{1.00} = 2.00d.$$

Clearly, it takes **more time** if the river is flowing!



However, velocities often change ... ! Think of a bouncing ball:



The *average velocity* from P_1 to P_2 is:

$$v_{1\to 2} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}.$$

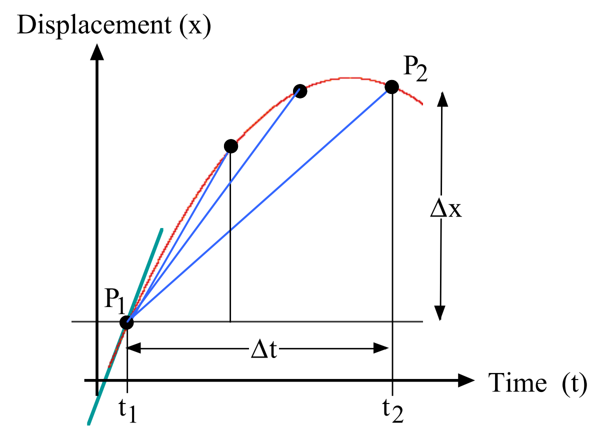
Note that this is less than:

$$v_{1\to 2'}$$

so the *average* velocity continually changes.

Instantaneous velocity

What is the *instantaneous velocity* at the point P?



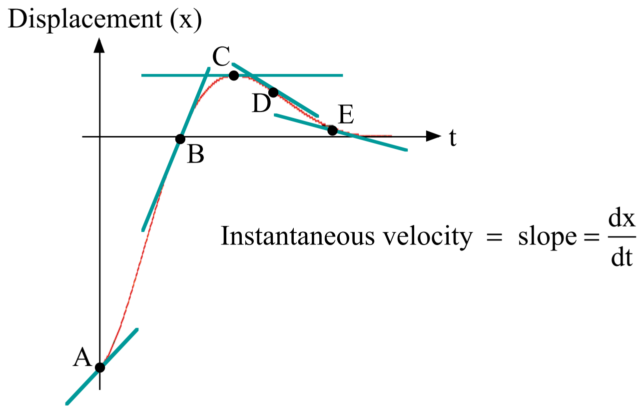
We define the instantaneous velocity at point $P_1\,(x_1, t_1)$ as the slope of the tangent at the point P_1.

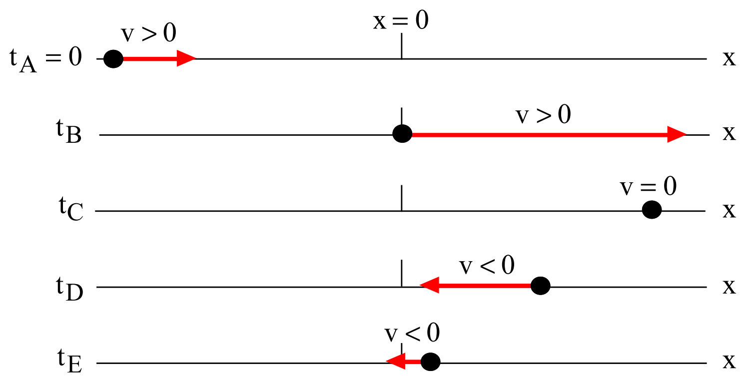
$$\therefore v(t) = \text{Limit}_{\Delta t\to 0}\frac{\Delta x}{\Delta t} = \left[\frac{dx}{dt}\right]_{P_1},$$

i.e., the *first derivative of* x *with respect to* t at the point P_1. Note that normally the instantaneous velocity (the slope) is a function of time, i.e.,

$$v \Rightarrow v(t).$$

“Reading” a *displacement - time* graph:

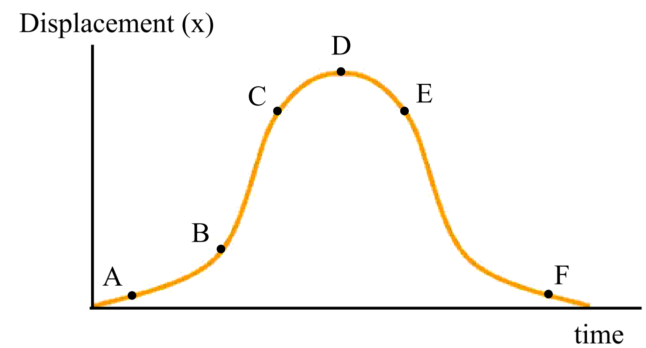




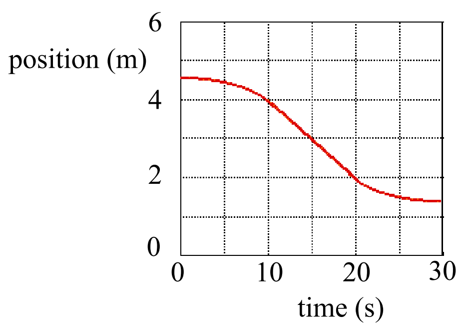
The length of the red line shows the *magnitude* (i.e., size) of the velocity

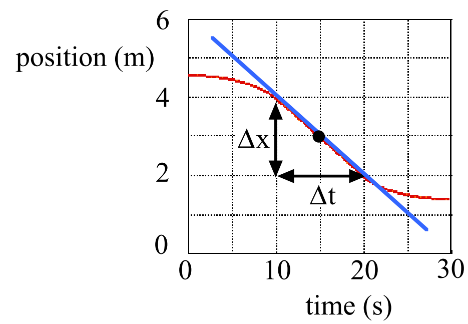
DISCUSSION PROBLEM [2.3]:

A physics professor is walking to campus when she realizes she’s forgotten copies of a test and she returns home. Her displacement as a function of time is shown below. At which point(s) is her velocity: (a) zero? (b) constant and positive? (c) constant and negative? (d) increasing in magnitude? (e) decreasing in magnitude?



