DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **32**, Number **12**, December **2012**

doi:10.3934/dcds.2012.32.4321

pp. **4321–4360**

PARAMETERIZATION OF INVARIANT MANIFOLDS BY REDUCIBILITY FOR VOLUME PRESERVING AND SYMPLECTIC MAPS

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(Communicated by Amadeu Delshams)

Abstract. We prove the existence of certain analytic invariant manifolds associated with fixed points of analytic symplectic and volume preserving diffeomorphisms. The manifolds we discuss are not defined in terms of either forward or backward asymptotic convergence to the fixed point, and are not required to be stable or unstable. Rather, the manifolds we consider are defined as being tangent to certain "mixed-stable" linear invariant subspaces of the differential (i.e linear subspace which are spanned by some combination of stable and unstable eigenvectors). Our method is constructive, but has to face small divisors. The small divisors are overcome via a quadratic convergent scheme which relies heavily on the geometry of the problem as well as assuming some Diophantine properties of the linearization restricted to the invariant subspace. The theorem proved has an *a-posteriori* format (i.e. given an approximate solution with good condition number, there is one exact solution close by). The method of proof also leads to efficient algorithms.

1. **Introduction.** Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a real analytic diffeomorphism, with a fixed point $p \in \mathbb{C}^n$. In this work we are concerned with the existence of analytic invariant manifolds associated to linear subspaces invariant under the differential $Df(p)$.

We do not require that the linear subspaces are either stable or unstable. Rather, we impose certain number theoretic assumptions on the eigenvalues of $Df(p)$ and that f preserves a geometric structure. Then we prove the existence of analytic manifolds on which the motion is conjugate to linear.

In the case that f is real and that the invariant spaces chosen are also real we obtain that the invariant manifolds we consider are real, but in the case that f is complex the theorem also allows elliptic invariant manifolds (the eigenvalues have modulus 1).

2000 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. Invariant manifolds, KAM theory, quadratic convergence, symplectic mappings, volume preserving mappings.

The first author is supported by NSF grant DMS 0901389, DMS 1162544, and Texas coordinating board NHRP 0223.

Our method is constructive. We study the functional equation expressing the fact that the manifold is invariant and that the motion on it is conjugate to linear. Given an approximate solution of this equation, we present a quadratically convergent iterative procedure which can be implemented effectively on a computer and show it converges to a solution. In addition we bound the difference between the initial approximation and the true solution in terms of the error of the initial approximation. The iterative procedure also leads to fast numerical algorithms.

Suppose that $Df(p)$ is diagonalizable and denote the stable and unstable eigenvalues by

$$\{\lambda_1^s, \ldots, \lambda_{n_s}^s\} \qquad \{\lambda_1^u, \ldots, \lambda_{n_u}^u\},$$

respectively, where $n_s + n_u = n$. Suppose also that we choose eigenvectors and denote them by

$$\{\xi_1^s, \ldots, \xi_{n_s}^s\} \qquad \{\xi_1^u, \ldots, \xi_{n_u}^u\}.$$

Let $E^s = \text{span}\{\xi_j^s\}_{j=1}^{n_s}$ and $E^u = \text{span}\{\xi_j^u\}_{j=1}^{n_u}$. Then it is well known that the sets

$$W^s(p) = \{x \in \mathbb{C}^n \mid f^n(x) \to p\},$$

and

$$W^u(p) = \{x \in \mathbb{C}^n \mid f^{-n}(x) \to p\},$$

are analytic invariant manifolds, tangent to E^s and E^u respectively at p. This is the celebrated Stable Manifold Theorem (see for example [19, 12]).

There are other results about existence of invariant submanifolds of stable manifolds. The strong stable manifold, which corresponds to the most contractive eigenspaces, has been known for a long time [12]. Manifolds corresponding to subspaces of the stable manifold with some non-resonance manifolds were established in in [6, 10, 8, 3].

A more delicate question is; if $\{\lambda_1, \ldots, \lambda_\ell\}$ is an arbitrary collection of eigenvalues having eigenvectors $\{\xi_1, \ldots, \xi_\ell\}$ and $E = \text{span}\{\xi_j\}_{j=1}^{\ell}$, does there exist an analytic invariant manifold $W^m(p)$ tangent to E at p? If the set $\{\lambda_j\}$ is of mixed-stability (some eigenvalues having magnitudes greater than one and some less or equal than one) and the answer to the previous question is *yes*, then we call $W^m(p)$ a *mixed-stable* invariant manifold. Note that in the mixed-stable case, the manifolds is not defined in terms of either forward or backward asymptotic to the fixed point p.

In the remainder of this work, we address the question of the existence of mixed-stable invariant manifolds in two special cases. Namely the case of co-dimension one manifolds for volume preserving diffeomorphisms f, and the case of Lagrangian manifolds for symplectic f. Even though the present work is concerned primarily with quite technical question of the existence of these mixed-stable invariant, we note that the manifolds will be of interest in applications.

For example in fluid dynamics the study of so called chaotic advection is often concerned with studying co-dimension one (un)stable invariant manifolds associated with fixed points, and intersections between these manifolds. Briefly, the reason for this is that co-dimension one invariant manifolds separate the phase space, and methods for numerically computing (un)stable manifolds are by now quite well known. The co-dimension one mixed-stability invariant manifolds studied here should be of interest in this setting as well, as any such invariant manifold forms separatrix.

Similarly, when studying symplectic dynamical systems, the Lagrangian submanifolds play an important role in organizing the dynamics. The mixed-stable invariant manifolds studied here could be of interest when studying transport phenomena such as Arnold Diffusion and Ballistic Capture in applied problems as, loosely speaking, these manifolds are composed of orbits which visit the fixed point from far away, spend an long time near it (in fact given any neighborhood U of the fixed point one could find orbits which stay in U for any finite number of iterations), and then move away. Numerical applications of the methods developed here will be the subject of a future work.

Let Λ denote the matrix with the eigenvalues $\lambda_1, \ldots, \lambda_\ell$ on the diagonal and zeros elsewhere. If $\alpha \in \mathbb{N}^\ell$ is a multi-index then we define $\Lambda^\alpha = \lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_\ell^{\alpha_\ell}$. We prove the following theorems.

Theorem 1.1 (Symplectic Diffeomorphisms). *Let $f : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be an analytic symplecticomorphism, p a fixed point of f with $Df(p)$ diagonalizable, and $S = \{\lambda_1, \ldots, \lambda_n\}$ a collection of n eigenvalues of $Df(p)$. Let $\{\xi_j\}_{j=1}^n$ be the associated eigenvectors and Λ be the $n \times n$ matrix with the elements of S on the diagonal and zeros elsewhere. Assume that there are constants $C_D, \tau > 0$ so that the eigenvalues satisfy the Diophantine conditions*

$$|\lambda_i - \Lambda^\alpha|^{-1} \leq C_D |\alpha|^\tau, \tag{1}$$

$$|\lambda_i^{-1} - \Lambda^\alpha|^{-1} \leq C_D |\alpha|^\tau, \tag{2}$$

for all $\alpha \in \mathbb{N}^n$ with $\alpha_1 + \ldots + \alpha_n \geq 2$, where $1 \leq i, j \leq 2n$.

Then, there is an n-dimensional analytic invariant manifold $W^m(p)$ tangent to $E = span\{\xi_j\}_{j=1}^n$ at p. Moreover, $W^m(p)$ is a Lagrangian sub-manifold and the dynamics restricted to $W^m(p)$ are analytically conjugate to dynamics given by Λ acting on $\mathbb{C}^n$.

In the case that $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is real analytic and that the invariant linear subspace is real, we obtain that the manifold is real.

In the volume preserving case we state the results for $f : \mathbb{R}^n \to \mathbb{R}^n$ a real analytic diffeomorphism and $p \in \mathbb{R}^n$ is a fixed point. (Volume preserving is made precise in Section 4). However this choice is largely cosmetic and a similar theorem could be stated for complex maps with the right notion of complex volume preserving (say for example a complex analyticomorphism with $\det(Df(p)) = 1$). We have the following Theorem.

Theorem 1.2 (Volume Preserving Diffeomorphisms). *Suppose that the real analytic diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ preserves volume and has a fixed point p with $Df(p)$ diagonalizable. Let $S = \{\lambda_1, \ldots, \lambda_{n-1}\}$ be any co-dimension one sub-collection of eigenvalues of $Df(p)$ closed under complex conjugation (i.e. if $\lambda_i \in S \cap (\mathbb{C} \backslash \mathbb{R})$ then $\bar{\lambda}_i \in S$ as well). Let $\{\xi_j\}_{j=1}^{n-1}$ be associated real eigenvectors, and Λ be the $n-1 \times n-1$ matrix having the elements of S on the diagonal and zeros elsewhere. Assume that there are constants $C_D, \tau > 0$ so that the eigenvalues satisfy the Diophantine conditions*

$$|\lambda_i - \Lambda^\alpha|^{-1} \leq C_D |\alpha|^\tau, \qquad 1 \leq i \leq n, \tag{3}$$

for all $\alpha \in \mathbb{N}^{n-1}$ with $\alpha_1 + \ldots + \alpha_{n-1} \geq 2$.

Then, there is an $n-1$ dimensional real analytic invariant manifold $W^m(p)$ tangent to $E = span\{\xi_j\}_{j=1}^{n-1}$ at p and the motion on this manifold is analytically conjugate to Λ.

The proofs are presented in Section 6.3.

1.1. Some comments on the hypotheses of Theorems 1.1, 1.2.

1. We will refer to the conditions (1), (2) as Diophantine conditions even if there are some important differences with the standard Diophantine conditions. Notably, our exponents α are required to be positive. So that for example, if $(\lambda_1, \lambda_2, \lambda_3) = (8, 16, 32)$, then, the set (λ_1, λ_2) satisfies (1), (2).
2. When f is symplectic, $\lambda_i \in \mathrm{spec}(Df(p))$ implies $\lambda_i^{-1} \in \mathrm{spec}(Df(p))$. Then Equations (1) and (2) imposes in particular that if $\lambda_i \in S$, then $\lambda_i^{-1} \notin S$. Then this places a constraint on the n dimensional sub-collections which are allowed.
3. Since the results of Theorem 1.1 are local, there is no loss of generality in making a change to Darboux coordinates near the fixed point and using the standard symplectic form. These coordinates are used in the proof of the theorem, but this is simply a convenience.
4. When f is real, then if λ is an eigenvalue in the collection we require that $\bar{\lambda}$ is also in the collection. This is in order to obtain a real manifold. (This is stated explicitly for the volume preserving case, but is necessary in the real analytic symplectic case as well). If $|\lambda| = 1$, then $\bar{\lambda} = \lambda^{-1}$, so that (1) cannot hold. Hence, in the real analytic case, Theorems 1.1 and 1.2 apply only to hyperbolic fixed points. Nevertheless, in the complex case, we can allow eigenvalues of modulus 1 and obtain complex elliptic manifolds.
5. Again, if f is real the methods developed here have no problem dealing with complex conjugate eigenvalues. The technical details for handling complex eigenvalues of real maps are discussed fully in [14].
6. The strategy of the proofs of Theorems 1.1 and 1.2 are based on combining the Parameterization Method of [3, 4, 5] with a KAM quadratic convergence argument. A KAM argument is needed because small divisors can arise as a consequence of the fact that we allow S a to be mixed-stability collection of eigenvalues.

   The quadratic convergence is made possible because the preservation of the geometric structure allows to transform the Newton method equation to a constant coefficient equation, which can be solved identifying coefficients.
7. Since the quadratically convergent iterative scheme requires only solving the constant coefficient linearized equation and manipulation of series, it leads to rather efficient numerical methods. We discuss algorithmic issues in Section 5, however the implementation of these algorithms is postponed for a later work.
8. Theorems 1.1 and 1.2 are in fact corollaries of a more general Theorem 2.2, which show that the desired manifolds exist as long as we have good enough approximate solutions. Then, in addition to establishing Theorems 1.2 and 1.1, Theorems 2.2 provide *a-posteriori* results which can be used to validate the results of numerical calculations.
9. The Diophantine conditions (1), (2), and (3) can be improved to the Rüssman-Brjuno conditions [22, 1]

$$\sum_{\ell} 2^{-\ell} \log \left(\sup_{2 \le |\alpha| \le 2^{\ell}} |\lambda_i^{\pm 1} - \Lambda^{\alpha}|^{-1} \right) < \infty. \tag{4}$$

   The changes needed in the proof presented here are rather standard. They are presented pedagogically in [7].

1.2. **Some comments on the literature.** Results similar to Theorems 1.1 and 1.2 appear in [20], which also considers the problem of obtaining a submanifold under non-resonance conditions. The non-resonance conditions of [20] are exactly the same as the conditions we assume, the paper in [20] does not require that the invariant manifolds have dimension n.

However, the method of proof of [20] is different from the one presented here. We use a KAM scheme whereas [20] uses a sophisticated majorant method. In [20, Ch. 3], one can find the question of whether the KAM method can be applied for these problems with exactly the same non-resonance conditions (1), (2). Here we provide an answer assuming that there is a symplectic or volume form preserved by the map.

It is known for experts even if it does not seem to have been written in detail in the literature (see the discussion in [20]), that KAM schemes for the problem can be implemented under stronger non-resonance conditions, namely

$$|\mu_i\mu_j^{-1} - \Lambda^\alpha|^{-1} \leq C|\alpha|^\tau, \qquad |\alpha| \geq 2. \tag{5}$$

where μ_i, μ_j are eigenvectors in the complement of S.

These conditions (5) (sometimes called *second Melnikov conditions*) allow one to reduce the linearized equation to a constant coefficient equation (up to a small error which one must show can be neglected). We call this strategy a *reducibility* argument. One then applies a quadratic method for improving reducibility, where in each step of the quadratic scheme only the reduced linearized equation (constant coefficient) must be solved (again see [20] for further details).

In this paper, we show that one can use the geometry of the system to obtain reducibility without the second Melnikov conditions and still obtain a KAM scheme. This is referred to as *automatic reducibility*, as the reduction to constant coefficients is enabled by the geometry of the problem rather than by hypothesizing the stronger second Melnikov conditions. A further difference between this paper and [20] is that our result is formulated in an a-posteriori way which provides an exact solution given an approximate solution. If the approximate solutions are produced numerically, the theorems presented here can be used to validate the numerics. Note however that in the symplectic case, $\mu_j^{-1} = \lambda_{j+n}$. Hence, we have $\mu_i\mu_j^{-1} = \lambda_{i+n}^{-1}\lambda_{j+n}$. In the volume preserving case, $\lambda_n^{-1} = \lambda_1 \cdots \lambda_{n-1}$.

2. The Parameterization Method.

In this section we review the main ideas behind the Parameterization Method for invariant manifolds [3, 4, 5].

Given a map $f : \mathbb{C}^n \to \mathbb{C}^n$, $f(0) = 0$, (in the sequel we assume without loss of generality that the fixed point is the origin) the parameterization method seeks an embedding $K : \mathbb{C}^\ell \to \mathbb{C}^n$ with $K(0) = 0$, and a polynomial map $P : \mathbb{C}^\ell \to \mathbb{C}^\ell$ in such a way that

$$f(K) = K(P). \tag{6}$$

It is clear from 6 and from $f(0) = 0$, $K(0) = 0$ that

$$Df(0) \circ DK(0) = DK(0)DP(0), \tag{7}$$

so that the range of $DK(0)$ is necessarily invariant under $Df(0)$.

Equation (6) clearly implies that the range of K is invariant under f. Similarly, Equation (7) implies that at a point $K(x)$ in range(K), the vector field X is in the tangent space to the range, so that the flow of X leaves range(K) invariant.

We also note that in (6) the interpretation of P is that P is the dynamics on the invariant manifold. The observation that the Parameterization function K solves a functional equation such as Equation (6) or Equation (8) leads to both efficient numerical algorithms for computing K, as well as existence proofs. The reader interested in numerical aspects of the Parameterization Method can consult [14] [15] [5] for results on maps and [23] [5] for results on flows. Theorem 2.2 provides a fairly general existence result for solutions of Equation (6), which does not require that either P or P^{-1} is a contraction.

Since K gives a parametric representation of the manifold, it has no difficulty in following the folds of the invariant manifold. This is in contrast with the representation of the manifold as a graph. Such a graph representation of the invariant manifold breaks down when the manifold starts to turn. From the practical point of view, it is interesting to remark that the unknowns K, P are functions of ℓ-variables, with ℓ in general smaller than d, the dimension of the phase space.

