CHAPTER 12

STATIC EQUILIBRIUM AND ELASTICITY

• Conditions for static equilibrium
  ◦ Center of gravity
  ◦ Examples of equilibrium

• Couples

• Elasticity
  ◦ Stress and strain
  ◦ Young’s modulus
  ◦ Hooke’s Law and the elastic limit
  ◦ Shear modulus

By definition, an object is in equilibrium when it is either at rest or moving with constant velocity, i.e., with no acceleration. The following are examples of objects in static equilibrium ...
Therefore, two conditions are necessary for a body to be in static equilibrium ...

- The net external force acting on the body is zero,
  i.e., \( \sum F_i = 0 \),
  so there is no translational motion

- The net external torque about any point is zero,
  i.e., \( \sum \tau_i = 0 \),
  so there is no rotational motion

Let’s do a problem as an illustration ...

**Question 12.1:** A see-saw consists of a board of length 4.0 m pivoted at its center. A 28 kg child sits at one end of the board. Where should a 40 kg child sit to balance the seesaw, i.e., have the see-saw in static equilibrium?
First, identify all the forces ...

The conditions for static and rotational equilibrium:

$$\sum_i F_{yi} = 0 \quad \text{and} \quad \sum_i \tau_i = 0.$$ 

Therefore, the pivot must supply an upward force so that the net force on the board is zero, i.e.,

$$F - (28 \text{ kg})g - (40 \text{ kg})g = 0 \quad \therefore F = (68 \text{ kg})g = 666.4 \text{ N}.$$ 

Define ccw torques as positive and taking torques about the pivot point we have:

$$(28 \text{ kg})g \times (2 \text{ m}) - (40 \text{ kg})g \times d = 0 \quad \therefore d = \frac{56 \text{ kg} \cdot \text{m}}{40 \text{ kg}} = 1.40 \text{ m}.$$ 

So, for equilibrium Cathy must stand 1.40 m from the pivot point.

**Question 12.2**: At airports you see people running around pulling their luggage on small trolleys. Sometimes they arrange their cases as shown on the left; sometimes as shown on the right.

(a) If $F_A$ and $F_B$ are the forces necessary to hold the carts in the two cases, which force is the smaller or are the equal?

(b) What difference, if anything, does increasing or decreasing the angle $\theta$ make?
(a) Take torques about the contact point with the ground. At equilibrium \( \sum_i \tau_i = 0 \), i.e.,

\[
F_3 x_3 - Mg x_1 - mg x_2 = 0.
\]

\[
\therefore F = \frac{(Mg x_1 + mg x_2)}{x_3}.
\]

\( M, m, x_1 \) and \( x_3 \) are the same on the left and on the right, but \( x_2 \) (left) > \( x_2 \) (right).

\[
\therefore F_A > F_B,
\]

so \( F_B \) is the smaller force.

(b) Note, \( x_i = d_i \cos \theta \). Substituting the equation for \( F \), \( \cos \theta \) cancels out. So, changing \( \theta \) makes no difference.

We assumed the weight force acts through the center of mass ... strictly speaking it should be the center of weight. What’s the connection between the two?

Take some origin, \( O \), and “break” the object into small elements of mass \( m_i \). The total torque about \( O \) is:

\[
\tau_o = \sum_i m_i g_i x_i = W x_{CG},
\]

where \( x_{CG} \) is the center of gravity (weight) defined as

\[
x_{CG} = \frac{\sum_i w_i x_i}{\sum_i w_i} = \frac{\sum_i w_i x_i}{W}.
\]

(This is similar to the definition of the center of mass.) The center of mass is at the same point as the center of weight, i.e., \( x_{CM} = x_{CG} \), only if \( g \) is constant, then

\[
\tau_o = (\sum_i m_i x_i) g = Mg x_{CM} = W x_{CM}.
\]
If the origin is taken at the center of gravity, then the object produces zero torque about that point.

The center of gravity is then ... the point about which the gravitational force on the object produces zero torque no matter what the orientation of the object. It could also be called ... the center of weight!

If g is constant over the object then the center of gravity and the center of mass occur at the same point.

Discussion problem 12.1: Why is it you cannot touch your toes without falling over if you have your heels against a wall as in (a) ... yet, ordinarily, you have no problem, as in (b)?

