RIGOROUS COMPUTATION OF (UN)STABLE MANIFOLDS FOR ANALYTIC MAPS AND ANALYTIC SHADOWING OF CONNECTING ORBITS

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Abstract. This work is concerned with high order polynomial approximation of stable and unstable manifolds for analytic discrete time dynamical systems. We develop 'a-posteriori' theorems for these polynomial approximations which allow us to obtain rigorous bounds on the truncation errors via a computer assisted argument. Moreover we represent the truncation error as an analytic function, so that that the derivatives of the truncation error can be bound using classical estimates of complex analysis. As an application of these ideas we combine the approximate manifolds and rigorous bounds with a standard Newton-Kantorovich argument in order to obtain a kind of 'analytic-shadowing' result for connecting orbits between fixed points of discrete time dynamical systems. Examples of the manifold computation are given for invariant manifolds which have dimension between two and ten. Examples of the a-posteriori error bounds and the analytic shadowing argument for connecting orbits are given for dynamical systems in dimension three and six.

1. Introduction. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is real analytic in some neighborhood $N \subset \mathbb{R}^n$ of a hyperbolic fixed point $p \in N$. Then f is a local real analyticomorphism of N. Let $n_s, n_u \in \mathbb{N}$ denote respectively the dimension of the stable and unstable eigenspaces of $Df(p)$, and note that $n_s + n_u = n$. It follows from the stable manifold theorem [24] that there are $\nu_s, \nu_u, > 0$ and analytic chart maps

$$P : B_{\nu_u}(0) \subset \mathbb{R}^{n_u} \to \mathbb{R}^n \qquad \text{and} \qquad Q : B_{\nu_s}(0) \subset R^{n_s} \to \mathbb{R}^n$$

for the local unstable and stable manifolds at p, so that

$$P[B_{\nu_u}(0)] = W^u_{\text{loc}}(p) \qquad \text{and} \qquad Q[B_{\nu_s}(0)] = W^s_{\text{loc}}(p).$$

The *Parameterization Method*, developed by Cabré, de la Llave, and Fontich in [9, 10, 11] (and reviewed in Sections 2 and 2.3), provides an efficient method for computing N-th order power series approximations P_N and Q_N for the chart maps P and Q, as well as a general framework for establishing the convergence of such series.

In the present work we assume that the differential $Df(p)$ is diagonalizable and denote by Λ_s the $n_s \times n_s$ diagonal matrix of stable eigenvalues and by Λ_u the $n_u \times n_u$ diagonal matrix of unstable eigenvalues. The parameterization method is based on the fact that the chart maps P and Q satisfy the functional equations

$$f[P(\theta)] = P(\Lambda_u\ \theta) \qquad \text{and} \qquad f[Q(\phi)] = Q(\Lambda_s\ \phi) \tag{1.1}$$

for any $\theta \in B_{\nu_u}(0) \subset \mathbb{R}^{n_u}$ and $\phi \in B_{\nu_s}(0) \subset \mathbb{R}^{n_s}$. The fact that the chart maps satisfy functional equations is essential in the development of both the formal series approximations P_N and Q_N, and in the convergence analysis of the formal series.

The main technical result of the present work is Theorem 4.1, which provides rigorous bounds on the truncation error $Q_N - Q$ (and similarly for the unstable

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manifolds). The estimates are 'a-posteriori' in the sense that the bounds we obtain are of the form

$$\sup_{\theta \in B_{\nu_s}} |Q(\theta) - Q_N(\theta)| \leq C(N) \sup_{\theta \in B_{\nu_s}} |f[Q_N(\theta)] - Q_N(\Lambda_s \theta)|, \tag{1.2}$$

where $C(N) \to 0$ as $N \to \infty$. The explicit form of $C(N)$ is given in Theorem 4.1. Since all terms on the righthand side of the Inequality (1.2) are explicitly know, and since the supremum on the righthand side of the inequality can be estimated using rigorous numerical methods, Theorem 4.1 can be used to obtain mathematically rigorous computer assisted bounds on the truncation errors associated with the polynomial approximations P_N and Q_N.