Question 2.4: The plot shows the variation of the position of a car with time. What was the velocity when $t = 15$ s?



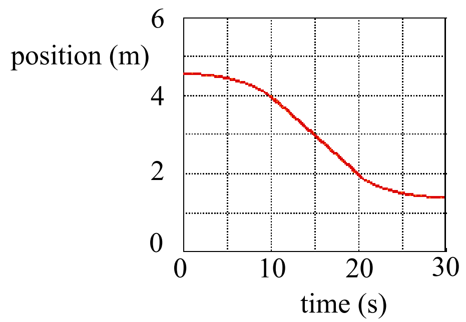


Velocity is defined as the slope at $t = 15$ s.

$$\text{i.e.,}\quad v = \left(\frac{dx}{dt}\right)_{t=15\text{ s}}$$

$$= \frac{\Delta x}{\Delta t} = \frac{(2\text{ m} - 4\text{ m})}{(20\text{ s} - 10\text{ s})} = -0.20\text{ m/s.}$$

DISCUSSION PROBLEM [2.4]:



During the time interval $0 < t < 10$ s was the car ...

A: speeding up,

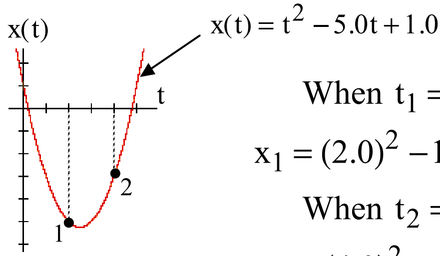
B: slowing down, or

C: moving with constant speed?

Question 2.5: The position of an object depends on time according to to the equation

$$x(t) = t^2 - 5.0t + 1.0,$$

where x is in meters and t in seconds. (a) Find the displacement and average velocity for the time interval $2.0\text{ s} \leq t \leq 4.0\text{ s}$. (b) Find the instantaneous velocities at $t = 2.0\text{ s}$ and $t = 4.0\text{ s}$. (c) At what time is the instantaneous velocity zero?



When $t_1 = 2.0$ s:

$$x_1 = (2.0)^2 - 10.0 + 1.0 = -5.0 \text{ m}.$$

When $t_2 = 4.0$ s:

$$x_2 = (4.0)^2 - 20.0 + 1.0 = -3.0 \text{ m}.$$

(a) $v_{av} = \dfrac{\Delta x}{\Delta t}$, where Δx = displacement

$$= x_2 - x_1 = -3.0 \text{ m} - (-5.0 \text{ m}) = 2.0 \text{ m}$$

and $\Delta t = t_2 - t_1 = 2.0$ s.

$$\therefore v_{av} = \frac{2.0 \text{ m}}{2.0 \text{ s}} = +1.0 \text{ m/s}.$$

(b) Instantaneous velocity $v = \dfrac{dx}{dt} = 2.0t - 5.0.$

At $t_1 = 2.0$ s: $v_1 = (4.0 - 5.0)$ m/s $= -1.0$ m/s.

At $t_2 = 4.0$s: $v_2 = (8.0 - 5.0)$ m/s $= 3.0$ m/s.

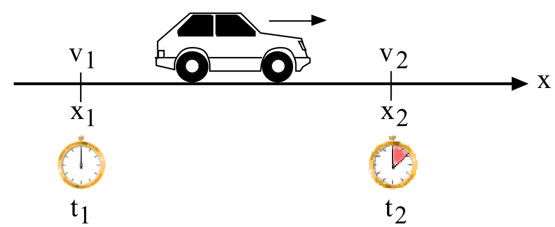
(c) When $v = 0$, $\dfrac{dx}{dt} = 0$, i.e., when $2.0t - 5.0 = 0$.

$$\therefore t = \frac{5.0}{2.0} = 2.5 \text{ s}$$

In the previous problem, the velocity changed with time,

$$v(t) = 2.0t - 5.0,$$

and whenever velocity changes we have ... *acceleration*.

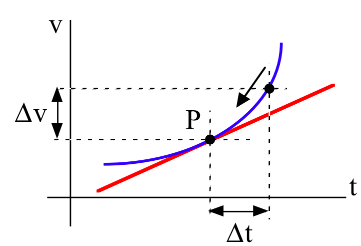


Average acceleration $\Rightarrow a = \dfrac{v_2 - v_1}{t_2 - t_1}$.

Dimension: $\dfrac{[L]}{[T]}\dfrac{1}{[T]} \Rightarrow [L][T]^{-2}$

Units: m/s^2, ft/s^2, etc.

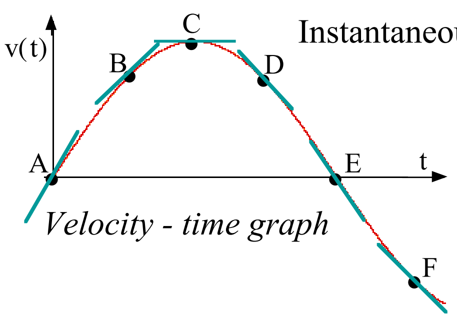
Instantaneous acceleration $a = \text{Limit}_{\Delta t \to 0} \dfrac{\Delta v}{\Delta t} = \dfrac{dv}{dt}$.