If you don’t believe me ... try it!
**Question 12.3:** A sign hangs in front of a store. The sign has a mass of 20 kg and it hangs at the end of a horizontal rod of length 2 m and mass 4 kg, which is hinged at the wall. The rod is supported by a wire attached to a point on the wall 1 m above the rod.

(a) What is the tension in the wire?
(b) What is the magnitude and direction of the force the rod exerts on the wall?

(a) The wall must supply a force on the rod ... *how do we know that?* However, we have no idea in what direction so choose it arbitrarily to begin with. Note, by Newton’s 3rd Law the force the wall exerts on the rod is equal and opposite to the force the rod exerts on the wall. For static and rotational equilibrium:

\[ \sum_i F_{xi} = 0, \sum_i F_{yi} = 0 \quad \text{and} \quad \sum_i \vec{r}_i = 0. \]

Take torques about the hinge ... *a great idea since we don’t know anything about $\vec{F}$. Then*

\[ T_y \times (2 \text{ m}) - (4 \text{ kg})g \times (1 \text{ m}) - (20 \text{ kg})g \times (2 \text{ m}) = 0 \]

\[ \therefore T_y = \frac{(44 \text{ kg} \cdot \text{m})g}{(2 \text{ m})} = 215.8 \text{ N.} \]
But \( \frac{|T_y|}{|T_x|} = \tan \theta = \frac{(1 \text{ m})}{(2 \text{ m})} = \frac{1}{2} \). 
\[ |T_x| = 2|T_y|, \text{ i.e., } T_x = -431.6 \text{ N}. \]
\[ \begin{align*}
T &= \sqrt{\left|T_x\right|^2 + \left|T_y\right|^2} \\
&= \sqrt{(-431.6 \text{ N})^2 + (215.8 \text{ N})^2} \\
&= 482.5 \text{ N}.
\end{align*} \]

Note ... we found \( \vec{T} \) even though we know nothing about \( \vec{F} \)! Of course, the wire must be strong enough to withstand a force of 482.5 N.

(b) For static and rotational equilibrium:
\[ \sum_i F_{xi} = 0, \sum_i F_{yi} = 0 \text{ and } \sum_i \vec{\tau}_i = 0. \]
So \[ \sum_i F_{xi} = F_x + T_x = F_x - 431.6 \text{ N} = 0 \]
\[ \therefore F_x = 431.6 \text{ N}, \]
and \[ \sum_i F_{yi} = F_y + T_y - (20 \text{ kg})g - (4 \text{ kg})g = 0 \]
\[ \therefore F_y = (24 \text{ kg})g -(215.8 \text{ N}) = 19.6 \text{ N}. \]
\[ F_y = (431.6 \hat{i} + 19.6 \hat{j}) \text{ N} \]
\[ \therefore \phi = \tan^{-1} \left( \frac{F_y}{F_x} \right) = 2.6^\circ \]
Therefore, the force of the rod on the wall is \[ \vec{F} = -(431.6 \hat{i} + 19.6 \hat{j}) \text{ N}. \]
Note: we could find \( \mathbf{F} \) another way ... (a) by taking torques about the right hand end of the rod, since the system is in static equilibrium, \( \sum_i \tau_i = 0 \) about any point, i.e., \( (4 \text{ kg})g \times (1 \text{ m}) - F_y \times (2 \text{ m}) = 0 \).

\[
\therefore F_y = \frac{(4 \text{ kg} \cdot \text{ m})g}{(2 \text{ m})} = 19.6 \text{ N}.
\]

(b) By taking torques about the top end of the wire, i.e., \( F_x \times (1 \text{ m}) - (4 \text{ kg})g \times (1 \text{ m}) - (20 \text{ kg})g \times (2 \text{ m}) = 0 \).

\[
\therefore F_x = \frac{(44 \text{ kg} \cdot \text{ m})g}{(1 \text{ m})} = 431.6 \text{ N}.
\]

**Question 12.4:** A square plate is made by welding together four smaller square plates, each of side \( \ell \). Each of the four squares is made from a different material so they have different weights, as shown in the figure.

(a) Find the position of the center of gravity, relative to the point \((0,0)\).

