Simple harmonic motion

Simple harmonic motion (SHM) is a very basic kind of oscillatory motion. The simplest example is a mass attached to a spring.

If a mass is displaced from the equilibrium position, in the absence of frictional forces it will oscillate with SHM. Let us analyze the motion of the mass on the vertical spring.
If the mass is displaced by $y$ from its equilibrium position, the restoring force acting on the mass is given by Hooke’s Law, i.e.,

$$F_y = -ky,$$

where $k$ is the spring constant.

By Newton’s 2nd Law

$$ma_y = F_y,$$

i.e.,

$$m \frac{d^2y}{dt^2} = -ky.$$

∴

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y.$$

This expression defines SHM, i.e.,

if the acceleration of an object is proportional to its displacement and is oppositely directed, the object will move with SHM.

A solution to the differential equation is

$$y(t) = A \cos(\omega t + \delta),$$

where $A$, $\omega$ and $\delta$ are constants.

The maximum displacement, $A$, is the amplitude, $\omega$ is the angular frequency, $\delta$ is the phase angle, which is determined by the displacement at $t = 0$. Take the first and second derivatives of $y$ with respect to $t$, then

$$\frac{dy}{dt} = v(t) = -\omega A \sin(\omega t + \delta),$$

and

$$\frac{d^2y}{dt^2} = a(t) = -\omega^2 A \cos(\omega t + \delta) = -\omega^2 y.$$

So, $y(t) = A \cos(\omega t + \delta)$ is a solution provided $\omega^2 = \frac{k}{m}$.

The frequency of oscillations is $f = \frac{\omega}{2\pi}$ and the periodic time (for one complete oscillation) is $T = \frac{1}{f}$. 

Note that the period of SHM is *independent of the amplitude*. (In music, it means that the pitch or frequency of a note struck on any stringed instrument, for example, is independent of how loudly the note is played.)

We would obtain the same type of solution for a horizontal spring, viz:

\[ x(t) = A \cos(\omega t + \delta). \]

**Question 14.1**: A 0.40 kg mass, attached to a spring with force constant 12.0 N/m, oscillates with an amplitude of 8.0 cm. Find

(a) the maximum speed of the mass,

(b) the speed and acceleration of the mass when it is at \( y = 4.0 \) cm from the equilibrium position, and

(c) the time it takes the mass to travel from \( y = 0 \) to \( y = 4.0 \) cm.
The solution for the displacement is \( y(t) = A \cos(\omega t + \delta) \).
The mass starts to oscillate at \( t = 0 \) when \( y = 8.0 \text{ cm} \), i.e., at maximum displacement (amplitude) so \( \delta = 0 \).

(a) The speed of the mass is \( v = \left| \frac{dy}{dt} \right| = |\omega A \sin(\omega t)| \), so the maximum speed is

\[
v_{\text{max}} = |\omega A| = A \sqrt{\frac{k}{m}} = (0.08 \text{ m}) \sqrt{\frac{12.0 \text{ N/m}}{0.40 \text{ kg}}} = 0.438 \text{ m/s}.
\]

Maximum speed occurs when \( \omega t = (2n + 1) \frac{\pi}{2} \), i.e., when \( y(t) = 0 \).

(b) When \( y = 0.04 \text{ m} \), we have

\[
0.04 \text{ m} = (0.08 \text{ m}) \cos \omega t \Rightarrow \omega t = \cos^{-1}(0.5) = \frac{\pi}{3}.
\]

\[
\therefore \; v = |\omega A \sin \omega t| = (0.438 \text{ m/s}) \sin \left( \frac{\pi}{3} \right) = (0.438 \text{ m/s}) \frac{\sqrt{3}}{2} = 0.379 \text{ m/s}.
\]

The acceleration is

\[
a = \frac{d^2y}{dt^2} = |\omega^2 A \cos \omega t| = v_{\text{max}} \omega \cos \left( \frac{\pi}{3} \right)
\]

\[
= (0.438 \text{ m/s}) \sqrt{\frac{12 \text{ N/m}}{0.40 \text{ kg}}} \times 0.5 = 1.20 \text{ m/s}^2.
\]

(c) When \( y = 0 \),

\[
A \cos \omega t = 0 \Rightarrow \omega t = \cos^{-1} 0 = \frac{\pi}{2}, \text{ i.e., } t = \frac{\pi}{2\omega}.
\]