While the a-posteriori bounds obtained in Theorem 4.1 are interesting in their own right, we also show how they can be applied to the problem of computer assisted proof of the existence connecting orbits in discrete time dynamical systems. This leads to a scheme, presented in Section 5, which is best thought of as an a-posteriori validation method for the method of *projected boundary conditions*. The method of projected boundary conditions was developed for numerical approximation of heteroclinic and homoclinic orbits by Beyn and Kleinkauf in [7, 8]. The idea is as follows.

Suppose for the moment that $f : \mathbb{R}^n \to \mathbb{R}^n$ is invertible (see however Remark 1.1 below). Define the *homoclinic operator equation* $F : \mathbb{R}^{nk} \to \mathbb{R}^{nk}$ by

$$F(\theta, x_1, x_2, \ldots, x_{k-2}, x_{k-1}, \phi) = \begin{bmatrix} f^{-1}(x_1) - P(\theta) \\ f^{-1}(x_2) - x_1 \\ f^{-1}(x_3) - x_2 \\ \vdots \\ f^{-1}(x_j) - x_{j-1} \\ f(x_j) - x_{j+1} \\ \vdots \\ f(x_{k-2}) - x_{k-1} \\ f(x_{k-1}) - Q(\phi) \end{bmatrix} \tag{1.3}$$

where $\theta \in \mathbb{R}^{n_u}$, $\phi \in \mathbb{R}^{n_s}$, and $x_i \in \mathbb{R}^n$ for each $1 \leq i \leq k-1$. Here j is some fixed integer with $1 \leq j \leq k-1$. Then x_j is a point whose inverse iterates lie on the local unstable manifold, and whose forward iterates lie on the local stable manifold. Let $\tilde{x} = (\tilde{\theta}, \tilde{x}_1, \ldots, \tilde{x}_{k-1}, \tilde{\phi})$ denote a zero of F, then $O = \{P(\tilde{\theta}), \tilde{x}_1, \ldots, \tilde{x}_{k-1}, Q(\tilde{\phi})\}$ is an orbit segment which begins on the local unstable manifold of p and ends, after k iterates, on the local stable manifold of p. It follows that orbit(q) is homoclinic to p for any $q \in O$.

Now, if P_N and Q_N are polynomial approximations of the chart maps P and Q, then one defines F_N in analogy with Equation 1.3 by replacing the the exact chart maps with their polynomial approximations. The method of projected boundary conditions consists of numerically solving $F_N(x) = 0$ using a Newton Scheme, and enables fast and accurate numerical computation of connecting orbits.

Now suppose that $\hat{x}$ is an approximate zero of F_N, computed numerically as just described. Then it is natural to try to invoke the Newton-Kantorivich Theorem (Thm 4.3) in order to prove the existence of an exact zero $\tilde{x}$ near $\hat{x}$ of the full map F. The possibility of this kind of computer assisted proof of the existence of a connecting orbit is in fact mentioned in [7, 8]. Note that the map F is on $\mathbb{R}^{nk}$ so that the Newton-Kantorovich argument finite dimensional.

The difficulty in implementing this argument is the fact that we only know explicitly the map F_N, yet we want to prove the existence of a zero of the map F. In order to overcome this difficulty we require;

(i) rigorous bounds on the truncation errors in the approximations $P \approx P_N$ and $Q \approx Q_N$, so that we can bound the true residual $\|F(\hat{x})\|$.

(ii) rigorous bounds on the derivative of the truncation errors at the approximate solution $\hat{x}$, so that we can bound the derivative of F at $\hat{x}$.

(iii) rigorous uniform bounds on the second derivative of the truncation errors in a neighborhood of the approximate solution $\hat{x}$, so that we can bound DF in a neighborhood of $\hat{x}$.

We note that these are precisely the difficulties overcome by our a-posteriori results on the parameterization truncation errors. Once we use Theorem 4.1 in order to bound the truncation error (as an analytic function) we obtain the necessary bounds on the first and second derivatives of the truncation using a c Cauchy Type Bound from KAM theory.

Remarks 1.1 (EXTENSIONS).

