CHAPTER 8

SYSTEMS OF PARTICLES, EXTENDED OBJECTS & CONSERVATION OF MOMENTUM

• Center of Mass
  ◦ finding the center of mass

• Motion of the center of mass

• Conservation of momentum

• Collisions
  ◦ impulse
  ◦ inelastic collisions
  ◦ elastic collisions
  ◦ coefficient of restitution

Center of mass

Up to now we have considered all objects as “point” masses, but what happens to the equations of motion, etc., if we have a collection of masses or objects with a complicated shape that not only have translational motion but rotational motion as well, for example,

2-d molecule (H₂O):

hammer:

It turns out that the concept of a center of mass becomes important. The translational and rotational motion of objects like those shown, is linked directly to the center of mass.
To locate the center of mass (cm) of a collection of masses, choose an origin, then the position vector of the cm relative to that origin is where

\[
\mathbf{r}_{cm} = \left( x_{cm}, y_{cm}, z_{cm} \right),
\]

where

\[
x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{M},
\]

\[
y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{\sum_i m_i y_i}{M},
\]

\[
z_{cm} = \frac{m_1 z_1 + m_2 z_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum_i m_i z_i}{\sum_i m_i} = \frac{\sum_i m_i z_i}{M},
\]

i.e., \[
\mathbf{r}_{cm} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{r}_i}{M},
\]

or \[
M \mathbf{r}_{cm} = \sum_i m_i \mathbf{r}_i,
\]

where \( M = \sum_i m_i \) is the total mass of the ensemble.

**Question 8.1:** Find the center of mass of the sheet of metal of uniform thickness, shown below, relative to the lower left hand corner. The mass of the sheet is 35 kg.
Consider the sheet as 3 separate object (masses), then
mass $1 + mass \ 2 + mass \ 3 = 35 \ kg$.
Since the sheet has constant thickness, mass $\propto$ area.
Area $1 + Area \ 2 + Area \ 3 = (3+1+3) \ m^2 = 7 \ m^2$.
\[
\therefore \ \text{mass} \ \frac{m}{m^2} = \frac{35 \ kg}{7 \ m^2} = 5.00 \ \text{kg/m}^2.
\]
mass $1 = mass \ 3 = 3 \ m^2 \times 5.00 \ \text{kg/m}^2 = 15.0 \ \text{kg},$
and mass $2 = 1 \ m^2 \times 5.00 \ \text{kg/m}^2 = 5.00 \ \text{kg}.$
Choose axes and an origin (0,0). By symmetry, the positions of the individual cm’s are:
$1 \ (0.5 \ m,1.5 \ m): \ 2 \ (1.5 \ m,0.5 \ m): \ 3 \ (2.5 \ m,1.5 \ m)$

The coordinates of the center of mass are
\[
x_{cm} = \frac{\sum m_i x_i}{M} \ \text{and} \ y_{cm} = \frac{\sum m_i y_i}{M}.
\]
\[
\therefore x_{cm} = \frac{(15.0 \ kg)(0.5 \ m) + (5.00 \ kg)(1.5 \ m) + (15.0 \ kg)(2.5 \ m)}{35.0 \ kg}
\]
\[
= \frac{52.5 \ \text{kg} \cdot \text{m}}{35.0 \ \text{kg}} = 1.50 \ \text{m}.
\]
and
\[
\therefore y_{cm} = \frac{(15.0 \ kg)(1.5 \ m) + (5.00 \ kg)(0.5 \ m) + (15.0 \ kg)(1.5 \ m)}{35.0 \ kg}
\]
\[
= \frac{47.5 \ \text{kg} \cdot \text{m}}{35.0 \ \text{kg}} = 1.36 \ \text{m}.
\]
\[
\therefore \ r_{cm} = (1.50 \ m,1.36 \ m),
\]
from the chosen origin.
DISCUSSION PROBLEM [8.1]:

A baseball bat is cut through its center of mass (cm) producing two parts. Which piece has the smaller mass?