From (b), when \( y = 4.0 \text{ cm} \), \( \omega' t = \frac{\pi}{3} \), i.e., \( t' = \frac{\pi}{3\omega} \).

\[
\therefore \; \Delta t = \frac{1}{\omega} \left( \frac{\pi}{2} - \frac{\pi}{3} \right) = \frac{\pi}{6\omega} = \frac{\pi}{6} \sqrt{\frac{0.40 \text{ kg}}{12.0 \text{ N/m}}} = 0.096 \text{ s} \quad (= 96 \text{ ms}).
\]
**Question 14.2:** Two identical springs each have a spring constant $k = 20 \text{ N/m}$. A mass, $M = 0.3 \text{ kg}$, is connected to the springs in two different configurations, as shown above. What is the periodic time for each configuration?

(a) Consider what happens when the mass is given a displacement $y (>0)$. Since each spring is stretched a distance $y$, the net restoring force (by Hooke’s Law) is

$$F = -(20 \text{ N/m})y - (20 \text{ N/m})y = -(40 \text{ N/m})y.$$  

So, the effective spring constant is $k' = 40 \text{ N/m}$. Hence,

$$T = 2\pi \sqrt{\frac{M}{k'}} = 2\pi \sqrt{\frac{0.3 \text{ kg}}{40 \text{ N/m}}} = 0.54 \text{ s}.$$  

(b) Give the mass a displacement $x$. One spring is stretched a distance $x$, the other is compressed a distance $x$ and so they each exert a force of magnitude $20 \text{ N/m})x$ on the mass in a direction opposite to the displacement. Hence, the total restoring force is

$$F = -(20 \text{ N/m})x - (20 \text{ N/m})x = -(40 \text{ N/m})x.$$  

Again, the effective spring constant $k' = 40 \text{ N/m}$, which is the same as in (a). So, the periodic times are the same.
Energy in simple harmonic motion

The total energy is the initial energy, i.e., \( E = \frac{1}{2} kA^2 \).

**Question 14.3:** A block of mass \( M \) is connected to a horizontal spring on a frictionless table. The spring is compressed a distance \( A \) and released so the mass executes simple harmonic motion.

(a) Deduce an expression for the velocity of the mass at any displacement \( x \).

(b) When the displacement is \( \frac{A}{2} \), what fraction of the mechanical energy is kinetic energy?

(c) At what displacement, as a fraction of \( A \), is the mechanical energy half kinetic and half elastic potential energy?
(a) The total energy of the system is
\[ E = \frac{1}{2} k x^2 + \frac{1}{2} M v^2. \]
But the total energy is the initial elastic potential energy,
\[ \text{i.e., } E = \frac{1}{2} k A^2. \]
\[ \therefore \frac{1}{2} M v^2 = \frac{1}{2} k A^2 - \frac{1}{2} k x^2, \]
\[ \text{i.e., } v = \sqrt{\frac{k}{M} (A^2 - x^2)}. \]

(b) Since the total energy is \( E = \frac{1}{2} kA^2 \), when the displacement is \( \frac{A}{\sqrt{2}} \), the elastic potential energy is
\[ U_E = \frac{1}{2} k \left( \frac{A}{\sqrt{2}} \right)^2 = \frac{1}{8} k A^2 = \frac{1}{4} E. \]
\[ \therefore K = E - U_E = \frac{3}{4} E. \]

(c) \[ U_E = \frac{1}{2} k x^2 = \frac{1}{2} E = \frac{1}{4} k A^2. \]
\[ \therefore x = \frac{A}{\sqrt{2}}. \]