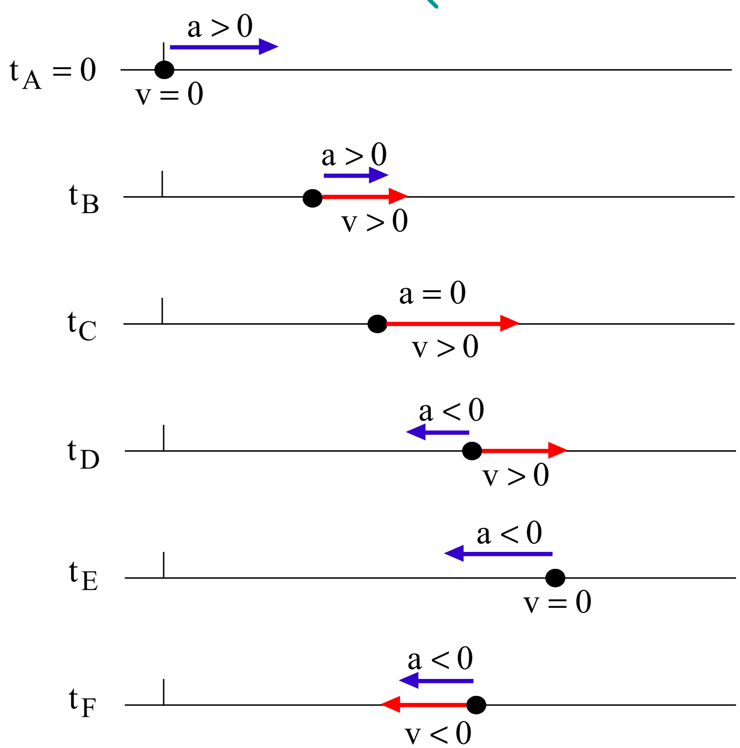
Also $v = \dfrac{dx}{dt}$.

$$\therefore a = \frac{d^2x}{dt^2}.$$

Instantaneous acceleration $a = \dfrac{dv}{dt}$.



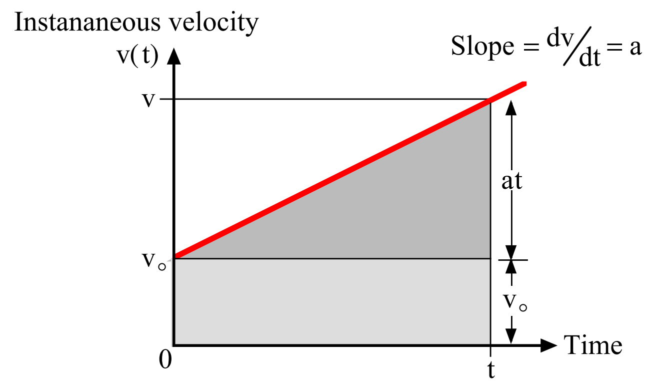
Velocity - time graph



In the previous example, the acceleration was a function of time, i.e., $a \Rightarrow a(t)$. However, we will only consider motion at *constant* (or *uniform*) acceleration:

$$\text{i.e., } a = \frac{dv}{dt} \Rightarrow \text{constant}$$

Example: *Free fall (with no air resistance).*



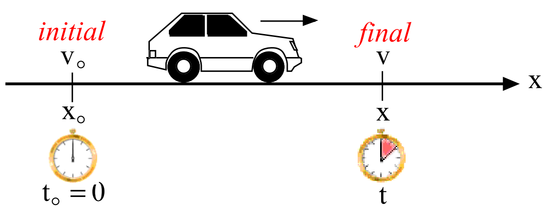
From earlier: $a = \dfrac{dv}{dt} = \dfrac{v - v_\circ}{t}$

$$\therefore \boxed{v = v_\circ + at}$$

Average velocity: $v_{av} = \dfrac{1}{2}(v + v_\circ)$*.

* Only true with constant acceleration.

Motion at constant acceleration



We have from before:

$$\Delta x = x - x_\circ = v_{av} t = \frac{1}{2}(v_\circ + v)t \quad \dots \quad \dots \quad (1)$$

But $v = v_\circ + at$.

$$\therefore x - x_\circ = \frac{1}{2}(2v_\circ + at)t,$$

$$\text{i.e., } \boxed{x - x_\circ = v_\circ t + \frac{1}{2}at^2}$$

Also $t = \dfrac{v - v_\circ}{a}$.

Substitute in (1), then $x - x_\circ = \dfrac{1}{2}(v_\circ + v)\dfrac{(v - v_\circ)}{a}$.