A: The piece on the left.
B: The piece on the right.
C: They both have the same mass.

The gravitational potential energy of an ensemble of particles is easily found:

\[ U_G = \sum_i U_i = \sum_i m_i g y_i = g \sum_i m_i y_i. \]

But \( y_{cm} = \frac{\sum_i m_i y_i}{M} \).

\[ \therefore U_G = Mg y_{cm}, \]

i.e., \( U_G \) depends on the total mass and the vertical coordinate of the cm only.

NOTE: We have assumed that the value of \( g \) is the same for all particles, i.e., \( g \) is constant.
The center of mass of an object can also be found by integration. The x-component for the object shown is

\[ x_{cm} = \frac{1}{M} \sum_{i} x_i (\Delta m_i), \]

where \( \Delta m_i \) is the mass of an element located at \((x_i, y_i)\) and \( M \) is the total mass of the object. If the number of elements \( \to \infty \), then \( \Delta m_i \to 0 \), and

\[ x_{cm} = \lim_{\Delta m_i \to 0} \frac{1}{M} \sum_{i} x_i (\Delta m_i) = \frac{1}{M} \int x dm. \]

Similarly, for the y- and z-components:

\[ y_{cm} = \frac{1}{M} \int y dm \quad \text{and} \quad z_{cm} = \frac{1}{M} \int z dm. \]

The limits of the integrals are determined by the x, y and z dimensions of the object. Also, generalizing an earlier expression, we can write:

\[ \vec{r}_{cm} = \frac{1}{M} \int \vec{r} dm. \]

**Question 8.2:** Find the position of the center of mass of the triangular sign shown below, from the point O.

Assume the sign is of uniform thickness.
Define $x$ and $y$ axes. Let the thickness of the triangle be $\delta$ and its density be $\rho$. Then, from before,

$$\vec{r}_{cm} = \frac{1}{M} \int \vec{r} \, dm.$$  

Clearly, by symmetry the center of mass lies on the $x$-axis, i.e., $y_{cm} = 0$, so we only need to concern ourselves with the $x$-coordinates. Therefore, we have to evaluate

$$x_{cm} = \frac{1}{M} \int x \, dm,$$

where $dm = 2y\delta\rho \, dx$ and $M = ab\delta\rho$. We need to express $y$ as a function of $x$,

i.e., $y = \frac{b}{a} x$.

Then, substituting for $M$ and $dm$, we get

$$x_{cm} = \frac{1}{ab\delta\rho} \int_0^a \left( \frac{2b\delta\rho}{a} \right) x^2 \, dx = \frac{2}{a^2} \int_0^a x^2 \, dx$$

$$= \frac{2}{a^2} \left[ \frac{x^3}{3} \right]_0^a = \frac{2a}{3}.$$
The center of mass of an irregular object can be found by experiment ... by suspending the object from two different points.

The center of mass lies on the vertical line through each suspension point. Therefore, the center of mass is where two such lines intersect.

**Question 8.3:** In question 8.1 we determined the coordinates of the center of mass of a sheet of metal relative to the origin (0,0). If the sheet is suspended from the origin, what will be the angle between the vertical and the original x-axis?
We found the coordinates of the center of mass of the sheet were (1.50 m, 1.36 m). So, since the center of mass lies on the vertical line through the suspension point, the angle, $\theta$, between the vertical and the x-axis is

$$\theta = \tan^{-1}\left(\frac{y_{\text{cm}}}{x_{\text{cm}}}\right) = \tan^{-1}\left(\frac{1.36 \text{ m}}{1.50 \text{ m}}\right)$$

$$= 42.2^\circ.$$
Newton’s Laws for the center of mass:

**First Law:**
If the *center of mass* of a system is at rest (or is moving with constant velocity) it will remain at rest (or moving with constant velocity) unless a net external force acts on the system.