Of course, there are, in general many invariant subspaces for $Df(0)$, and once we choose an invariant subspace there are many embeddings. This is due to the freedom in choosing the lengths of the eigenvectors. However, as discussed thoroughly in [3, 4, 5], the choice of the lengths of the eigenvectors corresponds only to a re-parameterization of the invariant manifold and the parameterization is unique modulo the freedom to rescale the the eigenvectors.

The choice of the invariant space under $Df(0)$ corresponds to the choice of which invariant manifold to study. The choice of embedding $K(0)$ corresponds to choosing the scale (basis of the eigenspace) and coordinates of the parameterization of the linear space. Once these choices are made, it is easy to see that the other terms in the expansion are determined uniquely. Indeed, proceeding by induction, we see that if $n \in \mathbb{N}^\ell$ is a multi-index with $|n| \geq 2$ then the power series coefficients K_n of K are determined by the linear system

$$[Df(0) - \Lambda^n I]K_n = S_n(K_1, \ldots, K_{n-1}),$$

where S_n is an explicit polynomial expression whose coefficients can be written in terms of the derivatives of f at 0 and some combinatorial numbers (for the derivation of this equation see [3]. For explicit examples of the computation of S_n in applications see [15, 14]). Hence, under the assumption that $\lambda_i \neq \Lambda^\alpha \neq 0$, (which is considerably weaker than our assumption (1)) we can determine recursively all the K_n and, hence obtain that the analytic solutions are unique.

Remark 1. In what follows we focus on the case when the spectrum of Df is composed of distinct eigenvalues $\{\lambda_i\}_{i=1}^n$, which satisfy certain non-resonance conditions. In this case we can take $P = \Lambda$ with Λ the $\ell \times \ell$ diagonal matrix having the λ_i on the diagonal.

Remark 2. The parameterization method can be applied to vector fields as well. Given a vector field X in $\mathbb{C}^n$ with $X(0) = 0$, the parameterization method seeks an embedding $K : \mathbb{C}^\ell \to \mathbb{C}^n$ with $K(0) = 0$ and a polynomial vector field $P : \mathbb{C}^\ell \to \mathbb{C}^\ell$, $P(0) = 0$ in such a way that

$$X \circ K(x) = DK(x)\, P(x). \tag{8}$$

The results for flows can be proved directly in a manner similar to the results for maps presented here. However it is worth pointing out that the results for maps imply the results for flows following an argument in [3]. To see this, denote by X_t

the time-t map of the flow and by P_t the time-t map of the vector field P. Then we can use the results for maps to obtain a map K and $P_1 = DX_1(0)$ in such a way that $X_1 \circ K = K \circ P_1$.

To show that K satisfies (8), we note that

$$X_1 \circ X_{-t} \circ K \circ P_t = X_{-t} \circ X_1 \circ K \circ P_t = X_{-t} \circ K \circ P_1 \circ P_t$$
$$= X_{-t} \circ K \circ P_t \circ P_1,$$

and that $D(X_{-t} \circ K P_t)(0) = DK(0)$. By the uniqueness of solutions of (6) satisfying the normalization condition, we obtain that $K = X_{-t} \circ K \circ P_t$, equivalently, $X_t \circ K = K \circ P_t$. Taking derivatives with respect to t in the above, we obtain (8).

2.1. **Functional Analytic Framework.** Let $(z_1, \dots, z_m) = z \in \mathbb{C}^m$ denote a complex m-vector. We endow $\mathbb{C}^m$ with the norm

$$|z|_{\mathbb{C}^m} = \sup_{1 \le i \le m} |z_i|, \tag{9}$$

where $|\cdot|$ is the usual Euclidean norm (or absolute value) on $\mathbb{C}$. We will denote the norm on $\mathbb{C}^m$ by $|\cdot|$ whenever there is no possibility of confusion. For fixed $z \in \mathbb{C}^m$, and $r > 0$, the *poly-disk of radius r about z in $\mathbb{C}^m$*, denoted $D_r^m(z)$, is the set

$$D_r^m(z) \equiv \{w \in \mathbb{C}^m \ : \ |w_i - z_i| \le r \ \text{ for all } 1 \le i \le m\}.$$

Note that the poly-disks are the closed balls generated by the norm (9) on $\mathbb{C}^m$.

For $\ell < n$, let $\mathcal{X}_r^{\ell,n} = \mathcal{X}_r$ be the space of analytic mappings of the interior of the poly-disk $D_r^\ell(0) \subset \mathbb{C}^\ell$ into $\mathbb{C}^n$ and continuous on the polydisk which also map the origin in $\mathbb{C}^\ell$ to the origin in $\mathbb{C}^n$.

While $\mathcal{X}_r$ depends explicitly on ℓ and n, we suppress this dependence since it will be fixed throughout the paper.

Note then that if $Q \in \mathcal{X}_r^{\ell,n}$, then $DQ(z)$ is of maximal rank ℓ for all $z \in D_r^\ell(0)$.

Define $\mathcal{Y}_r \subset \mathcal{X}_r$ be the set of functions in $\mathcal{X}_r$ whose first differential vanishes at the origin. Then any $Q \in \mathcal{X}_r$ has power series representation

$$Q(z) = \sum_{|\alpha| \ge 1} q_\alpha z^\alpha,$$

while $R \in \mathcal{Y}_r$ has a power series representation

$$R(z) = \sum_{|\alpha| \ge 2} r_\alpha z^\alpha,$$

where $\alpha \in \mathbb{N}^\ell$, $q_\alpha, r_\alpha \in \mathbb{C}^n$, $z \in \mathbb{C}^\ell$, and the power series converge on the poly-disk $|z_i| \le r$. When the dimension of the range is $n = 1$, then $\mathcal{X}_r$ and $\mathcal{Y}_r$ are Banach algebras for point-wise multiplication under the supremum norm

$$\| Q \|_r \equiv \sup_{|z_i| = r} |Q(z)|.$$

In applications this fact can often be applied component by component in order to obtain useful estimates even in the case that the image has dimension $n > 1$. We also have the bound

$$\| Q \|_r \le \sum_{|\alpha| \ge 1} |q_\alpha| r^{|\alpha|}. \tag{10}$$

The estimate given by Equation (10) is useful in applications, as it can be efficiently computed numerically (note however that the RHS of (10) may be undefined for functions in $\mathcal{X}_r$).

The following are standard results which follow almost immediately from the Cauchy integral representation of functions and their derivatives [2].

Proposition 1. *For Q analytic and bounded on $D^\ell_r(0)$ and $\delta > 0$, we have that*

$$\|DQ\|_{re^{-\delta}} \leq \frac{C_*}{|\delta|}\|Q\|_r \tag{11}$$

$$|q_\alpha| \leq \frac{\|Q\|_r}{r^{|\alpha|}}, \tag{12}$$

where C_* is a constant which depends only on the dimension ℓ of the domain.

We refer to the estimate given by Equation (11) as Cauchy Bounds, to distinguish them from the Cauchy Estimates given by Equation (12). These are essential in the sequel as they allow us to bound the norm of the derivative of an analytic function in terms of its supremum, albeit on a smaller domain.

Now let $f : \mathbb{C}^n \to \mathbb{C}^n$, $\{\lambda_1, \ldots, \lambda_\ell\}$, Λ, and $\{\xi_1, \ldots, \xi_\ell\}$ be as in Section 1. Let

$$A = [\xi_1 \,|\, \ldots \,|\, \xi_\ell],$$

denote the $n \times \ell$ matrix, whose columns are the eigenvectors ξ_i. We use the notation $[\cdot \,|\, \cdot]$ to indicate the juxtaposition of column vectors.

We now define $\mathcal{X}^0_r \subset \mathcal{X}_r$ to be the set

$$\mathcal{X}^0_r \equiv \{K \in \mathcal{X}_r \ : \ DK(0) = A\}.$$

This is the space of analytic embeddings which are tangent to $\mathrm{span}(\xi_1, \ldots, \xi_\ell)$ at the origin, and normalized to have i-th directional derivative of ξ_i at the origin.

Define the nonlinear operator $\Phi : \mathcal{X}^0_r \to \mathcal{Y}_r \subset \mathcal{X}_r$ by

$$\Phi(K) = f \circ K - K \circ \Lambda. \tag{13}$$

Note that finding a solution $K \in \mathcal{X}^0_r$ to the equation

$$\Phi(K) = 0, \tag{14}$$

is equivalent to finding a parameterization function K which satisfies the invariance Equation (6) with a normalized derivative at the origin.

Note also that $\Phi(K) \in \mathcal{Y}_r$ precisely because $K \in \mathcal{X}^0_r$ solves Equation (6) to first order. In the sequel we study Equation (14) when f leaves invariant a symplectic or a volume form.

2.1.1. *Composition Estimates for Parameterizations.* Assume that $\Phi(K) = f \circ K - K \circ \Lambda \in \mathcal{Y}_r$ and $\epsilon \equiv \|\Phi(K)\|_r$. Choose $\rho_* > 0$ so that $\mathrm{image}(K) \subset D^n_{\rho_*}(0)$. Then we have the following bound

$$\|K \circ \Lambda\|_r \leq \|f \circ K\|_r + \|\Phi(K)\|_r \leq \|f\|_{\rho_*} + \epsilon,$$

simply by considering the definition of Φ (13), and using that K is an approximate solution (i.e. $\Phi(K)$ is defined and in $\mathcal{Y}_r$). Similarly, suppose that $K' = K + \Delta$ with $\epsilon = \|\Phi(K)\|_r$ and $\|\Phi(K')\|_{r'} = \epsilon'$. Then

$$\Phi(K + \Delta) = \Phi(K) + Df(K)\Delta + R_K(\Delta) - \Delta \circ \Lambda,$$

as in Lemma A.1 and we have

$$\begin{aligned}\|\Delta \circ \Lambda\|_r &\leq \epsilon + \epsilon' + 2n\|Df\|_{\rho_*}\|\Delta\|_r + \|R_K(\Delta)\|_r \\ &\leq 2\epsilon + 2n\|Df\|_{\rho_*}\|\Delta\|_r + \|R_K(\Delta)\|_r.\end{aligned}$$

An explicit bound on R_K (obtained straightforwardly from the Taylor's theorem with remainder) can be found in Lemma A.1.

2.1.2. *Estimates on cohomology equations.* The following result is standard in KAM theory and we will use it repeatedly.

Lemma 2.1. *Let $\{\lambda_1, \dots, \lambda_\ell\} \subset \mathbb{C}$ be complex numbers and fix $\beta \in \mathbb{C}$. Let Λ be the matrix with diagonal entries λ_i. Let $\mathcal{X}_r = \mathcal{X}_r^{\ell,1}$ and $\mathcal{Y}_r$ be as above, so that if $q \in \mathcal{Y}_r$ then $q : D_r^\ell(0) \subset \mathbb{C}^\ell \to \mathbb{C}$, $q(0) = 0$, and $Dq(0) = 0$. Consider the bounded linear operator $\mathcal{L} : \mathcal{Y}_r \to \mathcal{Y}_r$ defined by*

$$\mathcal{L}(q) = \beta q - q \circ \Lambda. \tag{15}$$

Assume that for all multi-indices α with $|\alpha| \geq 2$, and all $\tau > 0$, β and Λ satisfy the Diophantine condition

$$|\beta - \Lambda^\alpha|^{-1} \leq C_D|\alpha|^\tau. \tag{16}$$

Then for any $p \in \mathcal{Y}_r$ having $\| p \|_r < \infty$, and any $\delta > 0$, the equation

$$\mathcal{L}(q) = p, \tag{17}$$

has a unique solution $q \in \mathcal{Y}_{re^{-\delta}}$. Furthermore there is a $\tilde{C}$ depending only on ℓ and C_D, so that

$$\begin{aligned}&||q||_{re^{-\delta}} \leq \tilde{C}|\delta|^{-\tau-\ell-1}||p||_r \\ &\|q \circ \Lambda\|_{re^{-\delta}} \leq \beta\|q\|_{re^{-\delta}} + \|p\|_r \leq 2\tilde{C}\delta^{-\tau-\ell-1}\|p\|_r.\end{aligned} \tag{18}$$

Remark 3. The possibility that $\beta = \lambda_i$ for some $1 \leq i \leq \ell$ is not excluded (in fact, this case will arise in the sequel). Note that because we consider only $|\alpha| \geq 2$, there is no contradiction between $\beta = \lambda_i$ and the Diophantine condition (16).

Proof. Let

$$p(\theta) = \sum_{|\alpha|\geq 2} p_\alpha \theta^\alpha,$$

with $||p||_r < \infty$. We seek $q(\theta) = \sum_{|\alpha|\geq 2} q_\alpha \theta^\alpha$, so that

$$\beta q(\theta) - q(\Lambda\,\theta) = p(\theta),$$

for all $\theta \in D_r^\ell(0) \subset \mathbb{C}^\ell$.

Formally, this gives

$$\begin{aligned}\beta q(\theta) - q(\Lambda\,\theta) &= \beta \sum_{|\alpha|\geq 2} q_\alpha \theta^\alpha - \sum_{|\alpha|\geq 2} q_\alpha (\Lambda\,\theta)^\alpha \\ &= \sum_{|\alpha|\geq 2} (\beta - \Lambda^\alpha) q_\alpha \theta^\alpha \\ &= \sum_{|\alpha|\geq 2} p_\alpha \theta^\alpha.\end{aligned}$$

Matching like powers of θ and solving for the unknown coefficients of q gives

$$q_\alpha = \frac{p_\alpha}{\beta - \Lambda^\alpha}. \tag{19}$$

To see that the formal solution is in the desired function function space, we estimate

$$\begin{aligned}
||q||_{re^{-\delta}} &= \left\| \sum_{|\alpha|\geq 2} q_\alpha \theta^\alpha \right\|_{re^{-\delta}} \leq \sum_{|\alpha|\geq 2} |q_\alpha| r^{|\alpha|} e^{-|\alpha|\delta} \\
&= \sum_{|\alpha|\geq 2} \left| \frac{p_\alpha}{\beta - \Lambda^\alpha} \right| r^{|\alpha|} e^{-|\alpha|\delta} \leq \sum_{|\alpha|\geq 2} C_D |\alpha|^\tau |p_\alpha| r^{|\alpha|} e^{-|\alpha|\delta} \\
&\leq \sum_{|\alpha|\geq 2} C_D |\alpha|^\tau \frac{\|p\|_r}{r^{|\alpha|}} r^{|\alpha|} e^{-|\alpha|\delta} = C_D \|p\|_r \sum_{|\alpha|\geq 2} |\alpha|^\tau e^{-|\alpha|\delta} \\
&= \tilde{C} |\delta|^{-\tau-\ell-1} \|p\|_r.
\end{aligned}$$

Here we have used the Diophantine condition and Cauchy Estimates on p (see Equation (12)). The last estimate is obtained by changing to polar coordinates and bounding the sum by the integral over the radial coordinate. If we call the resulting bound on the polar integeral M_ℓ, then we let $\tilde{C} \equiv C_D M_\ell$. Note that M_ℓ is a constant depending only on the dimension ℓ of parameter space and the surface measure of the unit ball in $\mathbb{C}^\ell$. From this follows the estimate claimed in the lemma, as well as the fact that q is analytic on $D^\ell_{re^{-\delta}}(0)$.

The second line in the estimates in (18) follows from the fact that q solves (17), we have $q \circ \Lambda = q + p$ and we have estimates for the two terms in the RHS. □

Remark 4. The bound above is not optimal. In [21] it is shown that the the exponent of $|\delta|$ in the bounds (18) can be reduced to $\tau + 1$, i.e. that the exponent need not depend on the dimension. Nevertheless, the bound given above is sufficient for our purposes and has the virtue of maintaining that the present work is self contained.

2.2. **Review of the Newton Procedure for the Parameterization Method.** In this section we review the Newton scheme, appropriate for solving Equation (14). The argument is a standard variation of the so called 'Nash-Moser' quadratic convergence schemes on a parameterized family of Banach Spaces [16, 18, 24].

A function $K \in \mathcal{X}_r$ is an "approximate solution" of Equation (6), whenever the error E, defined by

$$E \equiv \Phi(K) = f \circ K - K \circ \Lambda,$$

is small (in some norm, to be specified). More precisely, we will say that K is an ϵ-approximate solution of Equation (6) on the poly-disk $D^\ell_r(0) \subset \mathbb{C}^\ell$ whenever $||\Phi(K)||_r \leq \epsilon$. We find it suggestive to maintain the notation $E = \Phi(K)$ at various points throughout the sequel.

Given an approximate solution K_0 having $\Phi(K_0) = E_0$ small, we seek a Δ_* so that $K = K_0 + \Delta_*$ solves (6) exactly. Δ_* will be approximated iteratively, via a quadratic convergent scheme.