**Question 14.4:** A 3.0 kg mass on a frictionless horizontal surface attached to a spring, oscillates with an amplitude of 8.0 cm. If its maximum acceleration is 3.5 m/s\(^2\),
(a) what is the total mechanical energy?
(b) What is the maximum kinetic energy of the mass?
(a) The total energy is \( E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \). We need to determine the spring constant \( k \). The force exerted by the spring is
\[
F = ma = -kx, \text{ i.e., } k = -\frac{ma}{x},
\]
so the maximum acceleration occurs when \( x = -A \).
\[
\therefore E = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{ma_{\text{max}}}{A}\right)A^2,
\]
but \( v = 0 \) when \( x = \pm A \).
\[
\therefore E = \frac{1}{2}ma_{\text{max}}A
\]
\[
= \frac{1}{2}(3.0 \text{ kg})(3.5 \text{ m/s}^2)(0.080 \text{ m})
\]
\[
= 0.42 \text{ J}.
\]
(b) Maximum kinetic energy occurs when \( x = 0 \).
\[
\therefore K_{\text{max}} = \frac{1}{2}mv_{\text{max}}^2 = 0.42 \text{ J}
\]
i.e., \( v_{\text{max}} = \sqrt{\frac{2(0.42 \text{ J})}{3.0 \text{ kg}}} = 0.53 \text{ m/s} \).

**Physical pendulum**

Any rigid object that is free to rotate will oscillate about its equilibrium position when displaced and released. We show now that when such an object is released, its motion will be SHM. Using Newton’s 2nd Law for rotation
\[
I\ddot{\theta} = \tau,
\]
we have
\[
I\frac{d^2\theta}{dt^2} = -(D\sin \theta)Mg.
\]
If \( \theta \) is small, \( \sin \theta \to \theta \), then
\[
I\frac{d^2\theta}{dt^2} = -MgD\theta,
\]
i.e.,
\[
\frac{d^2\theta}{dt^2} = -\frac{MgD}{I} \theta.
\]
Note that this equation satisfies our definition of SHM. A suitable general solution is
\[
\theta(t) = \theta_{\text{max}} \cos(\omega t + \delta).
\]
\[ \frac{d\theta}{dt} = -\omega \theta_{\text{max}} \sin \omega t \]

and \[ \frac{d^2\theta}{dt^2} = -\omega^2 \theta_{\text{max}} \cos \omega t = -\omega^2 \theta. \]

Thus, if \( \theta = \theta_{\text{max}} \cos \omega t \) is a solution,

\[ \omega^2 = \frac{MgD}{I}, \quad \text{i.e.,} \quad \omega = \sqrt{\frac{MgD}{I}}. \]

But the swing frequency is \( f = \frac{\omega}{2\pi} \) and the period for one complete oscillation is \( T = \frac{1}{f} \).

\[ \therefore T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{MgD}}. \]

Note that providing \( \theta \) is small, the periodic time is independent of the amplitude. Note also, that \( I \) is the moment of inertia about the pivot point.

**Question 14.5:** A uniform right-angle square is suspended by a thin nail so that it pivots freely, as shown above. Each arm has length \( \ell = 25 \text{ cm} \) and mass \( m = 0.25 \text{ kg} \). If it is given a small angular displacement, in the plane of the square, and released, what is the periodic time of the oscillations?
The periodic time of oscillations of a physical pendulum is

\[ T = 2\pi \sqrt{\frac{I}{MgD}}, \]

where \( I \) is the moment of inertia about the rotation axis, \( M \) (\( = 2m \)) is the total mass and \( D \) is the distance from the rotation axis to the center of mass. The moment of inertia about the nail is

\[ I = \frac{m\ell^2}{3} + \frac{m\ell^2}{3} = \frac{2m\ell^2}{3}, \]

and the distance \( D \) is

\[ \ell \cos 45^\circ = \frac{\ell}{2\sqrt{2}}. \]

\[ \therefore T = 2\pi \sqrt{\frac{2m\ell^2 \cdot 2\sqrt{2}}{3(2m)g\ell}} = 2\pi \sqrt{\frac{2\ell\sqrt{2}}{3g}} \]

\[ = 2\pi \sqrt{\frac{2(0.25 \text{ m})\sqrt{2}}{3(9.81 \text{ m/s}^2)}} = 0.97 \text{ s}. \]

**Question 14.6:** A uniform disk with a radius \( R = 0.80 \text{ m} \) and a mass of 6.00 kg, has a small hole a distance \( D \) from the center of the disk. If the hole is a pivot point such that the disk can swing freely, what should be the distance \( D \) so that the periodic time of this physical pendulum in 2.50 s?
We know \( T = 2\pi \sqrt{\frac{1}{MgD}} \). Using the parallel axis theorem, the moment of inertia of the disk about the pivot point is

\[
I = I_{cm} + mD^2 = \frac{1}{2} mR^2 + mD^2.
\]

Squaring both sides we get

\[
T = 2\pi \sqrt{\frac{1}{2} \frac{mR^2 + mD^2}{mgD}} = 2\pi \sqrt{\frac{1}{2} \frac{R^2 + D^2}{gD}}.
\]

The two roots of the equation are

\[
D = 1.31 \text{ m and } D = 0.24 \text{ m.}
\]

Clearly, the latter is the appropriate solution.