1. The homoclinic operator equation given by Equation 1.3 can easily be modified to define an equation whose solution is a heteroclinic orbit between two distinct hyperbolic fixed points p_1 and p_2. This is done by taking P and Q to be respectively the parameterizations of the n_u dimensional unstable manifold at p_1 and the n_s dimensional stable manifold at p_2. As long as the manifolds satisfy the usual non-degeneracy conditions, namely $n_s + n_u = n$, then map F is non-degenerate and the method of projected boundary conditions is valid.

2. If f is not invertible, but p is a hyperbolic fixed point, then the local stable and unstable sets can still be defined and the stable manifold theorem generalizes as in [41]. Since p is hyperbolic, f is a local analyticomorphism and the parameterization method can still be used to compute the local manifolds. Of course the usual care must be take in globalizing the local stable manifold due to the non-existence of a unique inverse map. See [31, 32] for more complete discussion. At any rate, our a-posteriori scheme for computer assisted proof of the existence of connecting orbits can be applied to non-invertible maps as well. In the case that f is non-invertible, the parameterizations must be restricted to suitable neighborhoods of their fixed points, so that the image of the approximations P_N and Q_N do not intersect the singularity set of Df.

3. In the non-invertible case it is also natural to take $i = 1$ in the operator Equation 1.3 (or it's heteroclinic equivalent). That way only one application of an inverse map is needed. Of course the choice of inverse maps is dictated by the specific problem at hand (i.e. the approximate orbit whose existence is to be validated). For a more complete discussion of connecting orbits for non-invertible maps we refer to [42].

4. Finally note that the requirement that the map is real analytic can be lessened to piecewise real analytic (liner, polynomial, etc) and the methods presented here apply so long as all of the points $\hat{x}_i$, $\tilde{x}_i$, $1 \leq i \leq k-1$ are bounded away from the singularity set of Df.

The remainder of the paper is organized as follows. In Section 2 we discuss the background material used throughout the present work. We begin with a brief review of the parameterization method literature and a short discussion of the literature on computer assisted proof for the existence of connecting orbits in discrete time dynamical systems. In Section 2.2 we introduce the example dynamical systems used for the applications later in the paper.

In Section 2.3 we review the basic notions of the Parameterization Method for Stable and Unstable manifolds of fixed points of a local diffeomorphism f, including a brief discussion of the computation of the coefficients of the chart maps. Section 3 focuses on numerical issues for parameterization such as performance results for the numerical computation of the rigorous interval enclosure of the parameterization coefficients for a dimension dependent family of example maps which always have a one dimensional unstable manifold and a co-dimension one stable manifold. This allows us to examine the computational costs of computing the chart maps for invariant manifolds of dimension between two and ten. We also discuss a qualitative measure of the truncation error. (We note that while this error indicator is only qualitative, it is an essential input into the computer assisted arguments given later).

Section 4 is devoted to the proof of Theorem 4.1, the main technical result of the present work. The section is organized as follows. In Section 4.1 we review the functional analytic and complex variables theory which is needed for the proof of Theorem 4.1, and in Section 4.2 we sketch the proof while introducing a series of Lemmas. In Seciton 4.3 we prove the lemmas in order to complete the proof of Theorem 4.1. Section 4.4 shows how to obtain one of the bounds in the hypothesis of Theorem 4.1 if the case that f is polynomial.

In Section 5 we apply the a-posteriori estimates of Theorem 4.1 to the Newton-Kantorovich problem associated with zeros of Equation 1.3. The main result is Theorem 5.1; our analytic shadowing theorem. The proof of Theorem 5.1 is a straight forward application of the Newton-Kantorivich theorem and is given in Section 5.2

In Section 6 we present the results of several computer assisted proofs of the existence of transverse homoclinic orbits in the three dimensional Lomelí Map. Here the stable and unstable manifolds are one and two dimensional respectively. We provide examples of the use of high order approximations to the manifold (useful when proving the existence of many distinct homoclinic orbits at a single parameter set) and low order approximation of the manifold (useful when continuing a single orbit over a range of parameters). In order to demonstrate that the algorithms can be applied in dimensions higher than three, we also provide a six dimensional example computation for a pair of coupled Lomelí Maps. Here the proof involves establishing the existence of a transverse homoclinic orbit in the intersection of a four dimensional unstable manifold and a two dimensional stable manifold.