The first try is to find an approximate Δ that eliminates the error in the linear approximation. Referring to [3, 4, 5] one sees that

$$D\Phi(K)\Delta = Df(K)\Delta - \Delta \circ \Lambda.$$

Then the Newton method would consist in solving

$$Df(K)\Delta - \Delta \circ \Lambda = -E. \tag{20}$$

The Equation (20) is problematic. We do not know how to solve it when $Df(K)$ is a function of θ. Of course, even in the case when $Df(K)$ is replaced by a constant and we can apply Lemma 2.1, we only obtain estimates in a slightly smaller domain.

The key idea to accomplish the quadratic convergence scheme is that we can use the geometry of the problem to transform (20) into the constant coefficient equation (up to an small error) so that we can apply Lemma 2.1. Then, we obtain a procedure that reduces the error to its square, albeit in a slightly smaller domain and multiplied by a power of the loss of domain factor. The fact that such a procedure converges is rather standard, in KAM theory.

More precisely, rather that looking for inverses of $D\Phi$ (which is an unbounded operator), we look instead for a family of "approximate inverse" operators $\mathfrak{L}(K) : \mathcal{Y}_r \to \mathcal{Y}_{re^{-\delta}}$ having that

$$\|E - [D\Phi(K)][\mathfrak{L}(K)]E\|_{re^{-\delta}} \le C_1 \delta^{-\mu} \|E\|_r^2,$$

for all $\delta > 0$ when E is small enough. In other words $\Delta = -\mathfrak{L}(K)E$ approximately solves the equation $D\Phi(K)\Delta = -E$ up to a quadratic error in E, albeit on a smaller domain than the domain of K and E. (The appearance of $re^{-\delta}$ in the loss of domain is seen to be natural again by considering Lemma 2.1).

The sequel is largely concerned with studying the equation

$$D\Phi(K)\Delta = Df(K)\Delta - \Delta \circ \Lambda = -E, \tag{21}$$

and defining the approximate inverse operators.

Once we have the approximate inverses, we will define a quasi-Newton scheme

$$\Delta_n = -\mathfrak{L}(K_n)E_n, \qquad K_{n+1} = K_n + \Delta_n,$$

and we will show that it converges in a nontrivial domain.

To do so, we need to check that the loss of domain that we need at every step can still lead a nontrivial domain. Of course, we need to show that we can still define $\Phi(K_n)$. This mainly entails showing that the range of K_n is still contained in the domain of f. The fact that one can iterate such a quasi-Newton procedure if one starts from a sufficiently approximate solution is standard in KAM theory. In particular, the argument we present here is similar to that in [24].

We formulate the convergence of the Newton method given the existence of approximate inverses in the following result.

Theorem 2.2 (Newton Scheme for the Parameterization Method)**.** *Suppose that $f : D^n_{\rho_*}(0) \subset \mathbb{C}^n \to \mathbb{C}^n$ is an analytic function with $f(0) = 0$ a fixed point. Let $\{\lambda_1, \ldots, \lambda_\ell\}$ be a collection of ℓ distinct eigenvalues of $Df(0)$. Choose associated eigenvectors $\{\xi_1, \ldots, \xi_\ell\}$. Let Λ be the $\ell \times \ell$ matrix*

$$\Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_\ell \end{bmatrix},$$

and A be the $n \times \ell$ matrix

$$A = [\xi_1 | \ldots | \xi_\ell].$$

Let $\bar{C}$ be any bound

$$\|D^2 f\|_{\rho_*} \le \bar{C},$$

on the second derivative of f.

Assumptions;

*A1 (**Approximate Solution:**) Assume that we have $K_0 : D^\ell_{r_0}(0) \subset \mathbb{C}^\ell \to \mathbb{C}^n$ an analytic immersion so that*

$$K_0(0) = 0,$$

$$DK_0(0) = A,$$

and

$$\|f \circ K_0 - K_0 \circ \Lambda\|_{r_0} = \|\Phi(K_0)\|_{r_0} \equiv \epsilon_0.$$

Note that this implies that the image of K_0 is in the domain of f. We assume in addition that the image of K_0 is in the interior of the domain of f, i.e. that there is a ρ_0 so that

$$\|K_0\|_{r_0} \le \rho_0 < \rho_*.$$

*A2 (**Tame Estimates:**) Assume that there is an $\epsilon_* > 0$ so that for any $K \in \mathcal{X}^0_r$ having $\|\Phi(K)\|_r \le \epsilon_*$ there exists a family of linear operators $\mathfrak{L}(K) : \mathcal{Y}_r \to \mathcal{Y}_{re^{-\delta}}$ satisfying the estimates*

$$\|\mathfrak{L}(K)\, \Phi(K)\|_{re^{-\delta}} \le C_1 |\delta|^{-\nu} \|\Phi(K)\|_r,$$

and

$$\|\Phi(K) - D\Phi(K)[\mathfrak{L}(K)\, \Phi(K)]\|_{re^{-\delta}} \le C_2 |\delta|^{-\mu} \|\Phi(K)\|^2_r,$$

for every $\delta > 0$. Here C_1 and C_2 are positive constants which may depend on n, ℓ, and f, but which are required to be uniform in K as K ranges in a sup-norm neighborhood of K_0.

Fix $r_* < r_0$. Then if $\epsilon_0 < \epsilon_*(\nu, \mu, C_1, C_2, \bar{C}, r_0/r_*)$ – where $\epsilon_* > 0$ will be made explicit along the proof – there exists a functions $K_* \in \mathcal{X}^0_{r_*}$ such that

$$\Phi(K_*) = 0,$$

and

$$\|K_* - K_0\|_{r_*} \le s_*(\nu, \mu, C_1, C_2, \bar{C}, r_0/r_*)\epsilon_0,$$

where s_* will also be made explicit along the proof.

The proof of Theorem 2.2 is in the same vein as the proof of the Nash-Moser Implicit Function Theorems in [24]. It is also similar to Theorem 1 in [9]. We give the proof in Appendix 1 for the sake of completeness.

In symplectic case, we will take the dimension n on which f acts to be even and $\ell = n/2$. In the volume preserving case, n can be any integer and $\ell = n - 1$.

Remark 5. In practice the approximate solution K_0 might be defined by considering the linear approximation to the manifold, as in the proofs of Theorems 1.1 and 1.2. On the other hand, K_0 could be an approximate solution obtained numerically. If the numerical approximation is good enough, in the sense of [$A1$], and if the remaining conditions of Theorem 2.2 can be verified, then the theorem both allows us to conclude the existence of a true solution, and provides an explicit bound on the numerical error. It will also turn out that the steps used in the proof lead to a very efficient algorithm (low operation count, small storage requirements).

2.3. **Automatic Reducibility.** In order to apply Theorem 2.2 we must first define a suitable family of approximate inverse operators. In order to define the approximate inverse operators we must approximately solve a certain linear equation, namely Equation (21). By "approximately solve" we mean that we must solve the equation up to an error term which satisfies certain estimates. Automatic reducibility occurs when the geometry of the problem combined with approximate invariance forces the equation for the increment Δ to take a particularly simple or *reduced* form. The reducibility is said to be *automatic* when it is enabled by choice of a suitable system of coordinates, rather than by hypothesizing additional number theoretic constraints on the eigenvalues. These special coordinates are determined by the geometry of the problem, for example the volume or symplectic form in the phase space. For other examples and more thorough discussion of the literature see [11, 9, 7].

The idea is to make a change of variables $\Delta = MW$, which puts $Df(K)\Delta$ in Equation (21) in upper triangular form. Note that if we make this substitution in Equation (21), and multiply both sides by $[M \circ \Lambda]^{-1}$ we obtain

$$[M \circ \Lambda]^{-1}[Df(K)]M\,W - W \circ \Lambda = -[M \circ \Lambda]^{-1}\,E.$$

If the matrix $[M \circ \Lambda]^{-1}[Df(K)]M$ is upper triangular, then we can solve for W using Lemma 2.1 one component at a time using back substitution. The next lemma shows how the back-substitution procedure effects the bounds on the solution so obtained. The lemma uses the particular triangular form which will appear in the applications in the sequel.

Lemma 2.3. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

be an $n \times n$ matrix. Suppose that $n_1 + n_2 = n$ and that A_{11} and A_{22} are $n_1 \times n_1$ and $n_2 \times n_2$ diagonal constant matrices respectively, and that A_{12} is an $n_1 \times n_2$ matrix of bounded analytic functions on $D_r^\ell(0) \subset \mathbb{C}^\ell$ into $\mathbb{C}^n$, i.e. $[A_{12}]_{ij} \in \mathcal{X}_r^{\ell,n}$. Suppose that $\tilde{E} \in \mathcal{Y}_r^{\ell,n}$, and that Λ is an $\ell \times \ell$ matrix of complex numbers. Denote by β_i, $1 \le i \le n$ the nonzero entries of A_{11} and A_{12}. Assume that the n Diophantine conditions

$$|\beta_j - \Lambda^\alpha|^{-1} \le C_D |\alpha|^\tau, \qquad 1 \le j \le n,$$

are satisfied for all multi-indices $\alpha \in \mathbb{N}^\ell$ with $|\alpha| \ge 2$.

Then there is a unique solution $W \in \mathcal{Y}_{re^{-\delta}}$ to the equation

$$AW - W \circ \Lambda = \tilde{E},$$

with

$$\|W\|_{re^{-\delta}} \le C_W |\delta|^{-2(\tau+\ell+1)} \|\tilde{E}\|_r.$$

Note that since

$$W \circ \Lambda = AW - \tilde{E},$$

we also have

$$\|W \circ \Lambda\|_{re^{-\delta}} \le 2C_W |\delta|^{-2(\tau+\ell+1)} \|E\|_r.$$

Proof. Fix $\delta > 0$. The last n_2 rows of the matrix Equation (2.3) have the form

$$\beta_i W_i - W_i \circ \Lambda = \tilde{E}_i \qquad n_1 + 1 \le i \le n_1 + n_2 = n,$$

We take $\delta' = \delta/2$, $r' = re^{-\delta/2}$, and apply Lemma 2.1 n_2 times to obtain unique $W_i \in \mathcal{Y}_{r'}$ satisfying

$$\|W_i\|_{r'} \leq \tilde{C}|\delta'|^{-\tau-\ell-1}\|\tilde{E}_i\|_r \leq \tilde{C}2^{\tau+\ell+1}|\delta|^{-\tau-\ell-1}\|\tilde{E}\|_r, \qquad (22)$$

where $\tilde{C}$ is as in Lemma 2.1. The remaining component equations have the form

$$\left(\beta_i W_i + \sum_{j=1}^{n_2}[A_{12}]_{ij}W_{n_1+j}\right) - W_i \circ \Lambda = \tilde{E}_i \qquad 1 \leq i \leq n_1.$$

This is $\beta_i W_i - W_i \circ \Lambda = \overline{E}_i$, $1 \leq i \leq n_1$ where $\overline{E}_i \in \mathcal{Y}_{r'}$ is given by

$$\overline{E}_i = \tilde{E}_i - \sum_{j=1}^{n_2}[A_{12}]_{ij}W_{n_1+j},$$

and the W_{n_1+j}, $1 \leq j \leq n_2$ are already known, and satisfy Equation (22). Then

$$\|\overline{E}_i\|_{r'} \leq \|\tilde{E}\|_r + n_2\|A_{12}\|_{r'}\tilde{C}2^{\tau+\ell+1}|\delta|^{-\tau-\ell-1}\|\tilde{E}\|_r,$$

and we can solve for the first n_1 components of W using Lemma 2.1, now with $\delta' = \delta/2$. This gives $W_i \in \mathcal{Y}_{r'e^{-\delta/2}}$ with

$$\|W_i\|_{r'e^{-\delta/2}} \leq \tilde{C}|\delta/2|^{-\tau-\ell-1}\|\overline{E}_i\|_{r'}.$$

But $r'e^{-\delta/2} = re^{-\delta}$ so, taking supremums we have

$$\|W\|_{re^{-\delta}} \leq 2n_2\tilde{C}^2\|A_{12}\|_{r'}2^{2(\tau+\ell+1)}|\delta|^{-2(\tau+\ell+1)}\|\tilde{E}\|_r.$$

This gives an explicit expression for C_W. □

Under the hypotheses of Lemma 2.3 we can define a solution operator $\mathfrak{T} : \mathcal{Y}_r \to \mathcal{Y}_{re^{-\delta}}$ having

$$\|\mathfrak{T}\|_{(\mathcal{Y}_r, \mathcal{Y}_{re^{-\delta}})} \leq C_W|\delta|^{-2(\tau+\ell+1)}.$$

The next lemma shows that we can define the approximate inverse operators needed for Theorem 2.2, as long as we can find an M which puts Equation (21) into "approximately upper triangular form", i.e. upper triangular up to a linear error in E.

Lemma 2.4. *Suppose that $K \in \mathcal{X}_r^0$, $E = \Phi(K) \in \mathcal{Y}_r$. Let $r' = re^{-\delta/2}$ and assume that M and $[M \circ \Lambda]^{-1}$ are matrices with coefficients in $\mathcal{X}_{r'}$ and that*

$$[M \circ \Lambda]^{-1}Df(K)M = A + R,$$

where A has the upper triangular form hypothesized in Lemma 2.3 and R satisfies the bound

$$\|R\|_{re^{-\delta}} \leq C_R|\delta|^{-\sigma_R}\|E\|_r,$$

for all $\delta > 0$ and some positive constants C_R and σ_R. Then the family of linear operators $\mathfrak{L}(K) : \mathcal{Y}_r \to \mathcal{Y}_{re^{-\delta}}$ defined by

$$[\mathfrak{L}(K)]E = -M\mathfrak{T}([M \circ \Lambda]^{-1}E),$$

satisfies the tame estimates of Theorem 2.2.

Remark 6. The reason for taking $M_{ij} \in \mathcal{X}_{r'}$ is that in practice M will be a function of DK. Then we can bound M only on a strictly smaller domain than the domain of K.

Proof. Define $\tilde{E} \in \mathcal{Y}_{r'}$ by $\tilde{E} \equiv [M \circ \Lambda]^{-1}\Phi(K)$. By Lemma 2.3 and the comment thereafter we have a well defined and bounded solution operator $\mathfrak{T}(K) : \mathcal{Y}_{r'} \to \mathcal{Y}_{r'e^{-\delta/2}} = \mathcal{Y}_{re^{-\delta}}$ for equation

$$AW - W \circ \Lambda = \tilde{E},$$

having

$$\|\mathfrak{T}\|_{\left(\mathcal{Y}_r, \mathcal{Y}_{re^{-\delta}}\right)} \leq C_W 2^{2(\tau+\ell+1)}|\delta|^{-2(\tau+\ell+1)}.$$

Then the operator $\mathfrak{L}(K) : \mathcal{Y}_{re^{-\delta/2}} \to \mathcal{Y}_{re^{-\delta}}$ given by

$$\begin{aligned} \mathfrak{L}(K)\,\Phi(K) &= -M\,\mathfrak{T}([M \circ \Lambda]^{-1}\,E) \\ &= -MW, \end{aligned}$$

is well defined, and we have the bound

$$\begin{aligned} \|\mathfrak{L}(K)\,\Phi(K)\|_{re^{-\delta}} &\leq \|M\|_{re^{-\delta/2}}\|\mathfrak{T}\|_{\left(\mathcal{Y}_r, \mathcal{Y}_{re^{-\delta}}\right)}\|(M \circ \Lambda)^{-1}\|_{re^{-\delta/2}}\|E\|_r \\ &\leq \|M\|_{re^{-\delta}}\|(M \circ \Lambda)^{-1}\|_{re^{-\delta}} C_W 2^{2(\tau+\ell+1)}|\delta|^{-2(\tau+\ell+1)}\|E\|_r \\ &\leq C_1|\delta|^{\nu}\|\Phi(K)\|_r, \end{aligned}$$

where C_1 depends on both the coordinate transform M and the transformation $[M \circ \Lambda]^{-1}$.