After re-arrangement we get

$$\boxed{v^2 = v_\circ{}^2 + 2a(x - x_\circ)}$$

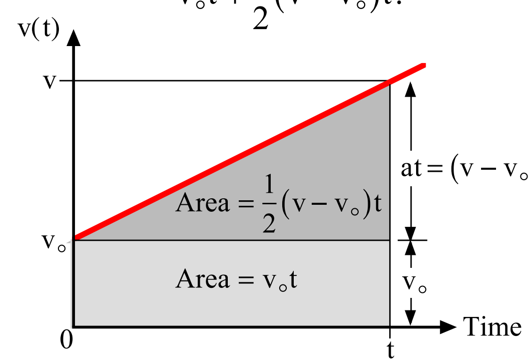
Let's look at

$$x - x_\circ = v_\circ t + \frac{1}{2}at^2,$$

in a little more detail. Since $a = \dfrac{v - v_\circ}{t}$, we get

$$v_\circ t + \frac{1}{2}at^2 = v_\circ t + \frac{1}{2}\left(\frac{v - v_\circ}{t}\right)t^2$$

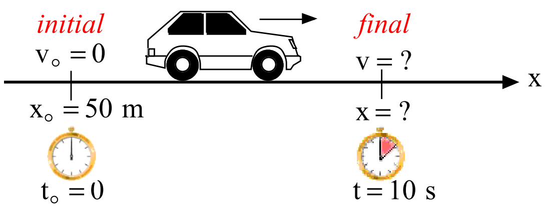
$$= v_\circ t + \frac{1}{2}(v - v_\circ)t.$$



But, from earlier, we see that this is simply the area under the velocity-time graph! So, the displacement

$$(x - x_\circ) \Rightarrow \text{area under the velocity-time graph.}$$

Question 2.6: A car, starting from rest at $x_\circ = 50$ m, accelerates at a constant rate of $8\ \text{m/s}^2$. (a) What is its velocity after 10 s? (b) How far has it traveled in those 10 s? (c) What is its average velocity over the interval $0 \le t \le 10$ s?



Using the equations of motion:

(a) $v = v_\circ + at = 0 + \left(8\ \text{m/s}^2 \times 10\ \text{s}\right) = 80\ \text{m/s}.$

(b) $x - x_\circ = v_\circ t + \frac{1}{2}at^2.$

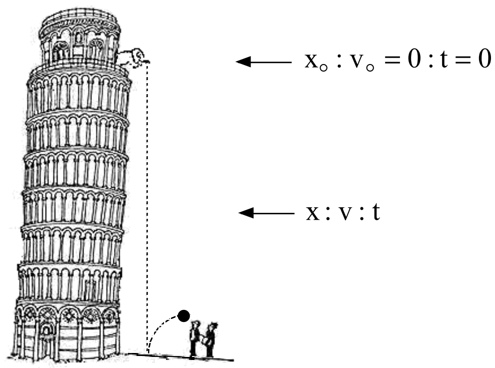
$$\therefore x = x_\circ + v_\circ t + \frac{1}{2}at^2$$

$$= 50\ \text{m} + \frac{1}{2}\left(8\ \text{m/s}^2\right) \times (10\ \text{s})^2 = 450\ \text{m}.$$

(c) $v_{av} = \frac{1}{2}\left(v_\circ + v\right) = 40\ \text{m/s}.$

*** Only true at constant acceleration ***

Galileo Galilei (1564-1642):



In “free fall”, i.e., in the absence of air resistance, *all* objects fall with a constant acceleration (downward).

$$a \Rightarrow g = 9.81\ \text{m/s}^2 \qquad (32.2\ \text{ft/s}^2)$$

Starting from rest $(v = v_\circ + gt)$:

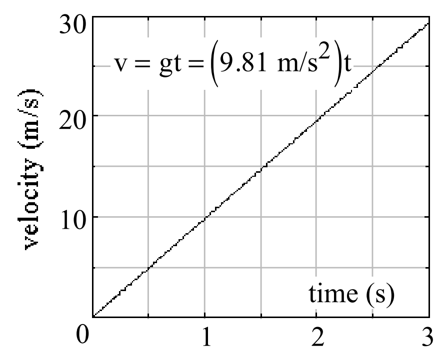
- after 1 s $\quad v = 9.81$ m/s (32.2 ft/s)
- after 2 s $\quad v = 19.6$ m/s (64.4 ft/s)
- after 3 s $\quad v = 29.4$ m/s (96.6 ft/s)

The velocity of the object increases by 9.81 m/s (32.2 ft/s) each second during free fall.

$$\text{Also } y - y_\circ = \overset{=0}{\cancel{v_\circ}}t + \frac{1}{2}at^2 = \frac{1}{2}gt^2.$$

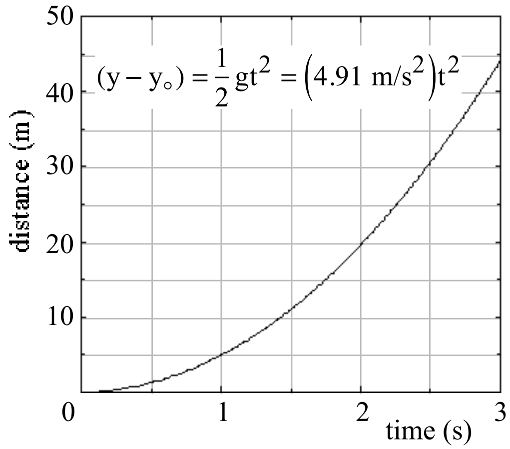
__CAREFUL__: If the y-axis is chosen so that y *increases upward*, then as the object falls, $y < y_\circ$, i.e., $g = -9.81\ \text{m/s}^2$. On the other hand, if you choose y to increase downward, then $y > y_\circ$, and $g = +9.81\ \text{m/s}^2$.

