CHAPTER 10

CONSERVATION OF ANGULAR MOMENTUM

- Vector nature of rotation
  - the cross product
  - the right hand rule

- Angular momentum

- Conservation of angular momentum
  - examples

Vector nature of rotation

Because rotation can be clockwise or counter clockwise, angular velocity and angular acceleration are actually vector quantities, with both direction and magnitude. The directions are given by the right-hand-rule.
\[ \omega = \frac{d\theta}{dt} \quad \text{and} \quad v = r\omega \]

\[ \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \quad \text{and} \quad a_t = r\alpha \]

Torque is also a vector. Expressed mathematically, the torque is defined as

\[ \tau = \vec{r} \times \vec{F}, \]

which is a vector (or cross) product. The direction of \( \tau \) is given by the right-hand-rule and the magnitude of \( \tau \) is given by

\[ |\tau| = |\vec{r}| |\vec{F}| \sin \phi = \ell |\vec{F}|. \]

Thus, in the case shown above, the direction of \( \tau \) is out of the page.

*Note: the torque depends on the choice of rotation axis and the direction of the force.*
We have already seen that the **scalar** or **dot product** of two vectors is defined as:

\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \]

If \( \vec{A} \Rightarrow (A_x, A_y, A_z) \) and \( \vec{B} \Rightarrow (B_x, B_y, B_z) \), then

\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z. \]

*~ see revision notes on website ~*

**Example:**

Work done (W) = \( \vec{F} \cdot \Delta \vec{x} \)

However, the cross (vector) product is very different ... for example: the result is a **vector**.

The **vector** or **cross product** of two vectors is defined as:

\[ \vec{C} = \vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta \hat{n} \]

where \( \hat{n} \) is a unit vector that is perpendicular to both \( \vec{A} \) and \( \vec{B} \) and in a direction given by the right-hand-rule.

*~ See web-site ~*
The Right hand rule for the cross product

\[ \vec{C} = \vec{A} \times \vec{B} \]

**Question 10.1:** Which of the following does the vector \( \vec{C} \) in the figure represent?

(a) \( \vec{A} \times \vec{B} \),
(b) \( \vec{A} \cdot \vec{B} \),
(c) \( \vec{B} \times \vec{A} \), or
(d) \( \vec{B} \cdot \vec{A} \).
Vector \( \vec{C} \) cannot be the result of a dot product, as the result of a dot product is a scalar. So (b) and (d) are eliminated. Using the right hand rule, \( \vec{C} \) is the result of \( \vec{B} \times \vec{A} \).

So, the answer is (c).

Since the unit vectors \( \hat{i}, \hat{j}, \hat{k} \) are orthogonal we have:

\[
\begin{align*}
\hat{i} \times \hat{j} &= \hat{k} & \hat{j} \times \hat{i} &= -\hat{k} \\
\hat{j} \times \hat{k} &= \hat{i} & \hat{k} \times \hat{j} &= -\hat{i} \\
\hat{k} \times \hat{i} &= \hat{j} & \hat{i} \times \hat{k} &= -\hat{j}
\end{align*}
\]

and

\[
\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0,
\]

~ Here’s an easy way to remember it ~


Question 10.2: If \( \vec{A} = 2\hat{i} + 3\hat{j} \) and \( \vec{B} = -4\hat{i} + \hat{j} \),

(a) sketch the two vectors.
(b) What is \( \vec{A} \times \vec{B} \), and
(c) what is the angle between \( \vec{A} \) and \( \vec{B} \)?