We note that W exactly solves the equation

$$[M \circ \Lambda]AW - [M \circ \Lambda][W \circ \Lambda] = -E,$$

and that

$$Df(K)M = [M \circ \Lambda][A + R],$$

so that

$$\begin{aligned} \Phi(K) - D\Phi(K)\,\mathfrak{L}(K)\,\Phi(K) &= E - [Df(K)\,[\mathfrak{L}(K)E] - [\mathfrak{L}(K)E] \circ \Lambda] \\ &= E - [Df(K)[-MW] - [-MW] \circ \Lambda] \\ &= E - [-[M \circ \Lambda][A + R]W + [MW] \circ \Lambda] \\ &= E + [M \circ \Lambda]AW - [M \circ \Lambda][W \circ \Lambda] + [M \circ \Lambda]RW \\ &= [M \circ \Lambda]RW. \end{aligned}$$

Then we obtain the estimate

$$\begin{aligned} &\|\Phi(K) - D\Phi(K)\,\mathfrak{L}(K)\,\Phi(K)\|_{re^{-\delta}} \\ &\leq \|M \circ \Lambda\|_{re^{-\delta/2}}\|R\|_{re^{-\delta}}\|W\|_{re^{-\delta}} \\ &\leq \|M \circ \Lambda\|_{r'}\,(C_R|\delta|^{\sigma_R}\|E\|_r)\left(\|M\|_{r'}C_W 2^{2(\tau+\ell+1)}|\delta|^{-2(\tau+\ell+1)}\|(M \circ \Lambda)^{-1}\|_{r'}\|E\|_r\right) \\ &\leq C_2|\delta|^{\mu}\|E\|_r^2, \end{aligned}$$

where C_2 depends both on the norm of the coordinate transform $M \circ \Lambda$ and its inverse $[M \circ \Lambda]^{-1}$. □

Then in order to apply Theorem 2.2 and prove the existence of the desired parameterization function and hence the desired invariant manifold, it is enough to be able to find the matrix M and remainder R described in Lemma 2.4. In Sections (3) and (4) we show how to compute the coordinate transform M for symplectic and volume preserving maps respectively, exploiting the geometry of the problem.

Remark 7 (Uniformity of C_1 and C_2)**.** In many applications of automatic reducibility (including the proofs of Theorems 1.1 and 1.2 discussed in the sequel) the coordinate transformation M is itself a function of the approximate solution K. In the context of the Newton scheme this raises an important concern; namely that Theorem 2.2 requires that the constants C_1 and C_2 are uniformly bounded in a neighborhood of K_0. However if C_1 and C_2 depend on K through M, then it is essential to bound M uniformly in a neighborhood of K_0, and to insure that the Newton iteration remains in this neighborhood. Obtaining such uniform estimates on M (and any auxiliary functions included in its definition) is a technical issue which we address in Section 6.

3. Symplectic Diffeomorphisms.

In this section we are concerned with maps $f : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$, having $f(0) = 0$, $Df(0)$ diagonalizable, and which preserve a symplectic structure ω. Let I_n denote the $n \times n$ identity matrix. By the Darboux theorem [13] we can assume that ω has the standard form, and denote it by

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

when expressed in coordinates.

The automatic reducibility argument in the symplectic case depends on the fact that the invariant manifolds we are studying are Lagrangian. The Lagrangian nature of the parameterization gives that the tangent space of K and its symplectic conjugate span $\mathbb{C}^{2n}$ at each point in the image of K. Since the manifold $M = \mathrm{Image}(K)$ is n-dimensional, this means that T_pM and its symplectic conjugate form a basis for $\mathbb{C}^{2n}$ for each $p \in M$.

3.1. Approximate Lagrangian Character of Approximate Solutions.

In this section we establish certain facts regarding the Lagrangian character of solutions, and approximate solutions of $\Phi(K) = 0$. In both the exact and approximate arguments, the following observation is crucial. Since a parameterization function K satisfies the equation

$$f \circ K = K \circ \Lambda,$$

and a symplectic map has that $f^*\omega = \omega$, then combining these facts gives

$$(f \circ K)^*\omega = K^* f^* \omega = K^*\omega,$$

by the functor property of pull-backs. However the invariance equation gives

$$(f \circ K)^*\omega = (K \circ \Lambda)^*\omega = \Lambda^* K^* \omega,$$

where again we have used the functoriality. Combining these gives

$$K^*\omega = \Lambda^* K^* \omega. \tag{23}$$

On the other hand, if K is an approximate solution with error E defined by

$$f \circ K - K \circ \Lambda = E,$$

then we define the *error form*

$$(f \circ K)^*\omega - (K \circ \Lambda)^*\omega = K^*\omega - \Lambda^* K^* \omega \equiv \omega_E.$$

Note that $\omega_E \equiv 0$ when K is an exact solution.

Theorem 3.1 (Exact Solutions are Lagrangian)**.** *Suppose that $K \in \mathcal{X}_r^0$ is an exact solution of Equation (14). Then*

$$K^*\omega = 0.$$

Proof. Rewriting Equation (23) in coordinates gives

$$[DK(\theta)]^\top J[DK(\theta)] = \Lambda^\top [DK(\Lambda\theta)]^\top J[DK(\Lambda\theta)]\Lambda.$$

Since this is cumbersome we define $L(K) : D_r^n(0) \subset \mathbb{C}^n \to GL(\mathbb{R}^n)$ by

$$L(K)(\theta) \equiv [DK(\theta)]^\top JDK(\theta).$$

Note that L just denotes $K^*\omega$ expressed in coordinates. Similarly we denote the composition with Λ by

$$L(K \circ \Lambda)(\theta) = L(K(\Lambda\theta)).$$

When it is clear from context that K is fixed (as inside a single step of the Newton Iteration) we sometimes write L for $L(K)(\theta)$ and $L(\Lambda)$ for $L(K \circ \Lambda)(\theta)$. Using these operators, Equation (23) becomes

$$\Lambda^{-1}[L(K(\theta))]\Lambda^{-1} - [L(K(\Lambda\theta))] = 0, \tag{24}$$

which is an $n \times n$ matrix equation (here we have also used that Λ is a diagonal matrix). Consider the scalar equations given by the components of Equation (24). The ij-th entry of the matrix is given by

$$\lambda_i^{-1}\lambda_j^{-1}L_{ij}(K(\theta)) - L_{ij}(K(\Lambda\theta)) = 0.$$

These are seen to be zero by applying Lemma 2.1 with a right hand side of zero. □

The theorem gives that $L(K) = 0$ when K solves Equation (14). In coordinates this says that for all $\theta \in D_r(0)$ we have

$$DK(\theta)^\top JDK(\theta) = 0.$$

Since $DK(\theta)$ is a $2n \times n$ matrix, the above says that $DK(\theta)$ and $JDK(\theta)$ span $\mathbb{C}^{2n}$. Then

$$\det([DK(\theta)|JDK(\theta)]) \neq 0,$$

as the columns constitute a basis.

The following theorem measures the size of $L(K)$, when K is only an approximate solution of Equation (14). This estimate is essential in Section 3.2 for establishing the bounds needed in order to apply Lemma 2.4. More specifically, in Section 3.2 we define, for the case of symplectic maps, the transformation M which is hypothesized Lemma 2.4. Then we apply Theorem 3.2 in order to show that the M so chosen diagonalizes $Df(K)$ up to a remainder which is linear in E.

Theorem 3.2 (Approximate Reducibility Estimate)**.** *Suppose that $K \in \mathcal{X}_r^0$ is an approximate solution for the functional Equation (14), so that $E = \Phi(K)$ is small. Suppose also that the eigenvalue collection $\{\lambda_1, \ldots, \lambda_n\}$ of n-eigenvalues of the symplectic matrix $Df(0)$ satisfy the Diophantine conditions*

$$|\lambda_i^{-1} - \Lambda^\alpha|^{-1} \leq C_D|\alpha|^\tau,$$

for some constants $C_D > 1$, $\tau > 0$, as in the hypotheses of Theorem 1.1.

Then ω_E is approximately zero, in the sense that for any $\delta > 0$ we have the bounds

$$\|L(K)\|_{re^{-\delta}} \leq C_L|\delta|^{-\mu_L}\|E\|_r,$$

where explicit expressions for C_L and μ_L are given in the proof. In addition, a similar bound exists for the composition $L(\Lambda)$.

Proof. First we note that in coordinates

$$\begin{aligned} & [D(f\circ K)]^\top J[D(f\circ K)] - [D(K\circ\Lambda)]^\top J[D(K\circ\Lambda)] \\ = \; & [DK(\Lambda)\Lambda + DE]^\top J[DK(\Lambda)\Lambda + DE] - [DK(\Lambda)\Lambda]^\top J[DK(\Lambda)\Lambda] \\ = \; & [\Lambda^\top DK(\Lambda)^\top + DE^\top]J[DK(\Lambda)\Lambda + DE] - \Lambda^\top DK(\Lambda)^\top J[DK(\Lambda)\Lambda] \\ = \; & \Lambda^\top DK(\Lambda)^\top J\; DE + DE^\top J\; DK(\Lambda)\Lambda + DE^\top J\; DE. \end{aligned}$$

Since we also have that, $\omega_E = K^*\omega - \Lambda^* K^*\omega$, or in coordinates

$$L - \Lambda^T L(\Lambda)\Lambda^T = \omega_E,$$

we obtain the matrix equation

$$\Lambda^{-1} L \Lambda^{-1} - L(\Lambda) = G,$$

where

$$G = \Lambda^{-1}[\Lambda^\top DK(\Lambda)^\top J\; DE + DE^\top J\; DK(\Lambda)\Lambda + DE^\top J\; DE]\Lambda^{-1}.$$

Then

$$\|G\|_{re^{-\delta/2}} \le \lambda_-^{-2}\lambda_+ 3\|DK\|_{re^{-\delta/2}} C_* |\delta/2|^{-1}\|E\|_r \le C_G|\delta|^{-1}\|E\|_r,$$

where λ_- and λ_+ denote the magnitude of the smallest and largest modulus of $\{\lambda_1, \ldots, \lambda_n\}$ respectively. Note that since it is assumed that the collection of eigenvalues is of mixed-stability, it is the case that $0 < \lambda_- < 1 < \lambda_+$.

Then we can bound L by applying Lemma 2.1 to its components. The ij-th component equation is

$$\lambda_i^{-1}\lambda_j^{-1} L_{ij} - L_{ij}(\Lambda) = (G)_{ij}.$$

But

$$\begin{aligned} |\lambda_i^{-1}\lambda_j^{-1} - \Lambda^\alpha|^{-1} &= |\lambda_j^{-1}(\lambda_i^{-1} - \Lambda^{\alpha+\chi_j})|^{-1} \\ &\le \lambda_+ |\lambda_i^{-1} - \Lambda^{\alpha+\chi_j}|^{-1} \\ &\le \lambda_+ C_D|\alpha+1|^\tau \\ &\le \lambda_+ C_D\left(\frac{3}{2}\right)^\tau |\alpha|^\tau, \end{aligned}$$

where the multi-index $\alpha + \chi_j \equiv (\alpha_1, \ldots, \alpha_j + 1, \ldots \alpha_n)$. This says that $|\lambda_i^{-1}\lambda_j^{-1} - \Lambda^\alpha|^{-1}$ is bound by a constant times the hypothesized Diophantine condition. Then, using Lemma 2.1 we obtain the bound

$$\begin{aligned} \|L_{ij}\|_{re^{-\delta}} &= \lambda_+\left(\frac{3}{2}\right)^\tau \tilde{C} 2^{\tau+n+1}|\delta|^{-\tau-n-1}\|G\|_{re^{-\delta/2}} \\ &\le \lambda_+\left(\frac{3}{2}\right)^\tau \tilde{C} C_G 2^{\tau+n+1}|\delta|^{-\tau-n-2}\|E\|_r \\ &\le C_L|\delta|^{-\mu_L}\|E\|_r. \end{aligned}$$

□

Remark 8. Note that C_L depends on DK (through C_G), but that in the context of the quadratic convergence scheme this can be bound uniformly in terms of DK_0 as in Section 6.2 (see also Remark (7)).

3.2. **Automatic Reducibility for Exact Solutions.** Now we are ready to define, for Lagrangian mixed-stable manifolds of symplectic maps, the coordinate transformation M which is needed in the definition of the approximate inverse operator of Lemma 2.4. For any $K \in \mathcal{X}_r^0$ we define the $n \times n$ matrix function $N : D_r^n(0) \subset \mathbb{R}^n \to GL(\mathbb{R}^n)$ by

$$N(\theta) = [DK(\theta)^\top DK(\theta)]^{-1}.$$

We note that since $DK(0) = [\xi_1| \ldots |\xi_n]$ is of maximal rank, $N(0)$ is defined for any $K \in \mathcal{X}_r^0$. Then for any choice of K, N in analytic on any small enough polydisk $D_r^\ell(0)$, and for all $\theta \in D_r^\ell(0)$ on has the normalization

$$DK(\theta)^\top DK(\theta)N(\theta) = \mathrm{Id}_n.$$

Next, we define the $2n \times 2n$ matrix function $M : D_r^n(0) \subset \mathbb{R}^n \to GL(\mathbb{R}^{2n})$ by

$$M(\theta) = [DK(\theta)|J^{-1}DK(\theta)N(\theta)].$$

We note that if K is an exact solution of $\Phi(K) = 0$ then by Theorem 3.1, K is Lagrangian. In that case, DK is normal to $J^{-1}DK$, and the normality is not disturbed by the factor of N. Then M is invertible.

Heuristically the core of the reducibility argument is that; since $M(\theta)$ is a basis for all θ, we have that $M(\Lambda\theta)$ is a basis as well. Then there is a $2n \times 2n$ matrix (of functions) $A(\theta)$ having

$$Df(K(\theta))\, M(\theta) = M(\Lambda\theta)A(\theta). \tag{25}$$

We will see that, when K is an exact solution, A is upper triangular. In order to compute A, we decompose into $n \times n$ blocks A_{ij} and write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Similarly, we decompose M into $2n \times n$ blocks and write

$$M = [M_1|M_2] = [DK|J^{-1}DK\, N].$$

In this decomposition, Equation (25) is equivalent to the two matrix equations

$$\begin{aligned} Df(K)M_1 &= M(\Lambda) \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} \\ Df(K)M_2 &= M(\Lambda) \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}. \end{aligned} \tag{26}$$

We begin by observing that

$$\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \begin{pmatrix} \Lambda \\ 0 \end{pmatrix}.$$

To see this, compute

$$\begin{aligned} Df(K)M_1 &= Df(K)\, DK \\ &= [DK](\Lambda)\Lambda \\ &= M(\Lambda) \begin{pmatrix} \Lambda \\ 0 \end{pmatrix}, \end{aligned}$$

where we have differentiated the invariance expression in order to pass from the first to the second line. (Note that $[DK](\Lambda)\Lambda$ means $[DK(\Lambda\theta)]\Lambda$).

The computations of A_{12} and A_{22} are somewhat more involved. To begin with, observe that Equation (26) can be rewritten as

$$\begin{aligned} Df(K)M_2 &= Df(K)J^{-1}\, DK\, N \\ &= M_1(\Lambda)\, A_{12} + M_2(\Lambda)\, A_{22}, \end{aligned}$$

or

$$Df(K)\, J^{-1}\, DK\, N = DK(\Lambda)\, A_{12} + J^{-1}\, DK(\Lambda)\, N(\Lambda)\, A_{22}. \tag{27}$$

To compute A_{12} we multiply both sides of Equation (27) by $DK(\Lambda)^\top$, giving

$$\begin{aligned} & DK(\Lambda)^\top\, Df(K)\, J^{-1}\, DK\, N \\ = \;& DK(\Lambda)^\top\, \left[DK(\Lambda)\, A_{12} + J^{-1}\, DK(\Lambda)\, N(\Lambda)\, A_{22}\right] \\ = \;& DK(\Lambda)^\top\, DK(\Lambda)\, A_{12} + DK(\Lambda)^\top\, J^{-1}\, DK(\Lambda)\, N(\Lambda)\, A_{22} \\ = \;& N(\Lambda)^{-1}\, A_{12} - L(\Lambda)N(\Lambda)A_{22} \\ = \;& N(\Lambda)^{-1}A_{12}, \end{aligned}$$

due to the Lagrangian property of the exact solution K. Then

$$A_{12} = N(\Lambda)DK(\Lambda)^T Df(K)J^{-1}DKN.$$

Finally, to obtain the expression for A_{22} we multiply both sides of Equation (27) by $DK(\Lambda)^\top\, J$. On the left this gives

$$\begin{aligned} DK(\Lambda)^\top\, J\, Df(K)\, J^{-1}\, DK\, N &= DK(\Lambda)^\top\, [Df(K)]^{-\top}\, DK\, N \\ &= \left[[Df(K)]^{-1}\, DK(\Lambda)\right]^{-\top}\, DK\, N \\ &= \left[DK\, \Lambda^{-1}\right]^\top\, DK\, N \\ &= \Lambda^{-T}\, DK^T\, DK\, N \\ &= \Lambda^{-T} I \\ &= \Lambda^{-1}, \end{aligned}$$

where we have used that $Df(K)$ is a symplectic matrix, the inverse function theorem, the fact that K is an exact solution of the invariance equation, the normalization $DK^\top DKN = I$, and the fact that Λ is diagonal.