**Simple pendulum**

The simple pendulum is a special case of the physical (or compound) pendulum. In this case, if we ignore the mass of the string, the moment of inertia of the bob of mass \( m \) about the pivot point is

\[
I = m\ell^2,
\]

and the distance of the center of mass of the bob from the pivot point is

\[
D = \ell.
\]

The solution is again

\[
\theta = \theta_{max} \cos \omega t,
\]

but with \( \omega = \sqrt{\frac{mgD}{I}} = \sqrt{\frac{mg\ell}{m\ell^2}} = \sqrt{\frac{g}{\ell}} \). The period for one entire oscillation is

\[
T = \frac{1}{f} = \frac{2\pi}{\omega}, \text{ i.e.,}
\]

\[
T = 2\pi \sqrt{\frac{\ell}{g}}.
\]

Note that we can write the arc length as \( s = \ell \theta \), so that
\[ s = \ell \theta_{\text{max}} \cos \omega t. \]

- The tangential speed of the bob is
  \[ v_t = \frac{ds}{dt} = -\omega \ell \theta_{\text{max}} \sin \omega t, \]
  which is zero when \( \omega t = n \pi \), i.e., when \( s = s_{\text{max}} = \ell \theta_{\text{max}} \).
  The tangential speed is a \textit{maximum} when \( \sin \omega t = 1 \), i.e., when \( \omega t = \frac{2n + 1}{2} \pi \). That occurs when \( s = 0 \).

- The tangential acceleration of the bob is
  \[ a_t = \frac{d^2 s}{dt^2} = -\omega^2 \ell \theta_{\text{max}} \cos \omega t, \]
  which is zero when \( \omega t = \frac{2n + 1}{2} \pi \), i.e., when \( s = 0 \). The tangential acceleration is a \textit{maximum} when \( \sin \omega t = 1 \), i.e., when \( \omega t = n \pi \). That occurs when \( s = s_{\text{max}} = \ell \theta_{\text{max}} \).

**Question 14.7**: The pendulum shown above is an “interrupted” pendulum. There is a peg \( P \) below the pivot point \( O \), so when the pendulum swings to the left, the string makes contact with the peg, which causes the bob to swing on a different arc. If the length of the string is 78.0 cm and the distance \( OP \) is one-half the length of the string, what is the periodic time of the pendulum?
We can consider the pendulum in two parts … when the bob swings from the right, the one-quarter period is

\[
T_1 = \left(\frac{1}{4}\right) 2\pi \sqrt{\frac{\ell}{g}} = \frac{\pi}{2} \sqrt{\frac{\ell}{g}},
\]

and when it swings past the vertical, the one-quarter period is

\[
T_2 = \left(\frac{1}{4}\right) 2\pi \sqrt{\frac{\ell/2}{g}} = \frac{\pi}{2} \sqrt{\frac{\ell}{2g}}.
\]

Hence, the total period is

\[
2(T_1 + T_2) = 2 \left( \frac{\pi}{2} \sqrt{\frac{\ell}{g}} + \frac{\pi}{2} \sqrt{\frac{\ell}{2g}} \right) = \pi \left(1 + \frac{1}{\sqrt{2}}\right) \sqrt{\frac{\ell}{g}}
\]

\[
= 1.71\pi \sqrt{\frac{0.78 \text{ m}}{9.81 \text{ m/s}^2}} = 1.51 \text{ s}.
\]

The behavior of the interrupted pendulum was discussed by Galileo … can you think why?