(b) \( \vec{C} = \vec{A} \times \vec{B} = (2\hat{i} + 3\hat{j}) \times (-4\hat{i} + \hat{j}) \)
\[ = -8(\hat{i} \times \hat{i}) + 2(\hat{i} \times \hat{j}) - 12(\hat{j} \times \hat{i}) + 3(\hat{j} \times \hat{j}) \]
\[ = 2\hat{k} - 12(-\hat{k}) = 14\hat{k}. \]

(c) Also, \( \vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta \hat{k} \),

but \( |\vec{A}| = \sqrt{2^2 + 3^2} = \sqrt{13} \)
and \( |\vec{B}| = \sqrt{(-4)^2 + 1^2} = \sqrt{17}. \)
So \( \vec{A} \times \vec{B} = (\sqrt{13} \times \sqrt{17} \sin \theta)\hat{k} = 14\hat{k}, \)

i.e., \( \theta = \sin^{-1}\left(\frac{14}{\sqrt{13} \times \sqrt{17}}\right) = \sin^{-1}(0.941). \)

The solution \( \theta = 70.3^\circ \) is clearly not appropriate, but we remember that \( \sin \alpha = \sin(180 - \alpha). \)

\[ \therefore \theta = 180^\circ - 70.3^\circ = 109.7^\circ. \]
The angular momentum about O is defined as:
\[ \mathbf{L} = \mathbf{r} \times \mathbf{p}, \]
where \( \mathbf{p} = m \mathbf{v} \) is the linear momentum of the object.
The magnitude of the angular momentum is;
\[ L = mr v \sin \phi = pr \sin \phi, \]
and the direction of \( \mathbf{L} \) is given by the right-hand-rule and is perpendicular to \( \mathbf{r} \) and \( \mathbf{v} \).

Note: a particle does not have to be rotating to have angular momentum.

**Dimension:** \( [M][L]^2/[T] \)

**Units:** kg \( \cdot \) m\(^2\)/s (or J.s)

**Question 10.3:** An object of mass 3 kg is moving with a velocity \( \mathbf{v} = (3.0 \text{ m/s}) \hat{i} \) along the line \( z = 0, \ y = 5.3 \text{ m} \).
What is the angular momentum of the object relative to the origin

(a) when it is at the point \( x = 0, \ y = 5.3 \text{ m} \)?
(b) When it is at the point \( x = 5.3 \text{ m}, \ y = 5.3 \text{ m} \)?
(c) When it is at the point \( x = 12 \text{ m}, \ y = 5.3 \text{ m} \)?
(d) If a force \( \mathbf{F} = (-3 \text{ N}) \hat{i} \) is applied to the object at the point \( x = 12 \text{ m}, \ y = 5.3 \text{ m} \), what is the torque of this force relative to the origin?
(a) Angular momentum about O is \( \mathbf{L}_1 = \mathbf{r}_1 \times \mathbf{p} \),
where the position vector is \( \mathbf{r}_1 = (5.3 \text{ m})\hat{j} \) and the
momentum is \( \mathbf{p} = m\mathbf{v} = (9 \text{ kg} \cdot \text{m/s})\hat{i} \).
∴ \( \mathbf{L}_1 = (5.3 \text{ m})\hat{j} \times (9 \text{ kg} \cdot \text{m/s})\hat{i} = (-47.7 \text{ kg} \cdot \text{m}^2/\text{s})\hat{k} \).
\( \text{(Note the direction)} \)

(b) For the position vector \( \mathbf{r}_2 = (5.3\hat{i} + 5.3\hat{j}) \text{ m} \),
\( \mathbf{L}_2 = \mathbf{r}_2 \times \mathbf{p} = (5.3\hat{i} + 5.3\hat{j})m \times (9 \text{ kg} \cdot \text{m/s})\hat{i} \)
\hphantom{\( \mathbf{L}_2 = \)} = (-47.7 \text{ kg} \cdot \text{m}^2/\text{s})\hat{k} \),
i.e., the same ... very interesting!

(c) For the position vector \( \mathbf{r}_3 = (12\hat{i} + 5.3\hat{j}) \text{ m} \),
\( \mathbf{L}_3 = \mathbf{r}_3 \times \mathbf{p} = (12\hat{i} + 5.3\hat{j})m \times (9 \text{ kg} \cdot \text{m/s})\hat{i} \)
\hphantom{\( \mathbf{L}_3 = \)} = (-47.7 \text{ kg} \cdot \text{m}^2/\text{s})\hat{k} \),
i.e., the same ... even more interesting!