2. Background.

2.1. Previous Work. The so called *Parameterization Method* of [9, 10, 11] provides a theoretical framework for studying the convergence of formal power series expansions of stable and unstable manifolds associated with fixed points of discrete and continuous time dynamical systems, under mild non-resonance conditions.

In [9] an existence theorem ([9] Theorem 1.1) is proved which gives, under quite general hypotheses, the existence of C^k chart maps for local stable and unstable manifolds of C^k local diffeomorphisms on Banach spaces. The proof is constructive and, as noted by the authors in the beginning of [9] Section 3, lends itself to a-posteriori analysis and computer assisted proof.

[11] gives a number of applications of the parameterization method, including some elementary proofs of theorems about invariant manifolds in the analytic category, C^0 invariant manifold theorems, and a rigorous treatment of "slow invariant

manifolds". The proofs in [11] rely on the use of the implicit function theorem. As a consequence they are not constructive (with the exception of [11] Theorem 5.4 on the existence of stable and unstable manifolds of hyperbolic periodic orbits of vector fields. This theorem is proven using the contraction mapping theorem, and explicit a-posteriori bounds are given). [10] develops optimal regularity results for the parameterization method with respect to system parameters in the C^k category.

In addition the parameterization method has been extend into a general method for studying a wide variety of invariant manifolds in dynamical systems theory. For example in [22, 23] a method is developed for computing invariant tori and their stable and unstable manifolds in quasiperiodic discrete time dynamical systems. In [28] the parameterization method is used to study KAM tori in symplectic maps without the use of the so called *action/angle coordinates*. In [30] the parameterization method is used to prove the existence of certain 'mixed-stability' invariant manifolds associated with hyperbolic fixed points of symplectic and volume preserving diffeomorphisms. These manifolds have some stable and some unstable directions and are not defined in terms of asymptotic behavior of the orbits. (Rather they are made up of orbits which 'spend a long time' near the fixed point before moving away). Some extensions to invariant tori of infinite dynamical systems are given in [18]. That the parameterization method can be extended to the study of center manifolds (at least in the case of a single eigenvalue of one) is shown in [5], while for example [21, 35, 6, 12, 13] give numerical applications of the theory. All of the work mentioned in the present paragraph are based on constructive arguments and can in principle be adapted for use in computer assisted proof.

The matter of obtaining enclosures of stable and unstable invariant manifolds using rigorous numerics has been investigated by a number of authors. The first results that we know of appear in [36], where one dimensional local stable and unstable manifolds are computed rigorously and used to give a proof of chaotic dynamics in the standard map. [49, 14, 19, 20] develop and implement general methods for validating the existence of hyperbolic invariant sets, and prove topological shadowing theorems for discrete and continuous time dynamical systems in an arbitrary number of finite dimensions. These methods study coverings of the invariant manifold by parallelograms satisfying some cone conditions. If the parallelograms and cones are mapped across one another by the dynamics the correct way then the existence of the hyperbolic invariant manifold is established. The efficiency and accuracy of these methods depends on the number and size of the parallelograms. Numerous applications of these methods can be found in [4, 2, 3, 48, 46, 15]

Methods for studying invariant manifolds using higher order representations and data structure know as *Taylor Models* has been developed and implemented by [37, 47]. A Taylor Model is a polynomial with floating point coefficients which approximates the invariant manifold combined with an interval remainder which bounds the C_0 error associated with the polynomial approximation. In practice once a polynomial approximation has been computed a topological argument similar in spirit to those of [49] must be employed in order to obtain rigorously the size of the error interval. In [37] this argument is implemented for one dimensional taylor models of the stable and unstable manifolds of the Hénon map using a covering argument which exploits nonlinear parallelogram coverings of the image of the polynomial approximation. In (CITE ALEX) the covering argument is extended to two dimensional manifolds, and rigorous enclosures of the so called Lorenz maniffold are given.