**Second Law:**
If a net external force acts on a system of particles, the *center of mass* is given an acceleration that is proportional to, and in the same direction as, the force, i.e., $\vec{F}_{net} = M\vec{a}_{cm}$.

What happens to the center of mass if the various parts of a system are moving relative to each other ... ?

**Question 8.4:** A 1500 kg car is moving west with a speed of 20 m/s, and a 3000 kg truck is traveling east with a speed of 16 m/s. What is the speed of the center of mass and in what direction is it moving?
In chapter 4, we stated Newton’s 2nd Law in the form:

\[ \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} \]

where \( \vec{p} = m\vec{v} \) is called the momentum of the particle of mass \( m \) and velocity \( \vec{v} \). This is a more general expression for the 2nd Law and is more like the way Newton stated it, i.e.,

**The net applied force equals the rate of change of momentum.**

But, as we saw earlier, \( \vec{F} = m\vec{a} \) is applicable *only when \( m \) is constant.*

Even though the car is traveling faster to the West than the truck, the greater product of mass and velocity of the truck means the center of mass is moving to the East.

In vector form \( \vec{v}_{\text{cm}} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} \).

**Dimension:** \( p \Rightarrow [M][L][T] \) (vector)

**Units:** kg.m/s
For a collection of objects, the total momentum is the vector sum of the individual momenta,

\[ \vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i. \]

Since \( \vec{v}_{cm} = \frac{\sum_i m_i \vec{v}_i}{M} \) then \( M \vec{v}_{cm} = \sum_i m_i \vec{v}_i = \vec{P}. \)

Take the time derivative,
\[
\frac{d\vec{P}}{dt} = \frac{d}{dt} \sum_i m_i \vec{v}_i = \sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_i = \vec{F}_{net},
\]
i.e., a net force acting on a collection of objects produces a change in the total momentum. This is Newton’s 2nd Law for a collection of objects. **NOTE: if there is no net external force acting on the objects**, i.e., \( \vec{F}_{net} = 0 \), then \( \vec{P} \) is constant and so is \( \vec{v}_{cm} \).

**Conservation of momentum:** If there is zero net external force acting on a system, the total momentum of the system remains constant (even if there’s a collision).

Since momentum is a vector, conservation of momentum means the x- and y-components are independently conserved, as in the following problem.

![Diagram](image)

**Question 8.5:** The two objects shown above, with masses \( m_1 = 3.0 \) kg and \( m_2 = 2.0 \) kg, and speeds \( v_1 = 4.0 \) m/s and \( v_2 = 5.0 \) m/s respectively, collide and stick together forming a composite object.

(a) What is the momentum of the composite object?
(b) Compare the kinetic energy of the system before and after the collision.
(a) Momentum is conserved in the collision, i.e., \( \bar{P}_i = \bar{P}_f \), where \( i \) and \( f \) refer to the initial and final conditions.

The momenta must be added vectorially, so we add the components in the \( x \) and \( y \) directions separately. Before collision, the total momentum in the \( x \)-direction is:

\[
m_1v_1 \cos 30^\circ + m_2v_2 \cos 60^\circ
= (3.0 \text{ kg})(4.0 \text{ m/s})(0.866) + (2.0 \text{ kg})(5.0 \text{ m/s})(0.50)
= 15.39 \text{ kg} \cdot \text{m/s}.
\]

After the collision the momentum in the \( x \)-direction is

\[
(m_1 + m_2)v_x = 15.39 \text{ kg} \cdot \text{m/s}.
\]

\( \therefore \) \( v_x = \frac{15.39 \text{ kg} \cdot \text{m/s}}{(3.0 \text{ kg} + 2.0 \text{ kg})} = 3.08 \text{ m/s} \).