(d) The torque about O is: \( \mathbf{\tau} = \mathbf{r} \times \mathbf{F} \)
\hphantom{\( \mathbf{L} = \)} = (12\hat{i} + 5.3\hat{j})m \times (-3\hat{i})N = (+15.9 \text{ N} \cdot \text{m})\hat{k} \).
\( \text{(Note the direction)} \)
The angular momentum of the object about the origin is
\[ \vec{L}_i = \vec{r}_i \times \vec{p} = r_i p \sin \theta_i \, \hat{k} \quad \text{(inward)} \]
Put \( \vec{r}_i = x_i \hat{i} + y_j \) and \( \vec{p} = \vec{p} \hat{i} \). Then
\[ \vec{L}_i = \vec{r}_i \times \vec{p} = (x_i \hat{i} + y_j) \times \vec{p} = y_j \times \vec{p} \hat{i} \]
\[ = -py \, \hat{k}. \]
Providing \( \vec{p} \) is unchanged, then \( \vec{L}_i \) is constant.

So, the angular momentum about the origin is constant provided there are no external forces to change \( \vec{p} \), i.e., no external torques.

\[ \sim \text{Rotation analog of the conservation of linear momentum} \sim \]

Consider a planar object rotating about O. If \( m_i \) is a small mass element, its angular momentum about the rotation axis is:
\[ \vec{L}_i = \vec{r}_i \times (m_i \vec{v}_i) = m_i r_i \vec{v}_i \, \hat{k} \]
But \( \vec{v}_i = r_i \omega \).
\[ \therefore \vec{L}_i = (m_i r_i^2) \omega \, \hat{k}. \]
So, the total angular momentum is
\[ \vec{L} = \sum_i \vec{L}_i = \sum_i (m_i r_i^2) \omega \, \hat{k} = I \omega \, \hat{k} \]
But \( \omega \, \hat{k} = \vec{\omega} \).
\[ \therefore \vec{L} = I \vec{\omega}, \]
i.e., in the direction of \( \vec{\omega} \). \textit{cf: linear momentum} \( \vec{p} \) is parallel to \( \vec{v} \).
... although a *centripetal force* acts on the object, it is directed through the rotation axis; it produces zero torque and so there’s no change in angular momentum!

If there is a net external torque acting on an object then, from Newton’s 2nd Law of rotation, we find

\[ \tau = I \alpha = I \frac{d\omega}{dt} = \frac{d(I\omega)}{dt} = \frac{dL}{dt}, \]

i.e., the rate of change of angular momentum of the object is equal to the net external torque. (Compare with Newton’s 2nd Law \( \vec{F} = \frac{d\vec{p}}{dt} \).) Note: if there is no net external torque, i.e., \( \tau \Rightarrow 0 \), then \( \frac{dL}{dt} = 0 \), as before.

If a torque is applied for time \( \Delta t \), then the impulse of the torque is

\[ \tau \Delta t = \Delta \vec{L}. \quad (\text{Units of } L \Rightarrow J \cdot s) \]

*Remember:* \( \vec{F} \Delta t = \Delta \vec{p} \), the force impulse equation?

*Let’s look some situations involving angular momentum ...*
DISCUSSION PROBLEM [10.1]:

A rotating cloud of gas begins to collapse under gravitational forces within. As it collapses, what happens to its:

• Moment of inertia?
• Angular momentum?
• Angular velocity?

DISCUSSION PROBLEM [10.2]:

A gymnast spinning freely in mid-air cannot change their ...

A: angular momentum,
B: moment of inertia,
C: rotational kinetic energy,
D: angular velocity.
**Question 10.4:** An ice skater starts her spin with arms outstretched, rotating at 1.5 rev/s. Estimate her rotational speed (in rev/s) when she brings her arms back against her body.

This is a conservation of angular momentum problem ...

\[ I_1 \omega_1 = I_2 \omega_2. \]

... but we need to make some assumptions ...