Multiplying the right hand side of Equation (27) by the same factor of $DK(\Lambda)^\top\, J$ gives

$$\begin{aligned} DK(\Lambda)^\top\, J\, \left[DK(\Lambda)\, A_{12} + J^{-1}\, DK(\Lambda)\, N(\Lambda)\, A_{22}\right] &= L(\Lambda)A_{12} + \mathrm{Id}_n A_{22} \\ &= A_{22}, \end{aligned}$$

or $A_{22} = \Lambda^{-1}$ which shows that

$$Df(K)\, M = M(\Lambda)A = M(\Lambda)\begin{pmatrix} \Lambda & A_{12} \\ 0 & \Lambda^{-1} \end{pmatrix}, \tag{28}$$

for an exact solution K.

3.3. **Automatic Reducibility for approximate solutions.** Next we want to measure the difference between $Df(K)M$ and the expression given by the far right hand side of Equation (28) when K is only an approximate solution of Equation

(14). We will show that when the image of K is only approximately invariant under f we have the decomposition

$$\begin{aligned}[M\circ\Lambda]^{-1}Df(K)M &= A+R \\ &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} \Lambda & \bar{A} \\ 0 & \Lambda^{-1} \end{pmatrix} + \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},\end{aligned}$$

where

$$\bar{A} = N(\Lambda)DK(\Lambda)^T Df(K)J^{-1}DKN. \tag{29}$$

The point of the computation is to obtain precisely the form of the remainder R, in order to show that we can bound $\|R\|$ linearly in $\|E\|$ as required by Lemma 2.4. Note however that when we perform the Quasi-Newton scheme, the R term is exactly the term which is ignored, or "thrown away". The point of the present discussion is to justify rigorously that ignoring the R term is allowable (in the sense of Lemma 2.4). That is, that the algorithm obtained by ignoring this term still leads to a quadratic improvement of the error with tame estimates.

Theorem 3.3 (Symplectic Remainder Estimate)**.** *Assume that K satisfies the hypotheses of Lemma 2.4 and Theorem 3.2, and in addition that*

$$\|N[\Lambda]\|^2_{re^{-\delta/2}}\|L[\Lambda]\|_{re^{-\delta/2}} < \sigma, \tag{30}$$

with $0<\sigma<1$ fixed. Then for any $\delta>0$ the remainder satisfies the estimate

$$\|R\|_{re^{-\delta}} \le C_R|\delta|^{-\mu_R}\|E\|_r,$$

where explicit bounds for C_R and μ_R are given in the proof.

We note that C_R will depend on K through N, M, and M^{-1}. However we will show in Section 6 that these (and hence C_R) can be bound uniformly in a small enough neighborhood of K_0, so that we can apply Lemma 2.4 and then Theorem 2.2. (See also Remark 7).

Proof. There are two parts to the proof. Part I is the derivation of explicit expressions for the R_{ij}, and Part II is estimating these quantities.

Part I: As in the case of exact an solution K we decompose M into $2n\times n$ blocks and write

$$M = [M_1|M_2] = [DK|J^{-1}DK\,N],$$

and obtain the matrix equations

$$\begin{aligned} Df(K)M_1 &= M(\Lambda)\begin{pmatrix} A_{11} \\ A_{21}\end{pmatrix}, \\ Df(K)M_2 &= M(\Lambda)\begin{pmatrix} A_{12} \\ A_{22}\end{pmatrix}.\end{aligned}$$

Now

$$\begin{pmatrix} A_{11} \\ A_{21}\end{pmatrix} = \begin{pmatrix} \Lambda \\ 0\end{pmatrix} + R_1,$$

where R_1 is bounded by E, as

$$\begin{aligned} Df(K)M_1 &= Df(K)\,DK \\ &= [DK](\Lambda)\Lambda + DE \\ &= M(\Lambda)\begin{pmatrix}\Lambda \\ 0\end{pmatrix} + DE, \end{aligned}$$

where we have used the differentiated form of the invariance expression. Then we multiply both sides by $[M \circ \Lambda]^{-1}$ to obtain

$$[M \circ \Lambda]^{-1} Df(K)M_1 = \begin{pmatrix}\Lambda \\ 0\end{pmatrix} + [M \circ \Lambda]^{-1}\, DE,$$

so that

$$\begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix} = [M \circ \Lambda]^{-1}\, DE.$$

More delicate is the computation of A_{12} and A_{22}. These terms are computed by multiplying

$$Df(K)J^{-1}DKN = DK(\Lambda)A_{12} + J^{-1}DK(\Lambda)N(\Lambda)A_{22},$$

by $DK(\Lambda)^\top$ and $DK(\Lambda)^\top J$ respectively. We will show that these multiplications lead to the exact equations

$$N(\Lambda)^{-1}\, A_{12} + L(\Lambda)\, N(\Lambda)\, A_{22} \;=\; G \tag{31}$$

$$L(\Lambda)\, A_{12} + A_{22} \;=\; \Lambda^{-1} + R_D, \tag{32}$$

where

$$G = DK(\Lambda)^\top\, Df(K)\, J^{-1}\, DK\, N.$$

To see that the right hand side of Equation (32) is correct and obtain the explicit form of R_D, consider $DK(\Lambda)^\top\, J\, Df(K)\, J^{-1}\, DK\, N$.

Note that, because $Df(K)$ is a symplectic matrix we have that

$$JDf(K)J^{-1} = Df(K)^{-\top} = [Df(K)^{-1}]^\top.$$

Now let $R_{f\circ K,1}(\cdot)$ denote the first order Taylor remainder of Df^{-1} expanded about $f \circ K$, so that

$$Df^{-1}(f \circ K - E) = Df^{-1}(f \circ K) + R_{f\circ K,1}(-E),$$

and there is a constant C_E so that $\|R_{f\circ K,1}(E)\|_r \le C_E \|E\|_r$. (An explicit bound on C_E can be obtained using the Lagrange remainder formula). The inverse function theorem applied to Df^{-1} gives

$$\begin{aligned} Df^{-1}(K \circ \Lambda) &= Df^{-1}(f \circ K - E) \\ &= Df^{-1}(f \circ K) + R_{f\circ K,1}(-E) \\ &= [Df(K)]^{-1} + R_{f\circ K,1}(-E), \end{aligned}$$

or

$$Df(K)^{-1} = Df^{-1}(K \circ \Lambda) - R_{f\circ K,1}(-E).$$

Then

$$DK(\Lambda)^\top\, J\, Df(K)\, J^{-1}\, DK\, N = DK(\Lambda)^\top \left[Df^{-1}(K \circ \Lambda) - R_{f\circ K,1}(-E)\right]^\top\, DK\, N$$

$$
\begin{aligned}
&= \left[[Df^{-1}(K\circ\Lambda)DK(\Lambda) - R_{f\circ K,1}(-E)]DK(\Lambda)\right]^\top DK\,N\\
&= \left[DK\,\Lambda^{-1} - [Df^{-1}(E)]DE\,\Lambda^{-1} - R_{f\circ K,1}(-E)\,DK(\Lambda)\right]^\top DK\,N\\
&= \left[\Lambda^{-\top}DK^\top - \Lambda^{-\top}DE^\top[Df^{-1}(E)]^\top - DK(\Lambda)^\top R_{f\circ K,1}(-E)^\top\right] DK\,N\\
&= \Lambda^{-1} + R_D,
\end{aligned}
$$

where we have used the invariance expression

$$Df^{-1}(K\circ\Lambda)DK(\Lambda) = DK\,\Lambda^{-1} - [Df^{-1}(E)]DE\,\Lambda^{-1},$$

to pass from line 2 to line 3, and have defined

$$R_D = \Lambda^{-1}DE^\top[Df^{-1}(E)]^\top\,DK\,N - DK(\Lambda)^\top R_{f\circ K,1}(-E)^\top\,DK\,N.$$

(Note that the invariance expression is obtained by applying f^{-1} to $\Phi(K) = E$ and differentiating).

Returning to equations (31) and (32) we have (in matrix form)

$$\begin{pmatrix} N(\Lambda)^{-1} & L(\Lambda)N(\Lambda)\\ L(\Lambda) & Id\end{pmatrix}\begin{bmatrix} A_{12}\\ A_{22}\end{bmatrix} = \begin{bmatrix} G\\ \Lambda^{-1}+R_D\end{bmatrix}.$$

Changing variables to $A_{12} - \bar{A} = R_{12}$ and $A_{22} - \Lambda^{-1} = R_{22}$ gives

$$
\begin{aligned}
\begin{pmatrix} N(\Lambda)^{-1} & L(\Lambda)N(\Lambda)\\ L(\Lambda) & Id\end{pmatrix}\begin{bmatrix} A_{12}-\bar{A}\\ A_{22}-\Lambda^{-1}\end{bmatrix} &= \begin{bmatrix} G - N(\Lambda)^{-1}\bar{A} - L(\Lambda)N(\Lambda)\Lambda^{-1}\\ R_D - L(\Lambda)\bar{A}\end{bmatrix}\\
&= \begin{bmatrix} -L(\Lambda)N(\Lambda)\Lambda^{-1}\\ R_D - L(\Lambda)\bar{A}\end{bmatrix},
\end{aligned}
$$

as $G - N(\Lambda)^{-1}\bar{A} = 0$. Now consider the matrix

$$
\begin{aligned}
\mathcal{A} &= \begin{pmatrix} N(\Lambda)^{-1} & L(\Lambda)N(\Lambda)\\ L(\Lambda) & Id\end{pmatrix}\\
&= \begin{pmatrix} N(\Lambda)^{-1} & 0\\ 0 & Id\end{pmatrix}\left[Id + \begin{pmatrix} 0 & N(\Lambda)L(\Lambda)N(\Lambda)\\ L(\Lambda) & 0\end{pmatrix}\right].
\end{aligned}
$$

Then

$$\mathcal{A}^{-1} = \left[Id + \begin{pmatrix} 0 & N(\Lambda)L(\Lambda)N(\Lambda)\\ L(\Lambda) & 0\end{pmatrix}\right]^{-1}\begin{pmatrix} N(\Lambda) & 0\\ 0 & Id\end{pmatrix},$$

exists by Neumann's Theorem and the assumption on σ (Equation (30)). Moreover we have the bound

$$\|\mathcal{A}^{-1}\|_r \le \frac{1}{1-\sigma}\|N[\Lambda]\|_r,$$

which allows us to control R_{12} and R_{22}.

Part II (Bounds):

1. Since $(R_{11}, R_{21})^\top = ([M\circ\Lambda]^{-1}DE)^\top$, it immediately follows that

   $$\sup(\|R_{11}\|_{re^{-\delta}}, \|R_{21}\|_{re^{-\delta}}) \le \|[M\circ\Lambda]^{-1}\|_{re^{-\delta/2}}\frac{C_*}{|\delta|}\|E\|_r \equiv \frac{C_{DE}}{|\delta|}\|E\|_r,$$

   by Cauchy Estimates. Again note that C_{DE} is a uniform bound as $[M\circ\Lambda]^{-1}$ will be shown in Section 6 to depend in a uniform way on K.

2. Consider R_D. We have

$$\|R_D\|_{re^{-\delta}} \quad \leq \quad C_{RD}|\delta|^{-1}\|E\|_r,$$

where

$$C_{RD} = 2\lambda_-^{-1}C_*C_E\|Df^{-1}\|_{\tilde{\rho}}\|\|DK\|^2_{re^{-\delta}}\|N\|_{re^{-\delta}},$$

where $0 < \epsilon < \tilde{\rho}$ (of course by taking $\tilde{\rho} > \epsilon_0$ this estimate is uniform throughout the Newton Iteration scheme).

3. We have

$$\|\bar{A}\|_{re^{-\delta/2}} \leq \|Df\|_{\rho_*}\|DK\|_{re^{-\delta/2}}\|DK[\Lambda]\|_{re^{-\delta/2}}\|N\|_{re^{-\delta/2}}\|N[\Lambda]\|_{re^{-\delta/2}},$$

directly from the definition.

4. The bound on $\mathcal{A}^{-1}$ allows us to estimate:

$$\begin{aligned}
&\left\|\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}\right\|_{re^{-\delta}} \\
&\leq \|\mathcal{A}\|^{-1}_{re^{-\delta/2}} \max\left(\lambda_-^{-1}\|L(\Lambda)\|_{re^{-\delta/2}}\|N(\Lambda)\|^2_{re^{-\delta/2}}, \|R_D\|_{re^{-\delta/2}} + \|L(\Lambda)\|_{re^{-\delta/2}}\|\bar{A}\|_{re^{-\delta/2}}\right) \\
&\leq \frac{\lambda_-^{-1}}{1-\sigma}\|N(\Lambda)\|_{re^{-\delta/2}}\|\bar{A}\|_{re^{-\delta/2}}(C_{RD} + C_L)|\delta|^{-\mu_L}\|E\|_r \\
&\leq C_{22}|\delta|^{-\mu_L}\|E\|_r,
\end{aligned}$$

where C_{22} depends on N, $\bar{A}$, λ_-, σ, C_{RD}, and C_L.

Finally we obtain that

$$\|R\|_{re^{-\delta}} \leq C_R|\delta|^{-\mu_R}\|E\|_r,$$

with

$$C_R = 2\max(C_{DE}, C_{22}) \qquad \text{and} \qquad \mu_R = \max(1, \mu_L).$$

□

Once again we remark that the constant C_R depends on K through N, M, and M^{-1}, but that these can be bound uniformly in a small neighborhood of the initial solution K_0 (as shown in Section 6. See also Remark 7). We also remark that the volume preserving remainder given in Section 4.1 can be shown to satisfy an estimate linear in $\|E\|_r$ by a similar computation, but that in the volume preserving case we will omit the details.

4. **Volume Preserving Diffeomorphisms.** Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is real analytic and invertible, that f preserves a volume form ω on $\mathbb{R}^n$, and that $f(0) = 0$ is hyperbolic fixed point of f. Let $\{\lambda_1, \ldots, \lambda_n\}$ denote the eigenvalues of $Df(0)$, written in some fixed order (and not necessarily ordered by magnitude). Fix the co-dimension one sub-collection of eigenvalues $\{\lambda_1, \ldots, \lambda_{n-1}\}$ and let Λ be the $(n-1) \times (n-1)$ matrix having the elements of the sub-collection on the diagonal and zeros elsewhere. We assume that λ_n is not the complex conjugate of any λ_i with $1 \leq i \leq n-1$; i.e. if there are complex conjugate eigenvalues, they are included in $\{\lambda_i\}$.

Let $\xi_1, \ldots, \xi_{n-1}$ be the associated real eigenvectors. Note that the linear subspace $E = \text{span}(\xi_1, \ldots, \xi_{n-1})$ is invariant under that action of $Df(0)$. This statement holds even in the case that the sub-collection $\{\lambda_1, \ldots, \lambda_{n-1}\}$ is a collection of *mixed-stability* eigenvalues (i.e. we are not asking that the elements of the collection all have absolute value either greater than or less than one). In this section we show that if the eigenvalues satisfy certain number theoretic hypothesis, then there exists an invariant manifold for f, which is tangent to E at the origin.

Throughout this section we assume (without loss of generality due to [17]) that the volume form preserved by f is the standard one

$$\omega = dx_1 \wedge \cdots \wedge dx_n.$$

We will also assume that $\mathbb{R}^n$ is endowed with the standard metric.

A construction that will play an important role in our discussion is the standard cross product of $n-1$ vectors to give another one. We will denote it

$$v = w_1 \times \cdots \times w_{n-1}.$$

In old-fashioned tensor notation v is defined by

$$v_i = \epsilon^i_{j_1 \cdots j_{n-1}} w_1^{j_1} \cdots w_{n-1}^{j_{n-1}},$$

where ϵ is the alternating Kronecker symbol [13]. The geometric meaning is that v is the vector perpendicular to all the $w_1, \cdots, w_{n-1}$ such that the volume of the parallelepiped spanned by $v, w_1, \cdots, w_{n-1}$ is equal to the area of the parallelogram spanned by $w_1, \cdots, w_{n-1}$.

In modern multi-linear algebra language, we contract ω with $w_1, \cdots, w_{n-1}$ to produce a 1-form. Using the metric, we can identify the 1-form with a vector v. That is, v is characterized by:

$$\omega(w_1, \cdots, w_{n-1}, \eta) = \langle v, \eta \rangle \qquad \forall \eta \in \mathbb{R}^n.$$

It is clear that $\times$ is multi-linear in all its arguments and antisymmetric.