Similarly, in the \( y \)-direction, before the collision

\[
-m_1v_1 \sin 30^\circ + m_2v_2 \sin 60^\circ
= -(3.0 \text{ kg})(4.0 \text{ m/s})(0.50) + (2.0 \text{ kg})(5.0 \text{ m/s})(0.866)
= 2.66 \text{ kg} \cdot \text{m/s}.
\]

After the collision

\[
(m_1 + m_2)v_y = 2.66 \text{ kg} \cdot \text{m/s}.
\]

\( \therefore \) \( v_y = \frac{2.66 \text{ kg} \cdot \text{m/s}}{(3.0 \text{ kg} + 2.0 \text{ kg})} = 0.53 \text{ m/s} \).

So, the momentum after the collision is

\[
\bar{P}_f = (m_1 + m_2)(v_x, v_y)
= (15.40 \hat{i} + 2.66 \hat{j}) \text{ kg} \cdot \text{m/s}.
\]

Also \( \theta = \tan^{-1}\left( \frac{P_y}{P_x} \right) = \tan^{-1}\left( \frac{2.66 \text{ kg} \cdot \text{m/s}}{15.40 \text{ kg} \cdot \text{m/s}} \right) = 9.80^\circ \).

(b) The kinetic energy before the collision is

\[
K_i = K_1 + K_2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2
= \frac{1}{2}(3.0 \text{ kg})(4.0 \text{ m/s})^2 + \frac{1}{2}(2.0 \text{ kg})(5.0 \text{ m/s})^2 = 49.0 \text{ J}.
\]
The kinetic energy after the collision is

\[ K_f = \frac{1}{2} (m_1 + m_2) v_f^2, \]

where

\[ v_f = \sqrt{(3.08 \text{ m/s})^2 + (0.53 \text{ m/s})^2} = 3.13 \text{ m/s}. \]

\[ : K_f = \frac{1}{2} (5.0 \text{ kg})(3.13 \text{ m/s})^2 = 24.5 \text{ J}. \]

Therefore, although momentum is conserved, kinetic energy is not conserved. Such a collision is called an \textit{inelastic collision}. A collision where both momentum and kinetic energy are conserved is called an \textit{elastic collision} (later).

Let’s look at some “collisions” ... i.e., when a force is applied to change the motion.

\textit{Examples:}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\textbf{Time (ms)} & \textbf{Force ($\times 10^5 \text{N}$)} \\
\hline
0 & 0 \\
50 & 3 \\
100 & 1 \\
\hline
\end{tabular}
\end{table}

“Crushing a Beetle”

“Hard” collision

“Soft” collision
Newton’s 2\textsuperscript{nd} Law is \( \vec{F} = \frac{d\vec{p}}{dt} \), i.e., \( \int_{t_1}^{t_2} \vec{F} dt = \Delta \vec{p} \). If the force is applied for a finite time \( \Delta t \) (\( = t_2 - t_1 \)), we define the \textit{impulse of the force} as
\[
\vec{I} = \int_{t_1}^{t_2} \vec{F} dt = \frac{2}{\Delta t} \int_{t_1}^{t_2} d\vec{p} = \vec{p}_2 - \vec{p}_1 = \Delta \vec{p},
\]
so the impulse equals the change in momentum. If the force varies, the \textit{average force} over the interval \( \Delta t = t_2 - t_1 \), is
\[
\vec{F}_{av} = \frac{1}{\Delta t} \int_{t_1}^{t_2} \vec{F}(t) dt = \frac{\vec{I}}{\Delta t} = \frac{\Delta \vec{p}}{\Delta t}.
\]

For a given change in momentum \( \Delta p \)
\[
F_{av} \propto \frac{1}{\Delta t}, \text{ i.e., } F_{av} \cdot \Delta t = \text{ constant.}
\]