A functional analytic argument for bounding the truncation errors associated with the parameterization method is developed and implemented in [6] for differential equations. A similar technique, which exploits majorant methods of complex analysis rather than fixed point arguments in function space, is developed and implemented for differential equations in [25] (such majorant arguments were used by Poincare, as is mentioned in [9]). In the present work we generalize the methods of [6] to discrete time dynamical systems. (The techniques of [25] could be adapted for studying discrete time systems as well).

Most of the methods described above can be used to obtain computer assisted proofs of the existence of connecting orbits. See for example [36, 37, 48, 6, 46] for examples of explicit computations. There are also rigorous shadowing methods for the computer assisted study of connecting orbits which exploit the theory of exponential dichotomies rather than computing first the stable and unstable manifolds. See for example [16, 39, 45] and the references therein.

2.2. Example Systems. For the numerical work in this paper we consider several generalizations of the classical Hénon map. The *delayed Hénon Map* is introduced in [44], and defined by

$$f(x_1, \ldots, x_n) = \begin{pmatrix} 1 - a\,x_1^2 + b\,x_n \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}. \tag{2.1}$$

The map is useful for producing examples of invariant manifolds of arbitrarily high dimension. The map has two fixed points $p_1, p_2 \in \mathbb{R}^n$ where

$$p_{1,2} = (x_\pm, \ldots, x_\pm) \qquad \text{with} \qquad x_\pm = b - 1 \pm \frac{\sqrt{(1-b)^2 + 4a}}{2a}.$$

We take $a = 1.6$ and $b = 0.1$ as in [44], so that p_1 has a one dimensional unstable manifold and an $n-1$-dimensional stable manifold for any phase space dimension n.

We also consider the five parameter family of (quadratic) volume preserving diffeomorphisms $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x, y, z) = f_{\alpha,\tau,a,b,c}(x, y, z) = \begin{pmatrix} z + Q_{\alpha,\tau,a,b,c}(x, y) \\ x \\ y \end{pmatrix}, \tag{2.2}$$

where Q is the quadratic function

$$Q_{\alpha,\tau,a,b,c}(x, y) = \alpha + \tau x + a x^2 + b x y + c y^2, \qquad \text{with} \qquad a + b + c = 1. \tag{2.3}$$

The family of maps was introduced in [33], as a volume preserving analog of the two dimensional area preserving Hénon map. We refer to the dynamical system determined by Equation (2.3) as the *Lomelí Map*.

In the present work we use the Lomelí map in order to find homoclinic orbits which make ‘long’ excursions. By a long excursion we mean a homoclinic orbit which, once it leaves a fixed fundamental domain on the local unstable manifold requires a large number of iterates before returning to a fixed fundamental domain for the local stable manifold. While this notion of a long excursion depends on the choice of fundamental domains (so that it would more correctly be called “long with respect to some choice of fundamental domains”), in the sequel the natural choice will be to consider maximal fundamental domains with respect to the parameterization polynomial error. Finding long orbits for the Lomelí Map by continuation is easy as there is a parameter which effects the length of the shortest excursion (see [17, 34]). Long orbits are good for testing how far our computer assisted arguments can be pushed before breaking down.

There is an extensive body of numerical evidence which suggests that the Lomelí map admits chaotic motions for many parameter values. See for example Figures 7, 8, 22, and 28 in [17]. The numerical studies in [34] suggest that the mechanism for much of this chaos is that a there is a homoclinic tangle associated with each of the two fixed points of the Lomelí Map. We rigorously establish the existence of such tangles for the Lomelí Map in Section 5.