(b) If the plate is suspended from the point P, what is the angle between the vertical and the left-hand side of the plate?
(a) By definition, the center of gravity (weight), with respect to \((0,0)\) is given by

\[
x_{CG} = \frac{\sum w_i x_i}{W} = \left(60 \, \text{N} \right) \times \frac{\ell}{2} + \left(50 \, \text{N} \right) \times \frac{\ell}{2} + \left(20 \, \text{N} \right) \times \frac{3 \ell}{2} + \left(30 \, \text{N} \right) \times \frac{3 \ell}{2}
\]

\[
= \frac{60 \, \text{N} \times \frac{\ell}{2} + 50 \, \text{N} \times \frac{\ell}{2} + 20 \, \text{N} \times \frac{3 \ell}{2} + 30 \, \text{N} \times \frac{3 \ell}{2}}{60 \, \text{N} + 50 \, \text{N} + 20 \, \text{N} + 30 \, \text{N}}
\]

\[
= 0.8125 \ell,
\]

and \(y_{CG} = \frac{\sum w_i y_i}{W}\)

\[
= \left(60 \, \text{N} \right) \times \frac{3 \ell}{2} + \left(50 \, \text{N} \right) \times \frac{\ell}{2} + \left(20 \, \text{N} \right) \times \frac{3 \ell}{2} + \left(30 \, \text{N} \right) \times \frac{\ell}{2}
\]

\[
= \frac{60 \, \text{N} \times \frac{3 \ell}{2} + 50 \, \text{N} \times \frac{\ell}{2} + 20 \, \text{N} \times \frac{3 \ell}{2} + 30 \, \text{N} \times \frac{\ell}{2}}{60 \, \text{N} + 50 \, \text{N} + 20 \, \text{N} + 30 \, \text{N}}
\]

\[
= 1.000 \ell.
\]

Therefore, the coordinates of the CG relative to \((0,0)\) are \((0.8125\ell, \ell)\).

(b) When hung from P, a vertical line through P passes through the CG.

\[
\therefore \tan \phi = \frac{0.8125\ell}{\ell},
\]

i.e., \(\phi = 39.1^\circ\).
**Question 12.5**: A uniform ladder rests against a frictionless, vertical wall. If the coefficient of static friction between the bottom of the ladder and the floor is 0.3, what is the smallest angle at which the ladder can rest against the wall without slipping?

Let the ladder, length \( \ell \), rest at an angle \( \theta \) against the wall. Identify all of the force acting on the ladder.

Since the wall is frictionless, the force of the wall acting on the ladder is \( \vec{F}_1 \), which is \( \perp \) to the wall, i.e., a normal force. The ground exerts two forces on the ladder; a normal force \( \vec{F}_n \) and a static frictional force \( \vec{f}_s \). The weight of the ladder is \( \vec{w} \), which acts through the CG, i.e., the middle of the ladder.

For static and rotational equilibrium:

\[
\sum_i F_{xi} = 0, \quad \sum_i F_{yi} = 0 \quad \text{and} \quad \sum_i \vec{\tau}_i = 0.
\]

\[
\therefore \sum_i F_{xi} = f_s - F_1 = 0 \quad \ldots \quad \ldots \quad (1)
\]

and

\[
\sum_i F_{yi} = F_n - w = 0 \quad \ldots \quad \ldots \quad (2)
\]
Taking torques about the foot of the ladder:

$$\sum \tau_i = -w\left(\frac{\ell}{2}\cos\theta\right) + F_1(\ell \sin \theta) = 0 \quad ... (3)$$

$$\therefore \tan \theta = \frac{w}{2F_1}.$$ 

But from (1) and (2):

$$F_1 = f_s = \mu_s F_n = \mu_s w.$$ 

$$\therefore \theta = \tan^{-1}\left(\frac{1}{2\mu_s}\right) = 59.0^\circ.$$ 

**Question 12.6**: In the previous problem there was no friction at the wall but there was static friction between the ground and the foot of the ladder. If, instead, there was friction at the wall but no friction at the ground, what difference would it make?
Again, identify all the forces acting on the ladder.

Summing the vertical and horizontal forces we get

\[ \sum F_y = F_n - w + f_s \ldots \]  \hspace{1cm} (1)

and

\[ \sum F_x = -F_1 \ldots \ldots \ldots \]  \hspace{1cm} (2).