\textit{Let’s take a closer look at ...}
\[
F_{av} \cdot \Delta t = \Delta p.
\]
If an object (truck, ball, you, etc.) is brought to a stop, then the longer the time \( \Delta t \) it takes, the smaller the required force \( F_{av} \) ... \textit{many examples in everyday life.}

\begin{itemize}
  \item \textbf{Same }\Delta p
    \begin{itemize}
      \item \textbf{Large }\Delta t: \text{ small } F_{av} \Rightarrow \text{ little damage}
      \item \textbf{Small }\Delta t: \text{ large } F_{av} \Rightarrow \text{ lots of damage!}
    \end{itemize}
  \end{itemize}

\textit{The impulse }\vec{I} = \Delta \vec{p} = \vec{F}_{av} \cdot \Delta t \text{ (vector)}

\textbf{Dimension}: same as momentum \( \Rightarrow [M][L][T] \)

\textbf{Units}: N\cdot s (alternatively kg \cdot m/s).
**Question 8.6:** A ball of mass 0.10 kg experiences the force shown above. If the ball was initially at rest, what is its speed after 8 ms?

We know that \( \int F \, dt = \Delta p \), i.e., the area under the F-t plot equals the change in momentum \( \Delta p = mv_f - mv_i \). Since the ball was at rest initially,

\[
\Delta p = mv_f = 0.12 \text{ N} \cdot \text{s},
\]

i.e., \( v_f = \frac{0.12 \text{ N} \cdot \text{s}}{0.10 \text{ kg}} = 1.20 \text{ m/s} \).
**Question 8.7:** A girl, of mass 55 kg, jumps from the bow of a boat, of mass 100 kg, onto a dock. If the boat is initially at rest and her velocity is 2.5 m/s to the right, what is the velocity of the boat after she jumps?

*Take the girl and the boat as the system ... there are no external forces, e.g., wind or a push, so the total momentum is conserved, i.e., constant.*

- **Before she jumped,** the initial momentum $P_i = 0$, since the girl and the boat are both stationary.
- **After she jumped,** the final momentum $P_f = 0$.

Taking the $+x$-direction as the $+ve$ direction and the velocity of the boat = $v$, we have:

$$P_i + P_f = 0$$

i.e., $[(55 \text{ kg}) \times (2.5 \text{ m/s})] + [(100 \text{ kg} \times v)] = 0$

$$\therefore v = -1.375 \text{ m/s}.$$  
Since $v < 0$ it is directed to the left. What about $v_{\text{cm}}$?
• Before she jumped, $v_{cm} = 0$, since the girl and the boat are both stationary.

• After she jumped, $v_{cm} = \frac{\sum_i m_i v_i}{M}$

  $$= \frac{(55 \text{ kg})(2.5 \text{ m/s}) - (100 \text{ kg})(1.375 \text{ m/s})}{100 \text{ kg} + 55 \text{ kg}}$$

  $$= 0,$$

i.e., $v_{cm}$ is unchanged.

**Question 8.8:** A ball with mass 100 g is thrown horizontally against a wall with a speed of 5.0 m/s. It strikes the wall at an angle of 40° and bounces off at the same angle with the same speed. If it is in contact with the wall for 2.0 ms, what is the average force exerted by the wall on the ball? Neglect the effect of gravity.
The change in momentum of the ball is
\[ \Delta \mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1 = \mathbf{I}_{\text{WB}} \]
\[ = \left[ (0.10 \text{ kg})(5.0 \text{ m/s}) \cos 40^\circ \right] \hat{i} \]
\[ - \left[ (0.10 \text{ kg})(-5.0 \text{ m/s}) \cos 40^\circ \right] \hat{i} \]
\[ = (0.766 \text{ kg} \cdot \text{m/s}) \hat{i}. \]

From the definition of impulse
\[ \mathbf{I}_{\text{WB}} = \mathbf{F}_{\text{av}} \Delta t, \]
i.e., \[ \mathbf{F}_{\text{av}} = \frac{\mathbf{I}_{\text{WB}}}{\Delta t} = \frac{(0.766 \text{ kg} \cdot \text{m/s}) \hat{i}}{2 \times 10^{-3} \text{s}} \]
\[ = (383 \text{ N}) \hat{i}. \]

By Newton’s 3rd Law this is equal and opposite to the force of the ball on the wall.