Note that the velocity, v, and displacement $(y - y_\circ)$ do not depend on the mass of the object! So, in the absence of air resistance, *__all objects fall at the same rate__*:

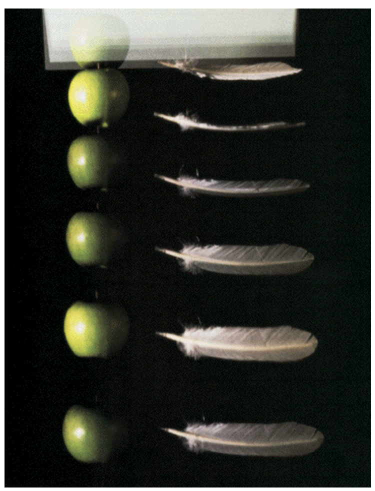


The velocity of an object in free-fall, starting from rest.

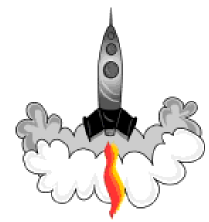
and all objects fall the same distance in the same time:



The distance fallen by an object in free-fall, starting from rest. (Same for all objects.)



Time elapse photographs of an apple and feather released at the same time, in the absence of air resistance (free fall).



Question 2.7: A rocket is fired vertically with a constant upward acceleration of 20.0 m/s^2. After 25.0 s, the engine is shut-off and the rocket continues to rise. The rocket eventually stops rising and then falls back to the ground. Find (a) the highest point the rocket reaches, (b) the total time the rocket is in the air, and (c) the speed of the rocket just before it hits the ground. Neglect air resistance,

Take upward direction as positive.

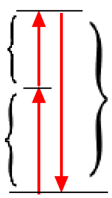
Maximum height $y_2 : v_2 = 0 : t_2$

Free fall: $a = -g = -9.81\ \text{m/s}^2$

Engine switch off $y_1 : v_1 : t_1$

Accelerated motion: $a = 20\ \text{m/s}^2$

Start $y_\circ = 0 : v_\circ = 0 : t_\circ = 0$



Free fall: $a = -g = -9.81\ \text{m/s}^2$

$y = 0 : v_3 : t_3$

(a) $(y_1 - y_\circ) = v_\circ(t_1 - t_\circ) + \frac{1}{2}a(t_1 - t_\circ)^2$,

$$\text{i.e.,}\quad y_1 = \frac{1}{2}\times\left(20.0\ \text{m/s}^2\right)\times(25.0\ \text{s})^2 = 6250\ \text{m}.$$

Also

$$v_1 = v_\circ + a(t_1 - t_\circ) = \left(20.0\ \text{m/s}^2\right)\times(25.0\ \text{s}) = 500\ \text{m/s}.$$

Between y_1 and y_2, $a = -g = -9.81\ \text{m/s}^2$ (*free fall*).

$$v_2^2 = v_1^2 + 2a(y_2 - y_1),$$

$$\text{i.e.,}\quad (y_2 - y_1) = \frac{v_2^2 - v_1^2}{2a} = \frac{0 - (500\ \text{m/s})^2}{2\times(-9.81\ \text{m/s}^2)}$$

$$= 12700\ \text{m}.$$

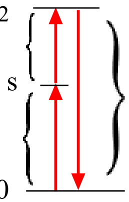
So, the maximum height is $y_2 = (12700\ \text{m}) + y_1$

$$= (12700 + 6250)\text{m} = 1.895\times10^4\,\text{m}\quad(\sim 19.0\ \text{km}).$$

$y_2 = 18950\ \text{m} : v_2 = 0 : t_2$

$y_1 = 6250\ \text{m} : v_1 = 500\ \text{m/s} : t_1 = 25\ \text{s}$

$y_\circ = 0 : v_\circ = 0 : t_\circ = 0$



$y = 0 : v_3 : t_3$

(b) Find t_2 (the total "climb time"):

$$v_2 = v_1 + a(t_2 - t_1),\ \text{i.e.,}\ (t_2 - t_1) = \frac{v_2 - v_1}{a}$$

$$= \frac{0 - 500\ \text{m/s}}{-9.81\ \text{m/s}^2} = 51.0\ \text{s}.$$

$$\therefore t_2 = (51.0 + 25.0)\ \text{s} = 76.0\ \text{s}.$$

To find t_3 (the "total time"):

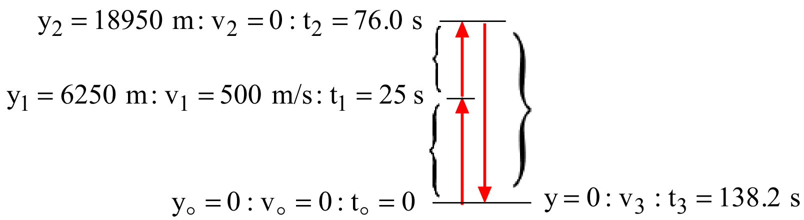
$$(y_3 - y_2) = v_2(t_3 - t_2) + \frac{1}{2}a(t_3 - t_2)^2,$$

$$\text{i.e.,}\quad -18950\ \text{m} = \frac{1}{2}\left(-9.81\ \text{m/s}^2\right)(t_3 - 76.0\ \text{s})^2.$$

$$\therefore (t_3 - 76.0\ \text{s}) = \sqrt{\frac{2\times(-18950\ \text{m})}{\left(-9.81\ \text{m/s}^2\right)}} = 62.2\ \text{s},$$

$$\text{i.e.,}\quad t_3 = (62.2 + 76.0)\ \text{s} = 138.2\ \text{s}.$$

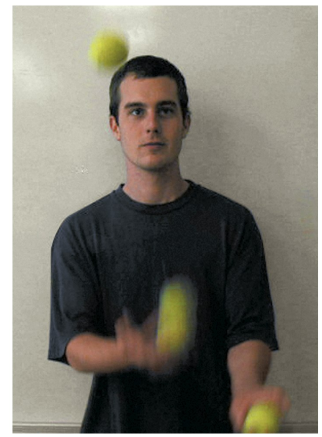
This is the total time the rocket is in the air.