Now eq (1) \textit{can} be zero (if \( f_s = w - F_n \)). However, eq (2) \textit{can never} be zero if the ladder is leaning against the wall as it consists of only one term (\( F_1 \)). Consequently, the ladder can never be in equilibrium, i.e., if there’s no friction with the ground it will \textit{always} slip!

We ignore the arrangement when the ladder is vertical!

\textbf{Question 12.7}: A box X, of mass 10 kg, is placed on a box Y, of mass 5 kg, so that the center of box X lies directly over the \textit{left hand edge} of box Y. If the two boxes are then placed over the edge of a table, what is the maximum possible overhang, \( h \), without the boxes falling, if the sides of the boxes is 0.50 m?
For ‘balance’ the CG of the two boxes combined must lie directly over the edge of the table, i.e., the ‘pivot’ point.

Let the length of the side of the boxes be \( L \), and find the CG with respect to the center line of the upper box (X), which is the same as the left hand edge of lower box (Y):

\[
x_{CG} = \frac{\sum m_i g x_i}{\sum m_i g} = \frac{(10 \text{ kg})g \times 0 + (5 \text{ kg})g \times \frac{L}{2}}{(15 \text{ kg})g} = \frac{L}{6}.
\]

Therefore, total overhang is,

\[
h = \frac{L}{6} + \frac{L}{2} = \frac{2L}{3} = 0.33 \text{ m}.
\]

**OR**

... take torques about O, above the edge of the table.

At equilibrium, \( \Sigma \tau_i = 0 \).

\[
(10 \text{ kg})g \times \left( h - \frac{L}{2} \right) - (5 \text{ kg})g \times (L - h) = 0,
\]

i.e., \( 10h - 5L - 5L + 5h = 0 \).

\[
\therefore h = \frac{10L}{15} = \frac{2(0.5 \text{ m})}{3} = 0.33 \text{ m}.
\]
**Question 12.8:** The figure shows a cart of mass 100 kg loaded with six identical packing cases, each of mass 50 kg. How is the total weight of the cart and packing cases distributed between the left hand and the right hand wheels?

At equilibrium: $\sum F_y = 0$ and $\sum \tau = 0$.

$\therefore (F_L + F_R) - (50 \text{ kg})g - (100 \text{ kg})g$

$- (150 \text{ kg})g - (100 \text{ kg})g = 0$

i.e., $\therefore (F_L + F_R) = 3924 \text{ N}$.

Taking torques about the axle of the right hand wheel (ccw is positive):

$(50 \text{ kg})g \times (2 \text{ m}) + (100 \text{ kg})g \times (1 \text{ m})$

$+ (100 \text{ kg})g \times (1 \text{ m}) - F_L \times (2 \text{ m}) = 0$.

$\therefore F_L = \frac{(300 \text{ kg} \cdot \text{m})g}{(2 \text{ m})} = 1471.5 \text{ N}$

and $F_R = 3924 \text{ N} - 1471.5 \text{ N} = 2452.5 \text{ N}$.
A couple

Two forces form a couple if they are equal but opposite and their line of action is separated (by a distance $\ell$):

i.e., $|\vec{F}_1| = |\vec{F}_2| = F$.

Then the torque about any point $O$ is:

$$\vec{\tau} = (\vec{x}_1 \times \vec{F}_1) + (\vec{x}_2 \times \vec{F}_2)$$

i.e., $|\vec{\tau}| = x_1 F - x_2 F = (x_1 - x_2)F = \ell F$

So the torque produced by a couple is the same about all points in space, i.e., no matter where $O$ is located ... it depends only on their perpendicular separation.

Stress and strain (Young’s modulus)

Objects deform when subjected to a force. In the case of the length of a metal bar or wire ...

The stress $\Rightarrow \frac{F}{A}$, and the strain $\Rightarrow \frac{\Delta L}{L}$.

Young’s modulus $\Rightarrow Y = \frac{\text{stress}}{\text{strain}} = \frac{(\frac{F}{A})}{(\frac{\Delta L}{L})}$

Units: Force/area $\Rightarrow N \cdot m^{-2}$ (scalar)

- Thomas Young (1773-1829)

The example above is tensile stress. There is also compressive stress. Young’s modulus is often the same in both cases, but there are exceptions (e.g., bone and concrete).
However, the strain is reversible over only a limited range of stress ... up to the elastic limit. Beyond that the material shows plastic behavior until it reaches the point of fracture (tensile strength).