**Question 8.9**: A baseball with mass 100 g is thrown horizontally towards you with a speed of 10.0 m/s. In catching the ball, your hand moves back 15.0 cm. Assuming your hand moves back parallel to the flight of the ball,
(a) what is the impulse of the ball on your hand, and
(b) what is the average force exerted by the ball on your hand?
(a) When you catch the ball, by Newton’s 3rd Law

\[ \vec{F}_{BH} = -\vec{F}_{HB}. \]

Since \( \Delta t \) is the same for both forces,

\[ I_{BH} = -I_{HB}, \]

i.e., \( |I_{BH}| = |I_{HB}| = |\Delta \vec{p}_{ball}| \)

\[ = |0 - [(0.1 \text{ kg})(10 \text{ m/s})]| = 1.0 \text{ N \cdot s} \quad \text{(kg \cdot m/s)}. \]

(b) If \( \Delta t \) is the time interval for the ball to stop, then

\[ I_{BH} = F_{BH} \cdot \Delta t = \Delta p = 1.0 \text{ N \cdot s.} \]

But, what is \( \Delta t \)?

- \( v^2 = v_0^2 + 2a\Delta x. \)
  \[ \therefore a = -\frac{v_0^2}{2\Delta x} = \frac{(10 \text{ m/s})^2}{2(0.15 \text{ m})} = -333.3 \text{ m/s}^2. \]
- \( v = v_0 + a\Delta t. \)
  \[ \therefore \Delta t = -\frac{v_0}{a} = -\frac{(10 \text{ m/s})}{(-333.3 \text{ m/s}^2)} = 0.030 \text{ s.} \]

\[ \therefore F_{BH} = \frac{\Delta p}{\Delta t} = \frac{1.0 \text{ N \cdot s}}{0.030 \text{ s}} = 33.3 \text{ N.} \]

Closer look at collisions:

\~ Momentum is always conserved \~

But there are two types of collisions ...

- non-elastic (inelastic) collisions
  (Kinetic energy not conserved).
- elastic collisions
  (Kinetic energy conserved).

1. Non-elastic (inelastic) collisions.
   (e.g., when objects “stick” together)

   \[ \vec{p}_1 \rightarrow \vec{p}_f \]
   \[ \vec{p}_1 + \vec{p}_2 = \vec{p}_f \]
   \[ \therefore \vec{v}_f (= \vec{v}_{cm}) = \frac{\vec{p}_f}{m_1 + m_2}. \]

   Examples:
   - bullet lodges in a target
   - book falling on a table

\~ What happens to the “lost” KE ? ~
Ballistic pendulum:

If the bullet lodges in the target, conservation of momentum tells us

\[ m_1 v_1 = (m_1 + m_2) v_f \]

\[ \therefore v_1 = \frac{m_1 + m_2}{m_1} v_f. \]

With conservation of mechanical energy, \( \Delta K = -\Delta U \),

\[ \frac{1}{2} (m_1 + m_2) v_f^2 = (m_1 + m_2)gh \]

\[ \therefore v_f = \sqrt{2gh} \]

i.e., \( v_1 = \frac{(m_1 + m_2)}{m_1} \sqrt{2gh} \).

*Can be used to determine the speed of a bullet.*

What about the kinetic energy?

Kinetic energy *before* collision:

\[ K_i = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} \left( \frac{m_1 v_1}{m_1} \right)^2 = \frac{p_1^2}{2m_1}. \]

Kinetic energy immediately *after* collision:

\[ K_f = \frac{1}{2} (m_1 + m_2) v_f^2 = \frac{p_2^2}{2(m_1 + m_2)} = \frac{p_1^2}{2(m_1 + m_2)}. \]

The change in kinetic energy:

\[ \Delta K = K_f - K_i = \frac{p_1^2}{2(m_1 + m_2)} - \frac{p_1^2}{2m_1} < 0. \]

Since \((m_1 + m_2) > m_1\), \( \Delta K \) is always negative, i.e., kinetic energy is never conserved.