4.1. **Approximate Reducibility for Volume Preserving Maps.** In this section we describe for the case of volume preserving maps the coordinate transformation M which allows us, via Lemmas 2.3 and 2.4, to define the approximate inverse operators of Theorem 2.2. If K is an approximate solution of the invariance equation (i.e., it satisfies Equation (6)), then taking derivatives of Equation (6) with respect to coordinates in the parameter space gives

$$Df(K(\theta))\, \partial_{\theta_i} K(\theta) - \lambda_i \partial_{\theta_i} K(\Lambda\theta) = \partial_{\theta_i} E(\theta).$$

Now we construct the vector

$$v(\theta) = N(\theta)\ \partial_{\theta_1} K(\theta) \times \cdots \times \partial_{\theta_{n-1}} K(\theta),$$

where $N(\theta)$ is a normalization factor chosen so that

$$\omega\left(\partial_{\theta_1} K(\theta), \cdots, \partial_{\theta_{n-1}} K(\theta), v(\theta)\right) = 1.$$

That is, we require that the volume of the parallelepiped spanned by the vectors is one. Then N is given explicitly by

$$N = \frac{1}{\omega\left(\partial_{\theta_1} K(\theta), \ldots, \partial_{\theta_{n-1}} K(\theta), \bar{v}(\theta)\right)},$$

and $\bar{v} = \partial_{\theta_1} K(\theta) \times \cdots \times \partial_{\theta_{n-1}} K(\theta)$.

For any $\theta \in D_r^{n-1}(0) \subset \mathbb{C}^{n-1}$ the vectors

$$\left\{\partial_{\theta_1} K(\theta), \ldots, \partial_{\theta_{n-1}} K(\theta), v(\theta)\right\},$$

form a basis for $\mathbb{C}^n$. It follows that the vector $Df(K)\, v(\theta)$ can be expressed as a linear combination

$$Df(K(\theta))\, v(\theta) = a_1(\theta) \partial_{\theta_1} K(\Lambda\theta) + \ldots + a_{n-1}(\theta) \partial_{\theta_{n-1}} K(\Lambda\theta) + a_n(\theta) v(\Lambda\theta).$$

Note that since $v(\Lambda)$ is orthonormal to $\partial_i K$, the coordinate function is the inner product

$$a_n(\theta) = \langle v(\Lambda\theta), Df(K)\, v(\theta)\rangle\,.$$

We will show that $a_n(\theta)$ has a particularly simple form. Namely, we show that the projection is approximately constant;

$$a_n(\theta) = \frac{1}{\lambda_1 \cdot \ldots \cdot \lambda_{n-1}} + R,$$

so that

$$Df(K(\theta))\, v(\theta) = \frac{1}{\lambda_1 \cdot \ldots \cdot \lambda_{n-1}} v(\Lambda\theta) + a_1(\theta)\partial_{\theta_1} K(\Lambda\theta) + \ldots + a_{n-1}(\theta)\partial_{\theta_{n-1}} K(\Lambda\theta) + R,$$

where R is just an algebraic expression involving derivatives of E and K. (In fact R is polynomial in E, but the dependance on K may be more complicated as $E = \Phi(K)$ depends on the form of f. On the other hand, if f is a polynomial map, then R depends polynomially on K as well). Note also that by the preservation of volume we have $(\lambda_1 \cdot \ldots \cdot \lambda_{n-1})^{-1} = \lambda_n$.

The geometric meaning of 4.1 is that the vector fields are transported. Since the volume is preserved, the height of the transformed parallelepiped should contact or expand by an amount that cancels the contraction or expansion of the base. A more precise argument which also gives the form of the error term R is as follows. Using the normalization of v and the volume preservation, we have;

$$\begin{aligned}
1 &= \omega\left(\partial_{\theta_1} K(\theta), \ \ldots\ , \partial_{\theta_{n-1}} K(\theta), v(\theta)\right) \\
&= \omega\left(Df(K(\theta))\, \partial_{\theta_1} K(\theta), \ \ldots, \ Df(K(\theta))\, \partial_{\theta_{n-1}} K(\theta), Df(K(\theta))\, v(\theta))\right) \\
&= \omega\left(\lambda_1 \partial_{\theta_1} K(\Lambda\theta) + \partial_{\theta_1} E(\theta), \ \ldots\ , \lambda_{n-1}\partial_{\theta_{n-1}} K(\Lambda\theta) + \partial_{\theta_{n-1}} E(\theta), Df(K(\theta))\, v(\theta)\right) \\
&= \omega\left(\lambda_1 \partial_{\theta_1} K(\Lambda\theta), \ \ldots\ , \lambda_{n-1}\partial_{\theta_{n-1}} K(\Lambda\theta), Df(K(\theta))\, v(\theta)\right) + R(\theta) \\
&= \lambda_1 \cdot \ldots \cdot \lambda_{n-1} \langle\, v(\Lambda\theta), \ DF(K(\theta))\, v(\theta)\,\rangle + R(\theta).
\end{aligned}$$

The expression or R is obtained by expanding the sums in the third expression above and collecting all the terms that include a derivative of E.

Define the matrix $M(\theta)$ by juxtaposing the column matrices $\partial_{\theta_i} K(\theta)$ and v

$$M(\theta) = \left[\ \partial_{\theta_1} K(\theta) \mid \ \cdots \ \mid \partial_{\theta_{n-1}} K(\theta) \mid v(\theta)\,\right],$$

where $\mid$ denotes that we are juxtaposing the column vectors. We note that the matrix M is invertible since it has determinant one (the normalization factor of v was chosen to have exactly this property).

The previous geometric argument shows that

$$DF(K(\theta))\, M(\theta) = M(\Lambda\theta) \begin{pmatrix} \lambda_1 & 0 & \cdots & a_1(\theta) \\ 0 & \lambda_2 & \cdots & a_2(\theta) \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & a_{n-1}(\theta) \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} R(\theta).$$

We will denote

$$A(\theta) = \begin{pmatrix} \lambda_1 & 0 & \cdots & a_1(\theta) \\ 0 & \lambda_2 & \cdots & a_2(\theta) \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & a_{n-1}(\theta) \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}. \tag{33}$$

Following the discussion in Section 2.3, we are required to solve

$$AW - W \circ \Lambda = -[M \circ \Lambda]^{-1} E,$$

which, when expressed in components, is

$$\begin{aligned}
\lambda_1 W_1(\theta) - W_1(\Lambda\theta) - a_1(\theta) W_n(\theta) &= \tilde{E}_1(\theta) \\
\lambda_2 W_2(\theta) - W_2(\Lambda\theta) - a_2(\theta) W_n(\theta) &= \tilde{E}_2(\theta) \\
&\vdots \\
\lambda_n W_n(\theta) - W_n(\Lambda\theta) &= \tilde{E}_n(\theta),
\end{aligned} \tag{34}$$

where $\tilde{E}(\theta) = -[M(\Lambda\theta)]^{-1} E(\theta)$. Then A has the exact form of Lemma 2.3 with A_{11} the $n-1 \times n-1$ matrix having $\{\lambda_1, \ldots \lambda_{n-1}\}$ as diagonal entries A_{12} is the $n-1\times 1$ matrix having $a_1, \ldots, a_{n-1}$ as entries, and A_{22} is the 1×1 matrix containing simply λ_n.

5. **Algorithms.**

The preceding discussion leads to the following algorithm for the Newton step in the Symplectic case. The inputs to the algorithm are the diffeomorphism f, the linear map Λ, and the current approximate solution K. The output is the improved solution.

Algorithm 5.1. [Symplectic Newton Step]

function `sympNewtonStep` (f, K, Λ);
$E = f \circ K - K \circ L$;
$N = [DK^\top DK]^{-1}$;
$M = [DK | J^{-1} DKN]$;
$\tilde{E} = -[M \circ \Lambda]^{-1} E$;
$\bar{A} = \text{fill}_S(\Lambda, f, M)$;
$W = \text{solve}_S(\bar{A}, \tilde{E})$;
$\Delta = MW$;
$K_{\text{new}} = K + \Delta$;
return $\{K_{\text{new}}\}$;

The function fill_S simply computes $\bar{A}$ using the formula given by Equation (29). The function solve_S caries out the back substitution solution of the upper triangular system as in the proof of Lemma 2.4. Each component equation can solved explicitly to any desired finite order using the formula of Equation (19). The algorithm converges as long as the assumptions of Theorem 2.2 are met.

Similarly, we have the following algorithm for the Newton Step in the volume preserving case. The inputs to the algorithm are the diffeomorphism f, the linear map Λ, and the current approximate solution K. The output is the improved solution.

Algorithm 5.2. [Volume Preserving Newton Step]

function `volNewtonStep` (f, K, Λ);
$E = f \circ K - K \circ L$;
$v = N \partial_{\theta_1} K \times \ldots \times \partial_{\theta_{n-1}} K$;
$M = [\partial_{\theta_1} K | \ldots | \partial_{\theta_{n-1}} | v]$;
$\tilde{E} = -[M \circ \Lambda]^{-1} E$;
$A = \text{fill}_V(\Lambda, f, M)$;

$W = \text{solve}_V(A, \tilde{E})$;
$\Delta = MW$;
$K_{\text{new}} = K + \Delta$;
return $\{K_{\text{new}}\}$;

The auxiliary function fill_V is a little more complicated than in the symplectic case. The fill_V assignment is defined by Equation (33), and requires the computation of the unknown functions a_i, $1 \leq i \leq n-1$. We discuss the computation of the a_i in the next section. The auxiliary function solve_V is similar to the symplectic case. The only difference is that upper triangular equation is given by the system of Equation (34) .

In both cases the Newton method simply consists of a while loop for the Newton step which terminates whenever $||\Phi(K_{\text{new}})||$ is below some desired tolerance. Here the norm can be bounded using Equation (10).

Computation of $M^{-1}(\theta)$ and $a_i(\theta)$. Here we discuss four possible methods for computing the $a_i(\theta)$, needed for the procedure $\text{fill}_V()$ discussed above. The methods are iterative, as in each step of the Newton Iteration the matrix inverse from the previous step can be used as a guess for the inverse in the current step. Of course in the first step of the Newton Scheme there is no previous step to exploit as an initial guess. Here a natural choice is to take just the constant term. For example $M^{-1}(\theta)$ could be approximated by $[M(0)]^{-1}$ which is just a matrix of numbers and can be inverted by standard numerics. A better initial guess could be obtained by a power matching scheme (i.e. computing the Taylor expansion of $M^{-1}(\theta)$ to some desired accuracy).

All three methods begin from the following observation. Given vectors $v_1(\theta)$, $\ldots$, $v_n(\theta)$ and another vector $e(\theta)$, we try to find $a_1(\theta), \ldots, a_n(\theta)$ so that

$$e(\theta) = a_1(\theta)v_1(\theta) + \ldots + a_n(\theta)v_n(\theta),$$

or

$$[v_1|\ldots|v_n]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = e.$$

We rewrite this last linear system simply as $A(\theta)\alpha(\theta) = e(\theta)$. It will also be important that we have fast ways to update the solution when the v changes.

Iterative Perturbation: Write

$$A(\theta) = A(0) + \hat{A}(\theta).$$

Then

$$e(\theta) = [A(0) + \hat{A}(\theta)]\alpha(\theta),$$

or

$$A(\theta) = A^{-1}(0)e(\theta) - A^{-1}(0)\hat{A}(\theta)\alpha(\theta). \tag{35}$$

Here it suffices to compute an LR or QR decomposition of $A^{-1}(0)$ and apply it to the vector.

We can iterate Equation (35) to obtain increasingly accurate approximations to $a(\theta)$. Note that every iteration gives the solution to a higher degree, and that for small θ this is a contraction.

Newton Scheme for the Inverse: Consider the constraints

$$B(\theta)A(\theta) = I,$$

and

$$B(0) = A^{-1}(0).$$

If $B_0(\theta)$ is an approximate solution, i.e. if

$$B_0(\theta)A(\theta) = I + E(\theta),$$

with E small, then we obtain a better solution by

$$B_1(\theta) = B_0(\theta) - E(\theta)B_0(\theta).$$

Note that if $E(\theta) = O(\theta^n)$, then

$$[I + E(\theta)]^{-1} = I - E(\theta) + O(||E(\theta)||^2)$$

$$I - E(\theta) + O(\theta^{2n}),$$

and this step requires n^2 multiplications. The full algorithm requires that at step k we keep B, which is an approximation of $A^{-1}(\theta)$ up to order 2^n. This can be used to approximate α up to error of order 2^n. The new step requires just one step of the update for B

QR Decomposition: This scheme is based on approximating a QR decomposition for A. A similar scheme could be developed using the singular value composition. To obtain the QR, we start with

$$QR = A + E$$

$$Q^T Q = I + F.$$

Here E and F are the errors. Since R is upper triangular (neglecting errors) we have

$$Q_{next} = Q_{this} + Q_{this}\Delta_Q$$

$$R_{next} = R_{this} + \Delta_R.$$

We remark that F is clearly symmetric. The Newton Equation becomes

$$Q\Delta_Q R + Q\Delta_R = -E$$

$$\Delta_Q^T Q^T Q + Q^T Q\Delta_Q = -F.$$

Multiplying on the right and left gives

$$\Delta_Q + \Delta_R R^{-1} = -Q^{-1}ER^{-1},$$

and the Newton Scheme becomes

$$\Delta_Q + \Delta_R R^{-1} = -Q^T ER^{-1}$$

$$\Delta_Q^T + \Delta_Q = -F.$$

Neglecting high order terms (which do not effect the quadratic convergence) the previous system can be solved.

Note that $\Delta_R R^{-1}$ is upper triangular. Then the First equation determines the below diagonal part of Δ_Q, and the second equation determines the upper diagonal part.

In practice, it is advantageous not to compute N and M^{-1} from scratch at each step. From the previous step we have already a good guess for N and M^{-1}. These can be corrected via numerical Newton Schemes.

6. Tame Estimates, and Existence of Mixed-Stable Invariant Manifolds.

In this section we make explicit the conditions under which there is an $\epsilon_* > 0$ so that the hypotheses of Theorem 2.2 hold. There are two steps. First we show that the increments in the normalization factors N and $N \circ \Lambda$, as well as the increments in the coordinate transformations M, $M \circ \Lambda$ and $[M \circ \Lambda]^{-1}$ satisfy certain "perturbative bounds". Next we show via an inductive argument that the normalization and coordinate transformations can be bound uniformly in terms of the initial data for the Newton procedure, as long as the perturbative bounds are satisfied. Finally we give the proof of Theorem 1.1.

6.1. Estimating the Increments.

In this section we treat the (somewhat technical) issue of obtaining uniform bounds on DK, N, N^{-1}, M, M^{-1}, and their compositions with the linear transformation Λ. The uniform bounds must hold in a small neighborhood of K_0 (see also Remark 7). The smallness of the neighborhood will be addressed explicitly in the proof. We note that while the statement of the estimates in Lemma 6.1 are somewhat awkward, the proof of the lemma is standard and quite straight forward. The idea is to bound the increments of the auxiliary quantities using the Neumann series and the invariance relations. The estimates for the composition terms are especially cumbersome as they require also the $\Delta \circ \Lambda$ estimates of Subsection 2.1.1.

Throughout this section let $K \in \mathcal{X}_r^0$ be an approximate solution of $\Phi(K) = 0$, $\Delta = -\mathfrak{L}(K)\Phi(K) \in \mathcal{Y}_{r'}$ with $r' = re^{-\delta}$ and $\delta > 0$, and $\|\Phi(K)\|_r = \epsilon$. Let $K' = K + \Delta \in \mathcal{X}_{r'}^0$ and $\|\Phi(K')\|_{r'} = \epsilon'$. Let M be as in Section 3.2, so that $\mathfrak{L}(K)\Phi(K) = -M\mathcal{T}([M \circ \Lambda]^{-1}\Phi(K))$.