The initial linear behavior is known as Hooke’s Law ... discovered by Robert Hooke (1635-1703).

Discussion problem 12.2:

An aluminum wire and a steel wire with the same lengths (L) and diameters (d), are joined to form a wire of length 2L. One end of the wire is fixed to the ceiling and the other is attached to a weight (w). Neglecting the weight of the wires, which one of the following statements is true?

A: The strains in the two wires are the same.
B: The stresses in the two wires are the same.
C: The stress in the aluminum wire is greater than the stress in the steel wire.
D: The stress in the aluminum wire is less than the stress in the steel wire.

\[ Y_{\text{steel}} = 200 \times 10^9 \text{ N} \cdot \text{m}^{-2} \text{ and } Y_{\text{Al}} = 70 \times 10^9 \text{ N} \cdot \text{m}^{-2}. \]
**Question 12.9:** A steel wire of length 1.50 m and diameter 1.00 mm is joined to an aluminum wire of identical dimensions to make a composite wire of length 3.00 m.

(a) What is the length of the composite wire if it is used to support a mass of 5 kg?

(b) What is the maximum load the composite wire could withstand? Assume the weights of the wires are negligible.

\( \text{Aluminum: } Y_{\text{Al}} = 70 \times 10^9 \text{ N} \cdot \text{m}^{-2}. \)

Tensile strength = \( 90 \times 10^6 \text{ N} \cdot \text{m}^{-2}. \)

\( \text{Steel: } Y_{\text{steel}} = 200 \times 10^9 \text{ N} \cdot \text{m}^{-2}. \)

Tensile strength = \( 520 \times 10^6 \text{ N} \cdot \text{m}^{-2}. \)

(a) The Young’s modulus is \( Y = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{(\Delta L/L)}. \)

The stress in each wire is:

\[ F/A = \frac{(5 \text{ kg})g}{\pi \times (0.5 \times 10^{-3} \text{ m})^2} = 6.25 \times 10^7 \text{ N} \cdot \text{m}^{-2}. \]

But \( \Delta L = (L/Y)(F/A). \)

\[ \therefore \Delta L = (6.25 \times 10^7 \text{ N} \cdot \text{m}^{-2})L/Y. \]

- **Aluminum:**

\[ \Delta L_1 = (6.25 \times 10^7 \text{ N} \cdot \text{m}^{-2}) \frac{(1.50 \text{ m})}{(70 \times 10^9 \text{ N} \cdot \text{m}^{-2})} = 1.34 \times 10^3 \text{ m} \Rightarrow 1.34 \text{ mm}. \]

- **Steel:**

\[ \Delta L_2 = (6.25 \times 10^7 \text{ N} \cdot \text{m}^{-2}) \frac{(1.50 \text{ m})}{(200 \times 10^9 \text{ N} \cdot \text{m}^{-2})} = 0.47 \times 10^3 \text{ m} \Rightarrow 0.47 \text{ mm}. \]

Therefore, the increase in length is 0.00181 m, so the new length is 3.00181 m.
(b) The tensile strength of aluminum is less than that of steel. Consequently, as the load increases, it will be the aluminum wire that fails.

The maximum stress is \( \frac{F}{A} = \frac{Mg}{A} = 90 \times 10^6 \text{ N} \cdot \text{m}^{-2} \).

\[
\therefore M = (90 \times 10^6 \text{ N} \cdot \text{m}^{-2}) \frac{A}{g} = (90 \times 10^6 \text{ N} \cdot \text{m}^{-2}) \frac{\pi (0.5 \times 10^{-3} \text{ m})^2}{9.81 \text{ m/s}^2} = 7.21 \text{ kg}.
\]

**Shear modulus**

The shear strain is \( \frac{\Delta x}{L} = \tan \theta \)

The stress is \( \frac{F}{A} \) (note which area!)

**Shear modulus** \( \Rightarrow M_s = \frac{\text{shear stress}}{\text{shear strain}} = \frac{(\frac{F}{A})}{(\frac{\Delta x}{L})} = \frac{(\frac{F}{A})}{\tan \theta} \).

**Units:** Force/area \( \Rightarrow \text{N} \cdot \text{m}^{-2} \) (scalar).