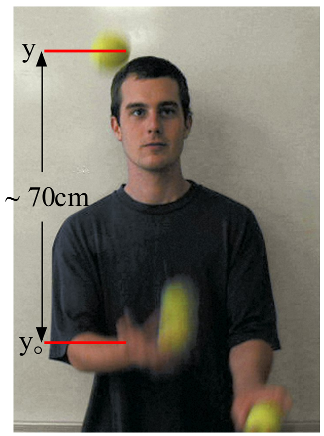
(c) Find v_3 (the final velocity):

$$v_3 = v_2 + a(t_3 - t_2)$$

$$= 0 + \left(-9.81 \text{ m/s}^2\right)(62.2 \text{ s}) = -610.2 \text{ m/s}.$$



Question 2.8: If, in the photograph, the upper ball is at the top of its trajectory, estimate the time it takes for a ball to loop from one hand to the other.



Take upward as positive. The maximum height the ball reaches above hand when thrown is ~ 70 cm.

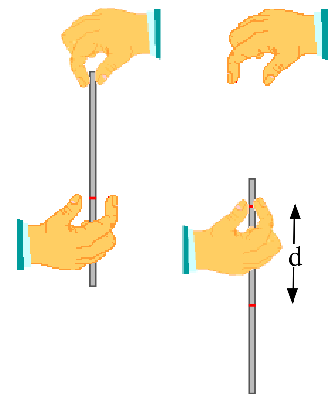
Since it falls back to his hand from 'rest' ... if t is time to fall we have

$$(y - y_\circ) = -0.70\text{ m} = \frac{1}{2}(-g)t^2 = \left(-4.91\text{ m/s}^2\right)t^2.$$

$$\therefore t = \sqrt{\frac{0.70\text{ m}}{4.91\text{ m/s}^2}} = 0.38\text{ s},$$

i.e., total time is $2 \times 0.38\text{ s} = 0.76\text{ s} \approx \tfrac{3}{4}\text{ s}.$

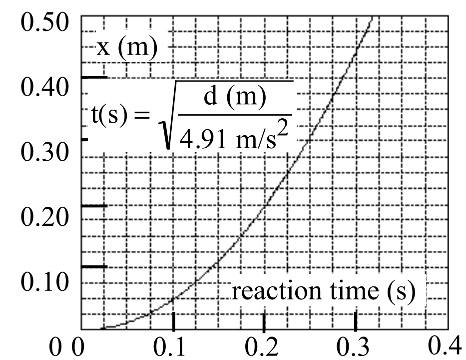
Measurement of reaction time ...



From earlier: $d = \frac{1}{2}gt^2$.

$$\therefore t(s) = \sqrt{\frac{d\text{ (m)}}{4.91\text{ m/s}^2}},$$

where t is the reaction time, i.e., the time between the release and "catch".



** Also "holds" for so-called *hang-time*
See useful notes on web-site

Relate reaction time to distance travelled driving a car

Reaction distance = speed × reaction time.

Measurement of reaction time ...

Reaction distance = speed × reaction time

mi/h	m/s
30	13.4
40	17.9
50	22.4
60	26.8
70	31.3

<u>Question 2.9</u>: It is often claimed that basketball superstars have hang times of at least 1 second. Is that reasonable?

If the hang time is 1 second, it means that the "rise" time and "fall" time are each $\frac{1}{2}$ s. But a fall time of $\frac{1}{2}$ s means the distance fallen is

$$d = \frac{1}{2}gt^2 = \frac{1}{2}\times\left(9.81 \text{ m/s}^2\right)\times(0.5 \text{ s})^2 = 1.23 \text{ m}$$

$$\Rightarrow 4 \text{ ft } \tfrac{1}{2} \text{ in}$$

which, is a very difficult "height" to jump.

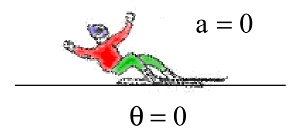
Measurements made on Michael Jordan give him a hang time of 0.92 s, which corresponds to a height of 1.04 m, i.e., 3 ft 5 in. So, a hang time close to 1 s *is* possible but rather unlikely.

For more details go to

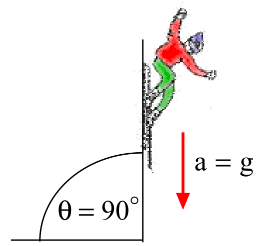
http://www.bizesor.com/brenkus/

and click on the *FSN sport* link.

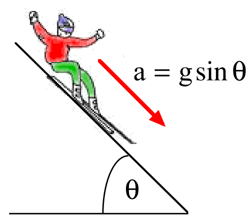
The acceleration due to gravity on a frictionless incline.



On a horizontal surface ($\theta = 0$), gravity does not produce an acceleration along the surface, i.e., $a = 0$.