- *Her body is a cylinder* \((r \approx 0.2 \text{ m}, M_b \approx 50 \text{ kg}).*
- *Each arm is a thin rod* \((\ell \approx 1 \text{ m}, M_a \approx 5 \text{ kg}).*

With arms “out” \( I_1 \Rightarrow \frac{1}{2} M_b R_1^2 + 2 \left( \frac{1}{3} M_a \ell^2 \right) \)

\[ = \frac{1}{2} (50 \text{ kg})(0.2 \text{ m})^2 + 2 \times \frac{1}{3} (5 \text{ kg})(1 \text{ m})^2 = 4.33 \text{ kg} \cdot \text{m}^2. \]

With arms “in” \( I_2 \Rightarrow \frac{1}{2} (M_b + 2M_a) R_2^2 \)

\[ = \frac{1}{2} (50 \text{ kg} + 10 \text{ kg})(0.22 \text{ m})^2 = 1.45 \text{ kg} \cdot \text{m}^2. \]

\[ \therefore \omega_2 = \frac{I_1 \omega_1}{I_2} = \frac{(4.33 \text{ kg} \cdot \text{m}^2)(1.5 \text{ rev/s})}{1.45 \text{ kg} \cdot \text{m}^2} = 4.5 \text{ rev/s}. \]
**Question 10.5**: Two disks, with identical masses but different radii (r and 2r) are spinning on frictionless bearings with the same angular speed $\omega_0$ but in opposite directions. The two disks are brought slowly together and the resulting frictional force between their surfaces eventually brings them to a common angular velocity.

(a) What is the magnitude of the final angular speed in terms of $\omega_0$?

(b) What is the change in the rotational kinetic energy?

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(a) The angular momenta are $L_1 = I_1\omega_0$ and $L_2 = I_2\omega_0$, but they are in opposite directions. Since there are no external torques angular momentum is conserved. The initial angular momentum is

$$L_1 + (-L_2) = I_1\omega_0 - I_2\omega_0 = (I_1 - I_2)\omega_0,$$

and the final angular momentum is

$$(I_1 + I_2)\omega.$$

∴ $$(I_1 + I_2)\omega = (I_1 - I_2)\omega_0.$$  

But $I_1 = \frac{1}{2}M(2r)^2 = 2Mr^2$ and $I_2 = \frac{1}{2}Mr^2$.

∴ $$(2 + \frac{1}{2})Mr^2\omega = \left(2 - \frac{1}{2}\right)Mr^2\omega_0,$$

i.e., $\frac{5}{2}Mr^2\omega = \frac{3}{2}Mr^2\omega_0$

∴ $\omega = \frac{3}{5}\omega_0.$
The total initial rotational kinetic energy:
\[ K_i = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} (I_1 + I_2) \omega_o^2 \]

The total final rotational kinetic energy:
\[ K_f = \frac{1}{2} (I_1 + I_2) \omega^2. \]

\[ \therefore \Delta K = \frac{K_f - K_i}{K_i} = \frac{\omega^2 - \omega_o^2}{\omega_o^2}, \]

but \( \omega = \frac{3}{5} \omega_o \), so \( (\omega^2 - \omega_o^2) < 0 \), i.e., \( \Delta K \) is always < 0.

\[ \therefore \Delta K = -0.64, \]

i.e., 64% of the kinetic energy is “lost”.

- What’s happened to it?

Note: this is the analog of the inelastic collision we studied in chapter 8.

**Question 10.6**: If all the ice at the North and South Poles melted and the resulting water was uniformly distributed over the Earth’s surface, would the length of the day change?
When the ice is concentrated at the poles it contributes very little to the moment of inertia.
\[ I_1 = \frac{2}{5} MR_1^2. \]

After it melts and the water is redistributed so
\[ I_2 = \frac{2}{5} MR_2^2. \]

Since \( R_2 > R_1 \) then \( I_2 > I_1 \).

There are no external torques so angular momentum is conserved,
\[ \text{i.e., } I_1\omega_1 = I_2\omega_2. \]
\[ \therefore \omega_2 < \omega_1, \]
which means the Earth rotates more slowly. As a result the length of the day is increased!

**Question 10.7**: (a) Assuming the Earth is a uniform sphere of radius \( R \) and mass \( M \), show that the period of rotation about its N-S axis is
\[ T = \left( \frac{4\pi M}{5L} \right) R^2, \]
where \( L \) is the angular momentum of the Earth about the N-S axis.