**Question 14:8:** Consider two pendulums, a simple pendulum of length 1 m and a 1 m ruler pivoted at one end. Which one, if either, has the smaller periodic time?
The periodic time of a simple pendulum is 
\[ T_s = 2\pi \sqrt{\frac{\ell}{g}}. \]

For a ruler of length \( \ell \) pivoted at one end, \( I = \frac{M\ell^2}{3} \) and \( D = \frac{\ell}{2} \).

\[ \therefore T_r = 2\pi \sqrt{\frac{1}{MgD}} = 2\pi \sqrt{\frac{M\ell^2}{3Mg} \frac{2}{\ell}} = 2\pi \sqrt{\frac{2\ell}{3g}}, \]

i.e., \( T_r = T_s \sqrt{\frac{2}{3}} = 0.816 T_s \). \( \therefore T_r < T_s. \)

\[ T_s = 2\pi \sqrt{\frac{1 \text{ m}}{9.81 \text{ m/s}^2}} = 2.00 \text{ s}, \text{ so } T_r = 1.64 \text{ s}. \]

So, the ruler pendulum has the shorter periodic time.

Torsional oscillator

A torsional oscillator, or torsional pendulum, consists of a disk-like mass suspended from a wire or thin rod. If the mass is twisted about the axis of the wire, the wire exerts a restoring torque. If twisted and released, the mass will oscillate back and forth with simple harmonic motion. If the angle \( \theta \) is sufficiently small so the wire is not plastically deformed, the restoring torque is \( \tau = -\kappa \theta \), where \( \kappa \) is the torsion constant. If the moment of inertia of the mass is \( I \), the angular acceleration of the mass is

\[ \alpha = \frac{\tau}{I}, \text{ i.e., } \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I} \theta, \]

which defines simple harmonic motion. The solution is

\[ \theta(t) = \theta_{\text{max}} \cos(\omega t + \delta), \]
where $\theta_{\text{max}}$ is the angular amplitude and $\delta$ is the phase angle, which are both determined by the initial conditions, and

$$\omega = \sqrt{\frac{\kappa}{I}} = \frac{2\pi}{T},$$

where $T$ is the periodic time of the oscillation. Therefore,

$$T = 2\pi \sqrt{\frac{I}{\kappa}}.$$

**Question 14.9**: What is the periodic time of torsional pendulum consisting of a uniform solid sphere of mass 400 g and radius 8.0 cm, which is suspended from a torsion wire 10 m long. The wire twists 90° when a torque of 0.04 N·m is applied to it.
First we need to determine the torsion constant.

\[ \tau = -\kappa \theta, \quad \text{i.e.,} \quad \kappa = \frac{\tau}{\theta} = \frac{0.40 \text{ N} \cdot \text{m}}{\pi/2} = 0.25 \text{ N} \cdot \text{m/rod}. \]

The moment of inertia of the sphere is

\[ I = \frac{2}{5} m r^2 = \frac{2}{5} (0.40 \text{ kg})(0.08 \text{ m})^2 = 1.02 \times 10^{-3} \text{ kg} \cdot \text{m}^2. \]

\[ \therefore T = 2\pi \sqrt{\frac{I}{\kappa}} = 2\pi \sqrt{\frac{1.02 \times 10^{-3} \text{ kg} \cdot \text{m}^2}{0.25 \text{ N} \cdot \text{m/rod}}} = 0.40 \text{ s}. \]

Note that it does not depend on the length of the wire.

Question 14.10: A uniform bar is suspended in a horizontal position by a vertical wire attached to its center. When a torque of 5.0 N \cdot m is applied to the bar, the bar rotates through an angle of 12°. When released, the bar oscillates as a torsion pendulum with a period of 0.50 s. Find the moment of inertia of the bar.
For a torsion oscillator $T = 2\pi \sqrt{\frac{I}{\kappa}}$.

\[ \therefore T^2 = (2\pi)^2 \left( \frac{I}{\kappa} \right) \Rightarrow I = \kappa \left( \frac{T}{2\pi} \right)^2. \]

In this case

\[ \kappa = \left| \frac{\tau}{\theta} \right| = \frac{5.0 \text{ N\cdotm}}{(12') \left( \frac{2\pi \text{ rad}}{360'} \right)} = 23.9 \text{ N\cdotm/rad.} \]

\[ \therefore I = (23.9 \text{ N\cdotm/rad}) \left( \frac{0.50 \text{ s}}{2\pi} \right)^2 \]

\[ = 0.151 \text{ kg\cdotm}^2. \]

If we know the length (or mass) of the bar we can find its mass (or length).