On a vertical surface ($\theta = 90^\circ$), the acceleration is the same as in free fall, i.e., $a = g$.



On a surface inclined at an angle of θ to the horizontal, the acceleration is $a = g\sin\theta$.

Equations of motion using calculus

Earlier we derived the equations of motion graphically. We can also use calculus. We start with the definition of acceleration, i.e., $\frac{dv}{dt} = a$, where v is the velocity and t is time. If we assume that a is constant, we can integrate this expression with respect to t to get v:

$$v = \int dv = \int a dt = at + C_1,$$

where C_1 is an integration contant. To get C_1 we consider the *initial conditions*, i.e., what is v when $t = 0$? Let us put $v = v_\circ$ when $t = 0$, then $C_1 = v_\circ$, so

$$v = v_\circ + at,$$

which is the same as an equation we derived earlier.

Also, we know that $v = \frac{dx}{dt}$. So,

$$x = \int dx = \int v dt = \int (v_\circ + at) dt$$
$$= v_\circ t + \frac{1}{2}at^2 + C_2,$$

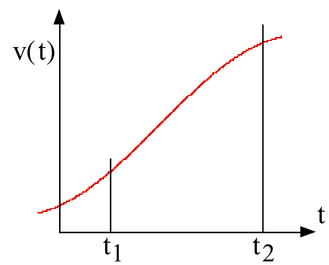
where C_2 is another constant.

To get C_2 we use the *initial conditions*, i.e., what is x when $t = 0$? Let us assume that $x = x_\circ$ when $t = 0$. Then $C_2 = x_\circ$. So, we get

$$x = x_\circ + v_\circ t + \frac{1}{2}at^2 \Rightarrow (x - x_\circ) = v_\circ t + \frac{1}{2}at^2,$$

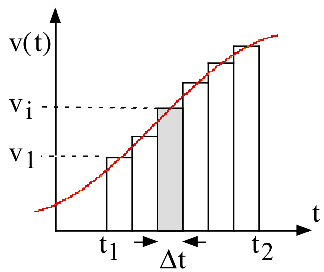
which is the same expression we obtained earlier.

If we know how the velocity of an object varies with time, we can also use calculus to determine the displacement over a period of time. For example,



given the velocity-time plot above, what is the displacement of the object over the time period $t_1 \rightarrow t_2$?

We split the time period $t_1 \rightarrow t_2$ into a *large number* of equal time intervals, Δt.



If Δt is very small, then the displacement from time t_1 to $t_1 + \Delta t$ is

$$\Delta x_1 \approx v_1 \Delta t.$$

So, the total displacement from $t_1 \rightarrow t_2$ is

$$x(t_2) - x(t_1) \approx v_1 \Delta t + v_2 \Delta t \ldots \approx \Sigma_i (v_i \Delta t).$$

The sum becomes *exact* if $\Delta t \rightarrow 0$. Then

$$x(t_2) - x(t_1) = \underset{\Delta t \rightarrow 0}{\text{limit}} \left(\Sigma_i v_i \Delta t\right) = \int_{t_1}^{t_2} v\,dt,$$

i.e., *the displacement is the area under the velocity-time curve from* t_1 *to* t_2. So, if we know the equation of the velocity-time curve we can determine the displacement over any time period.

Note, the above expression is true even if the acceleration is changing, i.e., when our earlier equations of motion cannot be applied.

Question 2.10: An object, starting from rest at $x_\circ = 1.00$ m, experiences a <u>non-constant</u> acceleration given by

$$a(t) = 1.50 + 0.20t \left(m/s^2\right).$$

At any time t, what is (a) the *instantaneous velocity*, and (b) the *position* of the object? (c) What is the *displacement* of the object over the time period from $2.0 \rightarrow 4.0$ s?

Because the acceleration is not constant we cannot use the simple equations of motion, we must use calculus.

(a) If $a(t) = 1.50 + 0.20t$, then $\dfrac{dv}{dt} = 1.50 + 0.20t$.

$$\therefore v = \int dv = \int (1.50 + 0.20t)dt$$

$$= 1.50t + 0.20\left(\frac{1}{2}t^2\right) + C_1,$$

$$\text{i.e.,} \quad v(t) = 1.50t + 0.10t^2,$$

since the initial velocity of object is zero.

(b) If $v = \dfrac{dx}{dt}$, the position is given by

$$x = \int dx = \int v dt = \int \left(1.50t + 0.10t^2\right)dt$$

$$= 1.50\left(\frac{1}{2}t^2\right) + 0.10\left(\frac{1}{3}t^3\right) + C_2,$$

$$\text{i.e.,} \quad x(t) = 1.00 + 0.75t^2 + 0.033t^3,$$

since the initial position of the object is $x_\circ = 1.00$ m.