(b) Suppose the radius changes by a small amount from \( R \) to \( R + \Delta R \). Show that the fractional change in the period of rotation is
\[ \frac{\Delta T}{T} = 2 \frac{\Delta R}{R}. \]

(c) By how many kilometers would the radius of the Earth have to increase for the period to change by 0.25 day/year so that leap years would no longer be necessary? The radius of the Earth is \( 6.37 \times 10^3 \) km.
(a) If the angular velocity is $\omega$,
- the angular momentum is $L = I\omega$,
- the moment of inertia is $I = \frac{2}{5}MR^2$.
- The period of rotation is $T = \frac{2\pi}{\omega} = \frac{2\pi LR}{I} = \frac{2\pi}{I} I = (\frac{4\pi M}{5L}) R^2$. QED.

(b) Let $R \Rightarrow R + \Delta R$ so $T \Rightarrow T + \Delta T$, then
\[
\Delta T = \frac{dT}{dR} = 2\left(\frac{4\pi M}{5L}\right) R = 2\left(\frac{T}{R^2}\right) R.
\]
\[\therefore \frac{\Delta T}{T} \approx 2 \frac{\Delta R}{R}.\]

(c) $\Delta R \approx \frac{R}{2} \frac{\Delta T}{T}$ and we need $\frac{\Delta T}{T} = \frac{0.25 \text{ day}}{1 \text{ year}} = \frac{0.25 \text{ day}}{365 \text{ day}}$.
\[\therefore \Delta R = \frac{(6.37 \times 10^3 \text{ km}) \times (0.25 \text{ day})}{2 \times (365 \text{ day})} = 2.18 \text{ km}.\]

**Question 10.8:** You have probably seen this demonstration in which a person, standing on a platform that is free to rotate, is handed a wheel that is spinning in the horizontal plane.

When he inverts the wheel (through $180^\circ$) he begins to rotate. Explain what is happening.
Change the scenario to a student sitting on a stool that is free to rotate. Assume the student is stationary. Take the student, the stool and the wheel as the system. Assume the student sees the wheel rotating clockwise. Then, the total initial angular momentum of the system is \(-L\) (down), due solely to the spinning wheel. To invert the wheel the student applies a torque, which is internal to the system. Since there are no external torques, angular momentum is conserved. After the wheel is inverted its angular momentum is now \(+L\) (up). Therefore, the change in the angular momentum of the wheel is

\[
\Delta L = L - (-L) = 2L.
\]

So, how is the total angular momentum conserved? ... by having the student (and stool) rotate so their combined angular momentum is \(-2L\)! So, she and the stool will rotate clockwise.

**Question 10.9:** A uniform rod of length 1.20 m and mass 0.80 kg rests on a horizontal, frictionless surface and is pivoted at its center. A bullet of mass \(m = 20 \text{ g}\) is fired with speed \(v = 400 \text{ m/s}\) perpendicular to the rod and strikes it at a point 0.30 m from one end. If the rod is initially stationary and the pivot is frictionless,

(a) what is the angular speed of the rod if the bullet embeds itself in the rod?

(b) What is the change in kinetic energy at the “collision”?
(a) There are no external forces or torques and so angular momentum is conserved, i.e., \( L_f = L_i \).

The initial angular momentum is due only to the bullet:
\[
L_i = \vec{r} \times \vec{p}_i = r p_i = m v_i r
\]
\[
= (20 \times 10^{-3} \text{ kg})(400 \text{ m/s})(0.30 \text{ m}) = 2.40 \text{ kg} \cdot \text{m}^2/\text{s}.
\]

The angular momentum after the collision is:
\[
L_f = I \omega_f = (I_{\text{bullet}} + I_{\text{rod}}) \omega_f.
\]

But
\[
I = (20 \times 10^{-3} \text{ kg})(0.30 \text{ m})^2 + \frac{1}{12} (0.80\text{ kg})(1.20 \text{ m})^2
\]
\[
= 0.0978 \text{ kg} \cdot \text{m}^2.
\]

\[
\therefore |L_f| = (0.0978 \text{ kg} \cdot \text{m}^2) \omega_f.
\]

Since \( L_f = L_i \),
\[
\therefore L_f = (0.0978 \text{ kg} \cdot \text{m}^2) \omega_f = 2.40 \text{ kg} \cdot \text{m}^2/\text{s},
\]
i.e., \( \omega_f = \frac{2.40 \text{ kg} \cdot \text{m}^2/\text{s}}{0.0978 \text{ kg} \cdot \text{m}^2} = 24.5 \text{ rad/s} \).