Lemma 6.1 (Symplectic Increment Estimates)**.** *Let $N = [DK^T DK]^{-1}$, and $M = [DK|J^{-1}DKN]$. Assume that N, $N \circ \Lambda$, M, $M \circ \Lambda$, and $[M \circ \Lambda]^{-1}$ are componentwise bounded on the poly-disk of radius $re^{-\delta/2}$. Let $0 < \delta' < \delta$. Assume also that $\|\Phi(K')\|_{r'} = \epsilon' < \epsilon$. Then*

1. $\|DK'\|_{r'e^{-\delta'/2}} \leq \|DK\|_{re^{-\delta/2}} + 2C_*|\delta'|^{-1}\|\Delta\|_{r'}$,
2. $\|D(K' \circ \Lambda)\|_{r'e^{-\delta'/2}} \leq \|D(K \circ \Lambda)\|_{re^{-\delta/2}} + 2C_*|\delta'|^{-1}[2\epsilon + 2n\|Df\|_{\rho_*}\|\Delta\|_{r'}]$
3. *There is a $C_N > 0$ so that if $0 < \sigma < 1$ and*
$$C_N\|N\|_{re^{-\delta/2}}\|DK\|_{re^{-\delta/2}}|\delta'|^{-1}\|\Delta\|_{r'} \leq \sigma,$$
then
$$\|N'\|_{r'e^{-\delta'/2}} < \frac{1}{1-\sigma}\|N\|_{re^{-\delta/2}}.$$
4. *There is a $C_{N_\Lambda} > 0$ so that if $0 < \sigma < 1$ and*
$$C_{N_\Lambda}\|N \circ \Lambda\|_{re^{-\delta/2}}\|DK \circ \Lambda\|_{re^{-\delta/2}}|\delta'|^{-1}[2\epsilon + 2n\|Df\|_{\rho_*}\|\Delta\|_{r'}] \leq \sigma.$$
Then
$$\|N' \circ \Lambda\|_{r'e^{\delta'/2}} < \frac{1}{1-\sigma}\|N \circ \Lambda\|_{re^{\delta/2}}.$$

5. *There is a* $C_M > 0$ *so that if* $0 < \sigma < 1$ *and*

$$C_M \|[M \circ \Lambda]^{-1}\|_{re^{-\delta/2}} \|DK \circ \Lambda\|^2_{re^{-\delta/2}} \|N \circ \Lambda\|^2_{re^{-\delta/2}} |\delta'|^{-1} \|\Delta\|_{r'} \leq \sigma.$$

Then

$$\left\|[M' \circ \Lambda]^{-1}\right\|_{r'e^{\delta'/2}} < \frac{1}{1-\sigma} \left\|[M \circ \Lambda]^{-1}\right\|_{re^{\delta/2}}.$$

Remark 9. We refer to these as perturbative bounds as the primed terms are bound in terms of the unprimed terms so long as the perturbation Δ is small enough. The perturbative bounds on DK' and $D(K' \circ \Lambda)$ are general, i.e. they make no use of the symplectic set up. We note that once we have bounds on N' and $N' \circ \Lambda$ it is trivial to bound M' and $M' \circ \Lambda$. We also note that while we are focusing on the symplectic case here, similar bounds for the volume preserving increments can be obtained.

Proof. The perturbative bound for DK' is simply the linearity of the differential and a Cauchy estimate on the $D\Delta$ term. The perturbative bound for $D(K' \circ \Lambda)$ is similar, but uses in addition the $\Delta \circ \Lambda$ estimate of Equation (15).

Choose $0 < \sigma < 1$ and consider the N' term. We have that

$$\begin{aligned} N' &= \left[\ [DK + D\Delta]^T [DK + D\Delta]\ \right]^{-1} \\ &= \left[\ I + N[DK^T D\Delta + D\Delta^T DK + D\Delta^T D\Delta]\ \right]^{-1} N \\ &= \left[\sum_{j=0}^{\infty} (-1)^j [N(DK^T D\Delta + D\Delta^T DK + D\Delta^T D\Delta)]^j \right] N, \end{aligned}$$

as long as the summand is norm less then one (in the $r'e^{-\delta'/2}$ norm). But this holds as long as we assume that

$$3\|N\|_{re^{-\delta/2}} \|DK\|_{re^{-\delta/2}} 2C_* |\delta'|^{-1} \|\Delta\|_{r'} < 1,$$

where C_* is the Cauchy bound constant. Let $C_N = 6C_*$. Assuming that $\|\Delta\|_{r'}$ is so small that

$$C_R \|N\|_{re^{-\delta/2}} \|DK\|_{re^{-\delta/2}} |\delta'|^{-1} \|\Delta\|_{r'} \leq \sigma,$$

we then have

$$\begin{aligned} \|N'\|_{r'e^{-\delta'/2}} &\leq \|N\|_{re^{-\delta/2}} \sum_{j=0}^{\infty} \|N\left[\ DK^T D\Delta + D\Delta^T DK + D\Delta^T D\Delta\ \right]\|^j_{r'e^{\delta'/2}} \\ &\leq \frac{1}{1-\sigma} \|N\|_{re^{-\delta/2}}, \end{aligned}$$

as desired. The perturbative bounds on $N' \circ \Lambda$ and $[M' \circ \Lambda]^{-1}$ are similar. □

6.2. **Inductive Proof of the Uniform Bounds.** In this section we refer repeatedly to the inductive argument in the proof of Theorem 2.2 given in Appendix A. The reader may need to consult the Appendix for the notation used here.

Lemma 6.2. *Assume the hypotheses of Theorem 2.2. In addition assume that* $\|DK_0\|_{r_0}$ *and* $\|D(K_0 \circ \Lambda)\|_{r_0}$ *are bounded. Then for small enough* $\epsilon_0 > 0$ *there are functions* $ds(\gamma)$ *and* $ds_\Lambda(\gamma)$ *(whose forms are given explicitly below) so that the derivatives of the approximates* K_n *satisfy the uniform bounds*

$$\|DK_n\|_{r_n e^{-\delta_n/2}} \leq \|DK_0\|_{r_0} + ds(\gamma),$$

and

$$\|D(K_n \circ \Lambda)\|_{r_n e^{-\delta_n/2}} \leq \|D(K_0 \circ \Lambda)\|_{r_0} + ds_\Lambda(\gamma).$$

Proof. Recall that under the hypotheses of Theorem 2.2 we have the inductive bounds

$$\|\Delta_n\|_{r_{n+1}} \leq C_1|\delta_0|^{-\nu}2^{n\nu}\gamma^{2^n}.$$

Then $K_{n+1} = K_n + \Delta_n \in \mathcal{X}^0_{r_{n+1}}$ and

$$\begin{aligned}
\|DK_{n+1}\|_{r_{n+1}e^{\frac{-\delta_{n+1}}{2}}} &\leq \|DK_0\|_{r_0} + \sum_{j=0}^{n}\|D\Delta_j\|_{r_je^{\frac{-\delta_j}{2}}} \\
&\leq \|DK_0\|_{r_0} + \sum_{j=0}^{n} C_* 2^{j+1}|\delta_0|^{-1}\|\Delta_j\|_{r_j} \\
&\leq \|DK_0\|_{r_0} + 2C_*C_1|\delta_0|^{-(\nu+1)}\sum_{j=0}^{\infty}2^{j(\nu+1)}\gamma^{2^j},
\end{aligned}$$

where the sum converges assuming that ϵ_0 and hence γ are small enough. Then $ds(\gamma) = 2C_*C_1|\delta_0|^{-(\nu+1)}\sum_{j=0}^{\infty}2^{j(\nu+1)}\gamma^{2^j}$, gives the desired uniform bound. The uniform bound on $D(K_{n+1}\circ\Lambda)$ is similar, but uses the estimate given by Equation (15). □

Lemma 6.3. *Let*

$$C_p = \prod_{j=1}^{\infty}\frac{j^2+1}{j^2},$$

and assume the hypotheses of Theorem 2.2 as well as Lemma 6.1 (so that, in particular we are working with symplectic f). Assume in addition that N_0, $N_0\circ\Lambda$, M_0, $M_0\circ\Lambda$, and $[M\circ\Lambda]^{-1}$ are bounded on the poly-disk of radius r_0. Then there is an $\epsilon_* > 0$ small enough so that if $\epsilon_0 < \epsilon_*$, then we have the uniform bounds

1. $$\|N_n\|_{r_ne^{-\frac{\delta_n}{2}}} \leq C_p\|N_0\|_{r_0},$$
2. $$\|N_n\circ\Lambda\|_{r_ne^{-\frac{\delta_n}{2}}} \leq C_p\|N_0\circ\Lambda\|_{r_0},$$
3. $$\|(M_n\circ\Lambda)^{-1}\|_{r_ne^{-\frac{\delta_n}{2}}} \leq C_p\|(M\circ\Lambda)^{-1}\|_{r_0}.$$

Proof. The key to obtaining the uniform bounds are the perturbative estimates of Lemma 6.1. We give the argument for N_n. The other bounds are similar.

Assume that ϵ_0 is small enough that

$$\gamma^{2^n} \leq \frac{|\delta_0|^{-(\nu+1)}}{2C_1C_NC_p\|N_0\|_{r_0}\left[\|DK_0\|_{r_0} + ds(\gamma)\right]\left[(n+1)^2+1\right]2^{n(\nu+1)}}, \tag{36}$$

for all $n \geq 0$. Note that this is possible as $\gamma < 1$, so that γ^{2^n} decreases faster than any exponential, and the right hand side only decreases like a polynomial times and exponential. The remainder of the proof is by induction.

For the base case we have that

$$\|N_1\|_{r_1e^{-\delta_1/2}} \leq 2\|N_0\|_{r_0},$$

as long as

$$C_N\|N_0\|_{r_0e^{-\delta_0/2}}\|DK_0\|_{r_0e^{-\delta_0/2}}|\delta_1|^{-1}\|\Delta_0\|_{r_1} \leq \frac{1}{2}.$$

But this holds, as

$$\begin{aligned} & C_N \|N_0\|_{r_0 e^{-\delta_0/2}} \|DK_0\|_{r_0 e^{-\delta_0/2}} |\delta_1|^{-1} \|\Delta_0\|_{r_1} \\ & \quad = C_N \|N_0\|_{r_0 e^{-\delta_0/2}} \|DK_0\|_{r_0 e^{-\delta_0/2}} |2^{-1}\delta_0|^{-1} \|\Delta_0\|_{r_1} \\ & \quad \leq 2C_N \|N_0\|_{r_0} \|DK_0\|_{r_0} |\delta_0|^{-1} \|\Delta_0\|_{r_1} \\ & \quad \leq 2C_N \|N_0\|_{r_0} \|DK_0\|_{r_0} |\delta_0|^{-1} \|\Delta_0\|_{r_1} \\ & \quad \leq 2C_1 C_N C_P \|N_0\|_{r_0} \left[\|DK_0\|_{r_0} + ds(\gamma)\right] |\delta_0|^{-(\nu+1)} \gamma, \end{aligned}$$

by the $n = 0$ case of Equation (36).

Now assume that for $1 \leq j \leq n$ we have that

$$\|N_j\|_{r_j e^{-\delta_j/2}} \leq \frac{j^2+1}{j^2} \|N_{j-1}\|_{r_{j-1} e^{-\delta_{j-1}/2}}. \tag{37}$$

It follows that for each $1 < j \leq n$ we also have that

$$\begin{aligned} \|N_j\|_{r_j e^{-\delta_j/2}} & \leq \frac{j^2+1}{j^2} \frac{(j-1)^2+1}{(j-1)^2} \|N_{j-1}\|_{r_{j-1} e^{-\delta_{j-1}/2}} \\ & \vdots \\ & \leq \prod_{i=1}^{j} \frac{i^2+1}{i^2} \|N_0\|_{r_0} \\ & \leq C_p \|N_0\|_{r_0}. \end{aligned}$$

Now we consider the $n+1$-th term and have that

$$\begin{aligned} \|N_{n+1}\|_{r_{n+1} e^{-\delta_{n+1}/2}} & \leq \frac{1}{1 - \frac{1}{(n+1)^2+1}} \|N_n\|_{r_n e^{-\delta_n/2}} \\ & \leq \frac{(n+1)^2+1}{(n+1)^2} \|N_n\|_{r_n e^{-\delta_n/2}} \leq C_p \|N_0\|_{r_0}, \end{aligned}$$

as long as

$$C_N \|N_n\|_{r_n e^{-\delta_n/2}} \|DK_n\|_{r_n e^{-\delta_n/2}} |\delta_{n+1}|^{-1} \|\Delta_n\|_{r_{n+1}} \leq \frac{1}{(n+1)^2+1}.$$

But Equation (6.2) holds as

$$C_N \|N_n\|_{r_n e^{-\delta_n/2}} \|DK_n\|_{r_n e^{-\delta_n/2}} \left|\delta_{n+1}\right|^{-1} \|\Delta_n\|_{r_{n+1}}$$

$$\begin{aligned} & \leq C_N C_p \|N_0\|_{r_0} \left[\|DK_0\|_{r_0} + ds(\gamma)\right] \left|\delta_0 2^{-(n+1)}\right|^{-1} \|\Delta_n\|_{r_{n+1}} \\ & = 2C_1 C_N C_p \|N_0\|_{r_0} \left[\|DK_0\|_{r_0} + ds(\gamma)\right] 2^{n(\nu+1)} |\delta_0|^{-(\nu+1)} \gamma^{2^n} \\ & \leq \frac{1}{(n+1)^2+1}, \end{aligned}$$

and the last inequality is just the $n+1$ case of the inequality of Equation (36).

This shows that the assumption that ϵ_0 is small enough to give Equation (36) implies uniform bounds on N_n. Similar arguments give the inductive bounds for $N_n \circ \Lambda$ and $(M_n \circ \Lambda)^{-1}$. However each of these arguments introduces an additional smallness condition on ϵ_0. If we let ϵ_* be the minimum of these three smallness conditions then $\epsilon_0 < \epsilon_*$ gives the theorem. □

Remark 10. The proof of uniform bounds in the volume preserving case is identical because the matrices N, M are just algebraic expressions involving K and its derivatives. So we can estimate the changes in N, M, M^{-1} (and their compositions) in terms of Δ, the change of K by using adding and subtracting and Neumann series as in here. We do not rewrite the arguments.

6.3. **Proof of Theorem 1.1.** Let $A = [\xi_1|\dots|\xi_n]$ and note that $K_0 \in \mathcal{X}^0_{r_0}$ given by

$$K_0(\theta) = A\theta \qquad \theta \in \mathbb{R}^n,$$

is analytic for any choice of $r_0 > 0$. Then the constant matrices $DK_0 = A$, $N_0 = [A^T A]^{-1}$, and $M_0 = [A|J^{-1}AN]$ are real analytic for any r_0. Note that N_0 exists as A is of full rank. M_0 is invertible as $J^{-1}AN$ is perpendicular to A. In addition $M_0 \circ \Lambda = M_0$ as M_0 is constant, so that $(M \circ \Lambda)^{-1}$ exists and is real analytic on any poly-disk.

Let $R_0(\cdot)$ be the Taylor remainder of f at the origin, and let C_{Taylor} be any constant so that $\|R_0\|_r \leq C_{\text{Taylor}}$ near the origin. Then

$$\begin{aligned}\Phi(K_0)(\theta) &= f(A\theta) - A[\Lambda\theta] \\ &= Df(0)[A\theta] + R_0(A\theta) - A\Lambda\theta \\ &= R_0(A\theta),\end{aligned}$$

as

$$Df(0)A\theta - A\Lambda\theta = Df(0)[\theta_1\xi_1 + \dots + \theta_n\xi_n] - [\lambda_1\theta_1\xi_1 + \dots + \lambda_n\theta_n\xi_n],$$

and $Df(0)\theta_i\xi_i - \lambda_i\theta_i\xi_i = 0$ for each $1 \leq i \leq n$ (as $\theta_i\xi_i$ is an eigenvector of λ_i for each value of θ_i).

Then

$$\epsilon_0 \equiv \|\Phi(K_0)\|_{r_0} = \|R_0 \circ A\|_{r_0} \leq C_{\text{Taylor}}\|A\|^2 r_0^2,$$

can be made as small as we wish, by choosing r_0 appropriately. Then we take r_0 so small that ϵ_0 satisfies the hypotheses of Lemma 6.2, in which case we have uniform bounds on K_n and Δ_n, as well as uniform bounds on their compositions with Λ, their derivatives, and the normalizations and coordinate transforms N_n, M_n, their compositions with Λ, and the needed inverse matrices $(M_n \circ \Lambda)^{-1}$.

Then we have the tame estimates needed in order to satisfy $A2$ of Theorem 2.2 in the symplectic case. Taking $\delta_0 = 2^{-1}$ and ϵ_0 also small enough to satisfy the hypotheses of Theorem 2.2, we have the existence of $\Delta_* \in \mathcal{Y}_{r_0 e^{-1}}$ so that $K_* = K_0 + \Delta_* \in \mathcal{X}^0_{r_0 e^{-1}}$ has $\Phi(K_*) = 0$.

Then image$(K_*) = K_*(D^n_{r_0 e^{-1}}(0)) \equiv W^b_{\text{loc}}(0)$ is the analytic embedding of an n dimensional poly-disk tangent to A at 0.

□

A similar argument, using analogous estimates gives the proof of Theorem 1.2. Moreover, the argument just given illustrates the steps which must be followed in order to obtain the existence of an invariant mixed-stable manifold when K_0 is any approximate solution of $\Phi(K) = 0$. For example, K_0 could be a numerical approximation obtained by applying the numerical algorithms of Section 5. In which case an argument similar to the one above can be given in order to establish *a-posteriori* convergence of the algorithm.