Finally, in Section 6 we couple two Lomelí maps

$$\begin{array}{ccc} f_1(x_1, y_1, z_1) & \equiv & f_{\alpha_1,\tau_1,a_1,b_1,c_1}(x_1, y_1, z_1) \\ & \text{and} & \\ f_2(x_2, y_2, z_2) & \equiv & f_{\alpha_2,\tau_2,a_2,b_2,c_2}(x_2, y_2, z_2) \end{array}$$

in order to obtain the six-dimensional dynamical system $G : \mathbb{R}^6 \to \mathbb{R}^6$ given by

$$G(x_1, y_1, z_1, x_2, y_2, z_2) \equiv \left[\begin{array}{c} f_1(x_1, y_1, z_1) + \varepsilon g_2(y_2,) \\ f_2(x_2, y_2, z_2) + \varepsilon g_1(y_1) \end{array} \right], \tag{2.4}$$

where

$$g_1(y_1) \equiv (y_1 - x_1^+)(y_1 - x_1^-) \quad \text{and} \quad g_2(y_2) \equiv (y_2 - x_2^+)(y_2 - x_2^-).$$

Here $x_{1,2}^{\pm}$ denotes a coordinate of the fixed points in the $f_{1,2}$ systems (the fixed points are on the $x = y = z$ line so that it is enough to specify only the x coordinate of the fixed point. See [33]). Note that this coupling does not move the fixed points in the $f_{1,2}$ systems, but does change the eigenspaces. When ε is small we can approximate a connecting orbit for G by taking the product of connecting orbits for $f_{1,2}$. This coupled map is useful for demonstrating that our methods are viable for higher-dimensional systems.

2.3. Overview of the Parameterization Method. In this section we review the Parameterization Method of [9, 10, 11]. We focus on the case where the map f is real analytic, the differential is diagonalizable, and there are no resonances between eigenvalues of like stability (these assumptions will be formalized below). For the general situation general reader should consult [9, 10, 11].

In order to formalize the discussion we take $p \in \mathbb{R}^n$ to be a hyperbolic saddle for the real analytic map $f : \mathbb{R}^n \to \mathbb{R}^n$. We assume that f is a local real analyticomorphism and uniformly bound on $B_\rho(p) \subset \mathbb{R}^n$ for some $\rho > 0$. We also assume that that $Df(p)$ is diagnolizable over $\mathbb{C}$. Then $Df(p)$ has n_s distinct stable eigenvalues $\{\lambda_1^s, \ldots, \lambda_{n_s}^s\}$ with $|\lambda_i^s| < 1$, and n_u distinct unstable eigenvalues $\{\lambda_1^u, \ldots, \lambda_{n_u}^u\}$ with $|\lambda_i^u| > 1$, and $n_s + n_u = n$ as p is a saddle. We choose eigenvectors $\{\xi_1^s, \ldots, \xi_{n_s}^s\}$ and $\{\xi_1^u, \ldots, \xi_{n_u}^u\}$ associated with the stable and unstable eigenvalues respectively. For the moment we leave the lengths of the eigenvectors unspecified.

As mentioned in the introduction, the stable manifold theorem gives that $W^s(p)$ and $W^u(p)$ are n_s and n_u dimensional manifolds, respectively tangent to span$\{\xi_i^{n_s}\}$ and span$\{\xi_i^{n_u}\}$ at p. The goal of the parameterization method is to determine real analytic mappings $Q : B(0, \nu_s) \subset \mathbb{R}^{n_s} \to \mathbb{R}^n$ and $P : B(0, \nu_u) \subset \mathbb{R}^{n_u} \to \mathbb{R}^n$ which parameterize the local stable and unstable manifolds $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(p)$ respectively at p. For the moment we focus our attention on the development for Q (the situation for P is similar, as discussed at the end of the section).

We simplify our notation a little by letting $B_s \equiv B(0, \nu_s) \subset \mathbb{R}^{n_s}$, and Λ denote the $n_s \times n_s$ matrix with λ_i^s in the i-th diagonal entry and zeros elsewhere (this was called Λ_s above). Then $Q[B_s]$ is a local stable manifold for p if and only of Q satisfies the following functional equation with initial data

1. $Q(0) = p$,
2. $DQ(0) = [\xi_1^s | \dots | \xi_{n_s}^s]$,
3. and

$$f[Q(\theta)] = Q(\Lambda\theta), \tag{2.5}$$

for all $\theta \in B_s$.

To see this note that for any Q satisfying these conditions, image(Q) is an immersed n_s-disk containing p and is tangent to span$\{\xi_i^{n_s}\}$ at p. Moreover Equation (2.5) implies that $(f \circ Q)(B_s) = Q[\Lambda B_s] \subset Q(B_s)$, so that the ω-limit set of image(Q) under f is p. Then

$$Q(B_s) = W^s_{\text{loc}}(p),$$

by definition.