*What happens to the “lost” energy?*
2. **Elastic collisions:**

*momentum and kinetic energy are both conserved.*

\[ \vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2' \]

and

\[ K_1 + K_2 = K_1' + K_2' \]

The analysis is more difficult in 2- and 3-dimensions because one needs to know *three* of the velocities involved in order to determine the *fourth.*

**Elastic collisions in 1-dimension**

Before collision

\[
\begin{align*}
\text{Speed } #1 &< \text{Speed } #2 \\
\text{Speed } #1 &< \text{Speed } #2
\end{align*}
\]

After collision

\[
\begin{align*}
\text{Speed } #1 &> \text{Speed } #2 \\
\text{Speed } #1 &> \text{Speed } #2
\end{align*}
\]

Using conservation of momentum and kinetic energy:

\[ m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \]

and

\[
\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2.
\]

Re-arranging the two equations gives,

\[ m_1 (v_{1i} - v_{1f}) = m_2 (v_{2f} - v_{2i}) \quad \ldots \quad (1) \]

and

\[ m_1 (v_{1i}^2 - v_{1f}^2) = m_2 (v_{2f}^2 - v_{2i}^2) \quad \ldots \quad (2) \]

Dividing (2) by (1), we get

\[
\frac{(v_{1i}^2 - v_{1f}^2)}{(v_{1i} - v_{1f})} = \frac{(v_{2f}^2 - v_{2i}^2)}{(v_{2f} - v_{2i})},
\]

i.e.,

\[
\frac{(v_{1i} + v_{1f})(v_{1i} - v_{1f})}{(v_{1i} - v_{1f})} = \frac{(v_{2f} + v_{2i})(v_{2f} - v_{2i})}{(v_{2f} - v_{2i})}.
\]

\[ \therefore v_{1i} + v_{1f} = v_{2f} + v_{2i}, \]

i.e.,

\[ -(v_{2i} - v_{1i}) = v_{2f} - v_{1f} \]

*speed #1 approaches #2 = speed #2 moves away from #1*
Note that this expression
\[-(v_{2i} - v_{1i}) = v_{2f} - v_{1f}\]
is true only for a 1-dimensional perfectly elastic collision, i.e., one in which kinetic energy is conserved.

In the macroscopic world, collisions are somewhere between perfectly inelastic and perfectly elastic; at the microscopic level elastic collisions are common, e.g., between atoms.

We define a coefficient of restitution \((e)\) to tell what type of a collision we have ...
\[e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}} = \frac{\text{speed of recession}}{\text{speed of approach}}\]

Note that ...
- if \(e = 1\) the collision is perfectly elastic.
- if \(e = 0\) the collision is perfectly inelastic*.

For a bouncing ball dropped vertically, if the speed at the instant before the collision is \(v_{1i}\) and the speed immediately after the collision is \(v_{1f}\), then
\[e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}} = \frac{v_{1f}}{v_{1i}},\]
since \(v_{1i}\) and \(v_{1f}\) are measured with respect to the ground (i.e., \(v_{2i} = v_{2f} = 0\)). Usually, a ball does not reach the original height, so \(|v_{1f}| < |v_{1i}|\), i.e., \(e < 1\).

Typical values (on concrete):
- \(e_{\text{baseball}} = 0.57\).
- \(e_{\text{basketball}} = 0.57\).
- \(e_{\text{racquetball}} = 0.86\).
- \(e_{\text{golfball}} = 0.89\).

Show for yourselves: if a ball is dropped from a height \(h\) onto a solid floor, the height of the first bounce is \(h_1 = e^2 h\).