7. **Acknowledgments.** Both authors would like to offer their sincere thanks to the two anonymous referees who carefully reviewed the manuscript. The suggestions and corrections of the both referees greatly improved the quality of the final version.

Appendix A. **Proof of Theorem 2.2.** Throughout the appendix we use the notation established in the hypotheses of Theorem 2.2. Fix $\delta_0 > 0$ and define

1. $\delta_n = 2^{-n}\delta_0$.
2. $r_{n+1} = r_n e^{-\delta_n}$, so that $r_n \to r_* = r_0 e^{-2\delta_0}$ as $n \to \infty$.
3. $\omega = \max(2\nu, \mu)$, $\hat{C} = 2^\omega$,
$$C_3 = \bar{C} \sum_{|\alpha|=2} \frac{2}{\alpha!},$$
and $C_4 \equiv C_1^2 C_3 + C_2$. We can arrange that $C_4|\delta_0|^{-\omega} > 1$ without loss of generality by adjusting up the values of C_1, C_2, and C_3,
4. $\gamma = \left(C_4 \hat{C} \delta_0^{-\omega}\right) \epsilon_0$
5. $$s(\gamma) = C_1 \delta_0^{-nu} \sum_0^\infty 2^{n\nu} \gamma^{2^n}.$$
Note that $s(\gamma) \to 0$ as $\gamma \to 0$,
6. (**Base Increments:**) $\Delta_0 \in \mathcal{Y}_{r_1}$, $K_1 \in \mathcal{X}^0_{r_1}$ by
$$\Delta_0 = -\mathfrak{L}(K_0)\,\Phi(K_0),$$
$$K_1 = K_0 + \Delta_0.$$
7. (**Inductive Increments:**) For $n \geq 1$ define the sequences $\Delta_n \in \mathcal{Y}_{r_{n+1}}$ and $K_{n+1} \in \mathcal{X}^0_{r_n}$ by
$$\Delta_n = -\mathfrak{L}(K_n)\,\Phi(K_n),$$
and
$$K_{n+1} = K_n + \Delta_n.$$

The proof of Theorem 2.2 depends on the following estimate, which we state as a Lemma.

Lemma A.1. *Suppose that $K \in \mathcal{X}^0_r$, and that C_1, C_2, C_3, C_4, ω and $\mathfrak{L}(K)$ are as in Theorem 2.2. Define $\epsilon \equiv \|\Phi(K)\|_r$. If*
$$\Delta = -\mathfrak{L}(K)\,\Phi(K),$$
and
$$\|K + \Delta\|_{re^{-\delta}} \leq \rho_*,$$
then for any $\delta > 0$ we have the estimate
$$\|\Phi(K + \Delta)\|_{re^{-\delta}} \leq C_4|\delta|^{-\omega}\epsilon^2.$$

Proof of Lemma A.1. Let $R_K(\cdot)$ denote the Taylor remainder for f at K, so that

$$R_K(\Delta) = f(K+\Delta) - Df(K)\Delta - f(K),$$

where the j-th component of R_K can be written as

$$R_K^j(\Delta) = \sum_{|\alpha|=2} \frac{2}{\alpha!} \Delta^\alpha \int_0^1 (1-t)\partial^\alpha f_j(K+t\Delta)\, dt.$$

Note that

$$\begin{aligned}
\|R_K(\Delta)\|_{re^{-\delta}} &\leq \sup_{1\leq j\leq n} \sum_{|\alpha|=2} \frac{2}{\alpha!} \|\Delta\|_{re^{-\delta}}^{|\alpha|} \int_0^1 |1-t||\partial^\alpha f_j(K+t\,\Delta)|\, dt \\
&\leq \|\Delta\|_{re^{-\delta}}^2 \|D^2 f\|_{\rho_*} \sum_{|\alpha|=2} \frac{2}{\alpha!} \\
&\leq C_3 \|\Delta\|_{re^{-\delta}}^2 \\
&\leq C_3\, |-\mathfrak{L}(K)\,\Phi(K)|_{re^{-\delta}}^2 \\
&\leq C_3 [C_1 |\delta|^{-\nu} \|\Phi(K)\|_r]^2 \\
&= C_1^2 C_3 |\delta|^{-2\nu} \epsilon^2,
\end{aligned}$$

where we have used the assumption that $\|K+\Delta\|_{re^{-\delta}} \leq \rho_*$, and numbered assumption 1 of Lemma A.1.

Consider

$$\Phi(K+\Delta) = f(K+\Delta) - (K+\Delta)\circ\Lambda.$$

Adding and subtracting the quantity $Df(K)\Delta + f(K)$ gives

$$f(K+\Delta) - (K+\Delta)\circ\Lambda + [Df(K)\Delta + f(K)] - [Df(K)\Delta + f(K)]$$

$$\begin{aligned}
&= (f(K+\Delta) - Df(K)\Delta - f(K)) + (Df(K)\Delta + f(K) - K\circ\Lambda - \Delta\circ\Lambda) \\
&= R_K(\Delta) + [\Phi(K) + D\Phi(K)\Delta] \\
&= R_K(\Delta) + [\Phi(K) - D\Phi(K)\,\mathfrak{L}(K)\,\Phi(K).]
\end{aligned}$$

Then

$$\begin{aligned}
\|\Phi(K+\Delta)\|_{re^{-\delta}} &\leq \|R_K(\Delta)\|_{re^{-\delta}} + \|\Phi(K) - D\Phi(K)\,\mathfrak{L}(K)\,\Phi(k)\|_{re^{-\delta}} \\
&= C_1^2 C_3 |\delta|^{-2\nu} \epsilon^2 + C_2 |\delta|^{-\mu} \epsilon^2 \\
&\leq C_4 |\delta|^{-\omega} \epsilon^2,
\end{aligned}$$

where we have used numbered assumption 2 of Theorem 2.2, and the Taylor Remainder of above. □

Proof of Theorem 2.2. Let $\epsilon_* > 0$ be small enough that the bounds C_1 and C_2 are uniform on the ϵ_* ball about K_0 in $\mathcal{X}_r^0$ and assume that ϵ_0 is so small that

- $\max\left(\epsilon_0, [C_4 \hat{C} \delta_0^{-\omega} \epsilon_0]^2\right) < \epsilon_*$,
- $\gamma < 1$,
- and $s < \rho_* - \rho_0$,

Most of the work in the proof is devoted to establishing the following inductive estimates, which we state as a final lemma.

Lemma A.2. *For each n we have that*

$$\|\Delta_n\|_{r_r e^{-\delta_n}} \le C_1|\delta_0|^{-\nu} 2^{n\nu}\gamma^{2^n},$$

that

$$\|K_n\|_{r_n} \le \rho_0 + C_1\delta_0^{-\nu}\sum_{j=0}^{n-1} 2^{j\nu}\gamma^{2^j} \le \rho_*,$$

and that the errors $\epsilon_{n+1} \equiv \|\Phi(K_{n+1})\|_{r_{n+1}}$ satisfy

$$\epsilon_{n+1} \le C_4|\delta_0|^{-\omega}\hat{C}^n\epsilon_n^2.$$

Proof of Lemma A.2. Define Δ_0, K_1, and r_1 as above. The base step of the induction begins with the estimate

$$\begin{aligned}
\|\Delta_0\|_{r_0 e^{-\delta_0}} &= \| - \mathfrak{L}(K_0)\,\Phi(K_0)\|_{r_1} \\
&\le C_1|\delta_0|^{-\nu}\|\Phi(K_0)\|_{r_0} \\
&\le C_1|\delta_0|^{-\nu}\ \epsilon_0 \\
&\le C_1|\delta_0|^{-\nu}\ C_4\hat{C}|\delta_0|^{-\omega}\ \epsilon_0 \\
&= C_1|\delta_0|^{-\nu}2^{0\nu}\gamma^{2^0},
\end{aligned}$$

where we have used numbered assumptions 1 and 6 of Theorem 2.2 (namely the definition of γ and that $C_4\hat{C}|\delta_0|^{-\omega} > 1$).

Then we have that

$$\begin{aligned}
\|K_1\|_{r_1} &= \|K_0 + \Delta_0\|_{r_0 e^{-\delta_0}} \\
&\le \|K_0\|_{r_0} + \|\Delta_0\|_{r_0 e^{-\delta_0}} \\
&\le \rho_0 + C_1|\delta_0|^{-\nu}2^{0\nu}\gamma^{2^0} \\
&\le \rho_0 + C_1|\delta_0|^{-\nu}\sum_{n=0}^{\infty} 2^{n\nu}\gamma^{2^n} \\
&\le \rho_0 + s \le \rho_*,
\end{aligned}$$

so that we can apply Lemma A.1 to obtain

$$\epsilon_1 \equiv \|\Phi(K_1)\|_{r_1} = \|\Phi(K_0 + \Delta_0)\|_{r_0 e^{-\delta_0}} \le C_4|\delta_0|^{-\omega}\epsilon_0^2 = C_4|\delta_0|^{-\omega}\hat{C}^0\epsilon_0^2.$$

Note that the bound on $\|K\|_{r_1}$ is essential in order to apply the Lemma.

Now assume for the inductive step assume that we have carried out $n > 1$ steps of the Newton iteration; so that for $0 \le j \le n-1$ the functions $\Delta_j \in \mathcal{Y}_{r_j e^{-\delta_j}}$, and for $0 \le j \le n$ the functions $K_j \in \mathcal{X}^0_{r_j}$ are well defined; and both sets of functions satisfy the inductive estimates above.

Define $\Delta_n \equiv -\mathfrak{L}(K_n)\,\Phi(K_n)$. Since $K_n \in \mathcal{X}^0_{r_n}$, $\Phi : \mathcal{X}^0_{r_n} \to \mathcal{Y}_{r_n}$, and $\mathfrak{L}(K_n) : \mathcal{Y}_{r_n} \to \mathcal{Y}_{r_{n+1}}$ is is clear that $\Delta_n \in \mathcal{Y}_{r_{n+1}}$ is well defined. Moreover by using numbered assumption 2 of Theorem 2.2, and the inductive hypothesis that for $1 \le j \le n$ we have

$$\epsilon_j \le C_4|\delta_0|^{-\omega}\hat{C}^{j-1}\epsilon_{j-1}^2,$$

and obtain

$$\begin{aligned}
\|\Delta_n\|_{r_{n+1}} &= \| - \mathfrak{L}(K_n)\,\Phi(K_n)\|_{r_{n+1}} \\
&\leq C_1|\delta_n|^{-\nu}\|\Phi(K_n)\|_{r_n} \\
&= C_1|\delta_0 2^{-n}|^{-\nu}\epsilon_n \\
&\leq C_1|\delta_0|^{-\nu}2^{n\nu}[C_4|\delta_0|^{-\omega}\hat{C}^{n-1}\epsilon_{n-1}^2] \\
&\vdots \\
&\leq C_1|\delta_0|^{-\nu}2^{n\nu}C_4^{2+2^2+\ldots+2^{n-1}}\hat{C}^{2^1(n-1)+\ldots+2^{n-1}}\epsilon_0^{2^n} \\
&\leq C_1|\delta_0|^{-\nu}2^{n\nu}(C_4|\delta_0|^{-\omega}\hat{C}^n\epsilon_0)^{2^n} \\
&\leq C_1|\delta_0|^{-\nu}2^{n\nu}\gamma^{2^n},
\end{aligned}$$

where we have used that

$$1 + 2 + 2^2 + \ldots + 2^n \leq 2^{n+1},$$

as well as that

$$2^0 n + 2^1(n-1) + 2^2(n-2) + \ldots + 2^{n-1}1$$

$$= 2^n(n2^{-n} + (n-1)2^{-n-1} + \ldots + 2^{-1}) \leq 2^n \sum_{k=1}^{\infty} k2^{-k} = 2^n 2 = 2^{n+1}.$$

Then we define $K_{n+1} \in \mathcal{K}^0_{r_{n+1}}$ by $K_{n+1} = K_n + \Delta_n$ and have by the inductive hypothesis on $\|K_j\|_{r_j}$ that

$$\begin{aligned}
\|K_{n+1}\|_{r_{n+1}} &= \|K_n + \Delta_n\|_{r_n e^{-\delta_n}} \\
&\leq \|K_n\|_{r_n e^{-\delta_n}} + \|\Delta_n\|_{r_n e^{-\delta_n}} \\
&\leq \|K_n\|_{r_n} + C_1|\delta_0|^{-\nu}2^{n\nu}\gamma^{2^n} \\
&\leq \rho_0 + C_1|\delta_0|^{-\nu}\sum_{j=0}^{n-1}2^{j\nu}\gamma^{2^j} \;+\; C_1|\delta_0|^{-\nu}\;2^{n\nu}\gamma^{2^n} \\
&\leq \rho_0 + C_1|\delta_0|^{-\nu}\sum_{n=0}^{\infty}2^{n\nu}\gamma^{2^n} \\
&\leq \rho_0 + s \leq \rho_*.
\end{aligned}$$

The bound $\|K_n + \Delta_n\|_{r_{n+1}} \leq \rho_*$ allows us to apply Lemma A.1 in conjunction with the inductive hypothesis on ϵ_j for $1 \leq j \leq n$ and assumption 1 of Theorem 2.2 to obtain

$$\begin{aligned}
\epsilon_{n+1} &= \|\Phi(K_{n+1})\|_{r_{n+1}} = \|\Phi(K_n + \Delta_n)\|_{r_n e^{-\delta_n}} \\
&\leq C_4|\delta_n|^{-\omega}\|\Phi(K_n)\|_{r_n}^2 \\
&= C_4|\delta_0|^{-\omega}2^{n\omega}\epsilon_n^2.
\end{aligned}$$

This concludes the inductive argument. □

In order to complete the proof of Theorem 2.2, consider the function Δ_* defined formally as

$$\Delta_* = \sum_{n=0}^{\infty}\Delta_n.$$

We claim that $\Delta_* \in \mathcal{Y}_{r*}$ where $r_* = r_0 e^{-2\delta_0} < r_n e^{-\delta_n}$ for all n. To see this, note that

$$\begin{aligned} \|\Delta_*\|_{r_*} &\leq \sum_{n=0}^{\infty} \|\Delta_n\|_{r_*} \\ &\leq \sum_{n=0}^{\infty} \|\Delta_n\|_{r_n} \\ &\leq \sum_{n=0}^{\infty} C_1 |\delta_0|^{-\nu} 2^{n\nu} \gamma^{2^n} \\ &= s < \infty, \end{aligned}$$

so that Δ_* is an analytic function on $D^{\ell}_{r_*}(0)$.

Finally we define $K_* = K_0 + \Delta_*$. Note that since

$$K_{n+1} = K_n + \Delta_n = \ldots = K_0 + \sum_{j=0}^{n} \Delta_j,$$

we have that $K_n \to K_*$ uniformly in $\mathcal{X}^0_{r_*}$. Applying the inductive estimate of ϵ_{n+1} repeatedly, in a fashion similar to as in the estimate of $\|K\|_{r_{n+1}}$, we obtain

$$\begin{aligned} \epsilon_{n+1} &\leq C_4 |\delta_0|^{-\omega} 2^{n\omega} \epsilon_n^2 \\ &\vdots \\ &= (C_4 |\delta_0|^{-\omega})^{1+2+2^2+\ldots+2^{n-1}} \hat{C}^{2^0 n + 2^1(n-1) + \ldots + 2^{n-1} 1} \epsilon_0^{2^{n+1}} \\ &\leq (C_4 |\delta_0|^{-\omega} \hat{C} \epsilon_0)^{2^{n+1}} = \gamma^{2^{n+1}}. \end{aligned}$$

Then

$$\|\Phi(K_*)\|_{r*} = \left\| \Phi\left(\lim_{n\to\infty} K_n\right) \right\|_{r_*} = \lim_{n\to\infty} \|\Phi(K_n)\|_{r*} \leq \lim_{n\to\infty} \gamma^{2^n} = 0,$$

as $\gamma < 1$, and $\Phi(K_*) = 0$ as desired. The bound on $\|K_* - K_0\|_{r_*}$ follows immediately from the fact that $K_* - K_0 = \Delta_*$. □

In practice, if we have some freedom in choosing ϵ_0, it is often sufficient to take $\delta_0 = 1/2$. On the other hand, for some computations it may be advantageous to fix ϵ_0 and optimize δ_0. For example if K_0 is a numerical solution of $\Phi = 0$ and ϵ_0 measures the numerical errors, then we can use Theorem 2.2 to prove there is a true solution nearby.

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Received June 2011; revised March 2012.

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