(b) The initial kinetic energy is
\[
K_i = \frac{1}{2} m v_i^2
\]
\[
= \frac{1}{2}(20 \times 10^{-3} \text{ kg})(400 \text{ m/s})^2 = 1600 \text{ J}.
\]

The final kinetic energy is
\[
K_f = \frac{1}{2} I \omega_f^2
\]
\[
= \frac{1}{2}(0.0978 \text{ kg} \cdot \text{m}^2)(24.5 \text{ rad/s})^2 = 29.4 \text{ J}.
\]

\[
\therefore \Delta K = K_f - K_i = -1570.6 \text{ J}.
\]
(Note this is the work done by the bullet to get the system rotating.)
**Question 10.10:** A projectile of mass 0.20 kg is traveling at a constant velocity of 2.00 m/s toward a stationary disk of mass 2.50 kg and radius 0.10 m that is free to rotate about its axis, O. Before impact, the projectile is traveling along a line such that it strikes and sticks to point B at the bottom of the disc. Neglecting gravity,

(a) what is the total angular momentum of the disk-projectile system about the axis before impact?

(b) What is the angular speed of the disk-projectile system just after impact?

(c) How much mechanical energy is lost in the collision?

(a) There are no external torques to the disc-projectile system, so angular momentum is conserved in this perfectly inelastic collision. Before impact, the only contribution to the angular momentum is the projectile,

\[ L_i = m_p v_p R \]

\[ = (0.20 \text{ kg})(2.00 \text{ m/s})(0.10 \text{ m}) = 0.04 \text{ kg} \cdot \text{m}^2/\text{s}. \]

(b) After impact, the moment of inertia of the system is

\[ I_f = \frac{1}{2} M_{\text{disk}} R^2 + m_p R^2 = \left( \frac{1}{2} M_{\text{disk}} + m_p \right) R^2 \]

\[ = \left( \frac{1}{2}(2.50 \text{ kg}) + (0.20 \text{ kg}) \right)(0.10 \text{ m})^2 = 0.0145 \text{ kg} \cdot \text{m}^2. \]

So, the final angular momentum is

\[ L_f = I_f \omega_f = (0.0145 \text{ kg} \cdot \text{m}^2)\omega_f. \]
Because angular momentum is conserved \( L_f = L_i \),

i.e., \( (0.0145 \text{ kg} \cdot \text{m}^2)\omega_f = 0.04 \text{ kg} \cdot \text{m}^2/\text{s} \).

\[ \therefore \omega_f = \frac{0.04 \text{ kg} \cdot \text{m}^2/\text{s}}{0.0145 \text{ kg} \cdot \text{m}^2} = 2.76 \text{ rad/s}. \]

(c) The kinetic energy before the collision is

\[ K_i = \frac{1}{2} m_p (v_p)^2 = \frac{1}{2} (0.2 \text{ kg})(2.00 \text{ m/s})^2 = 0.40 \text{ J}. \]

The kinetic energy after the collision is

\[ K_f = \frac{1}{2} I_f (\omega_f)^2 = \frac{1}{2} (0.0145 \text{ kg} \cdot \text{m}^2)(2.76 \text{ rad/s})^2 \]

\[ = 5.52 \times 10^{-2} \text{ J}. \]

\[ \therefore \Delta K = K_f - K_i = -0.345 \text{ J}, \]

and \[ \frac{\Delta K}{K_i} = \frac{-0.345 \text{ J}}{0.40 \text{ J}} = -0.862, \]

i.e., an 86.2% “loss”. 