* In a perfectly inelastic collision the two colliding objects stick together.
**Question 8.10:** A 0.10 kg rubber ball strikes a solid wall horizontally with a speed of 5.00 m/s and rebounds in the opposite direction. Above is shown the force the wall exerts on the ball. Neglecting the effect of gravity during the impact,

(a) what is the speed of the ball when it rebounds, and

(b) what is the coefficient of restitution for the impact?

(a) We need to find the speed of the ball after the rebound. During the impact, the change in momentum equals the impulse of the wall on the ball, i.e.,

\[ \Delta p = p_2 - p_1 = \int F \, dt = \text{area under the force-time curve}. \]

\[ \therefore m v_2 - m(-v_1) = \frac{1}{2} F_{\text{max}} \Delta t = \frac{1}{2} (225 \text{ N})(0.0065\text{ s}), \]

i.e., \[ v_2 + v_1 = \frac{0.731 \text{ kg} \cdot \text{m/s}}{0.10 \text{ kg}} = 7.31 \text{ m/s}. \]

\[ \therefore v_2 = 7.31 \text{ m/s} - 5.00 \text{ m/s} = 2.31 \text{ m/s} \]

(b) The coefficient of restitution is

\[ e = \left| \frac{v_2}{v_1} \right| = \frac{2.31 \text{ m/s}}{5.00 \text{ m/s}} = 0.462. \]
**Question 8.11**: A 2.0 kg object moving at 6.0 m/s collides with a 4.0 kg object that is initially at rest. After the collision, the 2.0 kg object moves *backwards* at 1.0 m/s.

Find

(a) the velocity of the 4.0 kg object after the collision,
(b) the velocity of the center of mass before and after the collision,
(c) the coefficient of restitution of the collision,
(d) the kinetic energy change in the collision.

Since momentum is conserved:

\[ p_{1f} + p_{2f} = p_{1i} + p_{2i}, \]

i.e., \((2.0 \text{ kg})(-1.0 \text{ m/s}) + (4.0 \text{ kg})v_{2f} = 12.0 \text{ kg} \cdot \text{m/s}.\)

\[ v_{2f} = \frac{14.0 \text{ kg} \cdot \text{m/s}}{4.0 \text{ kg}} = 3.5 \text{ m/s}. \]

(b) From earlier \(v_{cm} = \frac{\sum_i m_i v_i}{M}.\)

- Before the collision:
  \[ v_{cm} = \frac{(2.0 \text{ kg})(6.0 \text{ m/s}) + (4.0 \text{ kg})(0)}{(2.0 \text{ kg} + 4.0 \text{ kg})} = 2.0 \text{ m/s}. \]
- After the collision:
  \[ v_{cm} = \frac{(2.0 \text{ kg})(-1.0 \text{ m/s}) + (4.0 \text{ kg})(3.5 \text{ m/s})}{(2.0 \text{ kg} + 4.0 \text{ kg})} = 2.0 \text{ m/s}. \]
Therefore, the velocity of the center of mass is conserved in the collision.

\[(c) \ e = \left| \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}} \right| = \left| \frac{(3.5 \text{ m/s}) - (-1.0 \text{ m/s})}{(6.0 \text{ m/s}) - 0} \right| = 0.75.\]

So, the collision is \textit{non-elastic} (but \textbf{not} inelastic).

(d) Change in kinetic energy \(\Delta K = K_f - K_i\), i.e.,

\[
\begin{align*}
\Delta K &= \left( \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \right) - \left( \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 \right) \\
&= \frac{1}{2} \left( (2.0 \text{ kg})(-1.0 \text{ m/s})^2 + (4.0 \text{ kg})(3.5 \text{ m/s})^2 \right) \\
&\quad - \frac{1}{2} \left( (2.0 \text{ kg})(6.0 \text{ m/s})^2 \right) \\
&= 25.5 \text{ J} - 36.0 \text{ J} = -10.5 \text{ J}. \\
&\text{“lost”}
\end{align*}
\]