(c) By definition, the displacement is

$$\Delta x_{t_1 \to t_2} = \int_{t_1}^{t2} v dt = \int_{t_1}^{t2} \left(1.50t + 0.10t^2\right)dt.$$

Using the result in (b)

$$\int_{t_1}^{t2} \left(1.50t + 0.10t^2\right)dt = \left[0.75t^2 + 0.033t^3 + C_2\right]_2^4$$

$$= [(0.75 \times 16 + 0.033 \times 64) - (0.75 \times 4 + 0.033 \times 8)]$$

$$= 10.85 \text{ m.}$$

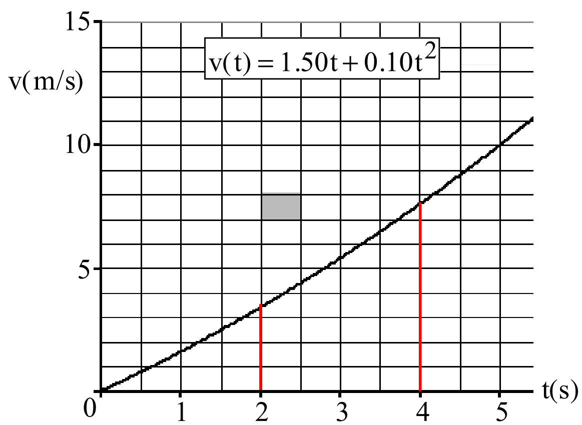
Check using (b):

$$x(t = 2) = 1.00 + 0.75 \times 2^2 + 0.033 \times 2^3 = 4.264 \text{ m.}$$

$$x(t = 4) = 1.00 + 0.75 \times 4^2 + 0.033 \times 4^3 = 15.112 \text{ m}$$

$$\therefore \Delta x = (15.112 - 4.264)\text{m} = 10.85 \text{ m.}$$

We can also determine the area under the curve by "counting rectangles" on the velocity-time graph ...



Area of each rectangle $= 1.0 \text{ m/s} \times 0.50 \text{ s} = 0.50 \text{ m}$. In the range $2.0 \text{ s} \leq t \leq 4.0 \text{ s}$, there are 18 complete rectangles plus parts that make approximately 4 more. Therefore, there are a total of ~ 22 rectangles. So the area under the curve is

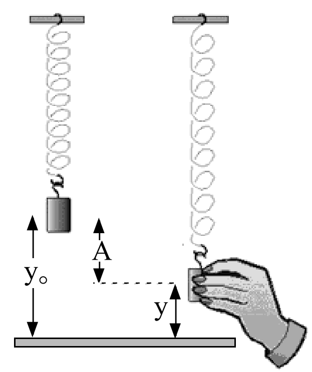
$$\sim 22 \times 0.50 \text{ m} \approx 11 \text{ m}.$$

Actual area by integration is 10.85 m.

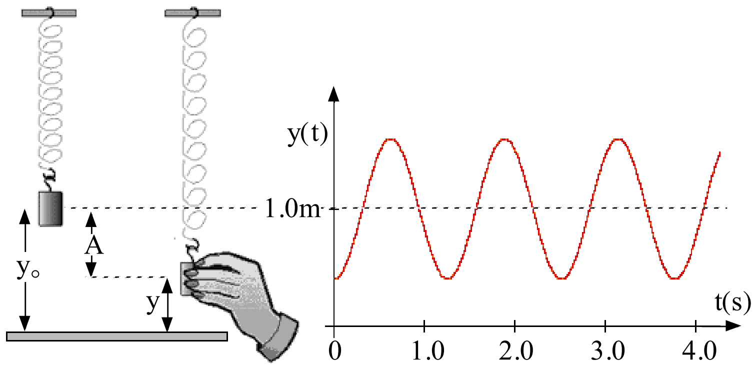
<u>Question 2.11</u>: A mass on the end of a spring is oscillating up-and-down and its position relative to the tabletop is

$$y = y_\circ - A\cos(\omega t).$$

Find the general expressions for (a) the instantaneous velocity, and (b) the instantaneous acceleration of the mass. If $y_\circ = 0.50$ m, $A = 0.30$ m and $\omega = 5.00 \text{ s}^{-1}$, what are (c) the maximum velocity, and (d) the maximum acceleration of the mass?



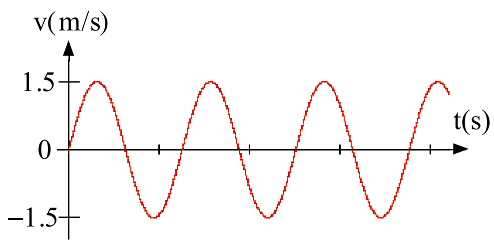
intitial conditions



intitial conditions

(a) the instantaneous velocity is $v(t) = \dfrac{dy}{dt}$

$$= +A\omega \sin(\omega t).$$

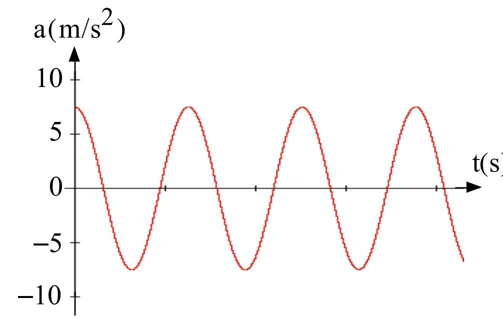


(c) Maximum velocity when $\omega t = \pi/2$,

$$\therefore v_{max} = A\omega = 1.50 \text{ m/s}.$$

(b) the instantaneous acceleration is $a(t) = \dfrac{dv}{dt} = \dfrac{d^2y}{dt^2}$

$$= A\omega^2 \cos(\omega t).$$



(d) Maximum acceleration when $\omega t = 0$,

$$\therefore a_{max} = A\omega^2 = 7.50 \text{ m/s}^2.$$

This type of oscillatory motion is called ***simple harmonic motion*** (SHM). You will see a number of examples of SHM in various physics courses.