In general it is impossible to compute Q in closed form. Instead, we note that Q satisfies a (functional) initial value problem with analytic data. Then it is natural to seek a power series expansion for Q of the form

$$Q(\theta) = \sum_{|\alpha| \geq 0} a_\alpha \theta^\alpha \qquad a_n \in \mathbb{R}^n, \quad \theta \in \mathbb{R}^{n_s}, \quad \alpha \in \mathbb{N}^{n_s} \tag{2.6}$$

convergent on B_s. Note that the first order constraints on Q demand that $a_{(0,\dots,0)} = p$ and $a_{e_i} = \xi_i^s$ (here e_i is the multi-index with one in the i-th component and zeros elsewhere). Then the problem is to try to determine the unknown coefficients a_α for $|\alpha| \geq 2$.

Remark 2.1. [Uniqueness] Note that the choice of the lengths of the eigenvectors ξ_i is free in the above formulation. This corresponds to the freedom in the choice of scaling of the parameterization of any manifold. Nevertheless, it is shown in [9] (and we will see again in Section 4) that the solution of Equation 2.5 is unique once the scale of the eigenvectors is fixed.

A formal solution of Equation (2.5) can be obtained by inserting the power series given by Equation (2.6) into Equation (2.5), expanding f as a power series, and computing recurrence relations for the coefficients of Q by matching like powers of θ. This approach, which is discussed further in Section 3 is sometimes called *automatic differentiation* and works well when f is built up out of elementary functions. See [34] for explicit derivation of the recurrence relations for the power series coefficients of Q

for the Lomelí Map. Iterative approaches for solving Equation 2.5 are discussed in [9], and numerical implementations of such iterative algorithms are found in [35, 47, 37].

Finally, we note that the parameterization P of the local unstable manifold for f at p parameterizes the local stable manifold for f^{-1} at p, so that P must satisfy the functional equation

$$f^{-1} \circ P = P \circ \Omega^{-1}, \tag{2.7}$$

where Ω is the matrix of unstable eigenvaules of $Df(p)$. But if we right compose Equation (2.7) with Ω and left compose with f then we obtain

$$P \circ \Omega = f \circ P,$$

which is identical to Equatoin 2.5. Then P and Q solve the same functional equation, modulo the appropriate choose of linear map Λ or Ω.

3. Parameterization Method Numerics.

3.1. Computation of the Power Series Coefficients. If $P : B \subset \mathbb{R}^k \to \mathbb{R}^n$ parameterizes a k dimensional (either stable or unstable) invariant manifold of $f : \mathbb{R}^n \to \mathbb{R}^n$ and, using the notation of Section 2.3, we suppose that

$$P(\theta) = \sum_{|\alpha|>0} a_\alpha \theta^\alpha,$$

then a formal computation shows that for any $|\alpha| \geq 2$ coefficient a_α satisfies the so called *homological equation*

$$[Df(p_0) - \Lambda^\alpha I]a_\alpha = s(\alpha'). \tag{3.1}$$

Here

$$\Lambda^\alpha = \lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_k^{\alpha_k} \in \mathbb{C},$$

and s is a function of the coefficients $a_{\alpha'}$ with $|\alpha'| < |\alpha|$. The form the function s depends only on the the nonlinearity of the function f. Then, for a fixed function f, computing the coefficient a_α requires only the local information p_0, $Df(p_0)$, and Λ, as well as the recursive computation of all the lower order coefficients.

The homological equation is discussed abstractly in [9], and is derived for the concrete example of the Lomelí Map in [35, 34] (in particular the explicit form of s is derived). The function s can be derived in a similar way for all the examples studied in this paper. See also [6] for similar developments for the Gray-Scott differential equation.

Consider the set $\{\lambda_1, \ldots, \lambda_k\}$ of either all stable or all unstable eigenvalues (depending on wether we are parameterizing the stable or unstable manifold). Let μ_- and μ_+ denote the magnitudes of the eigenvalues which are closet to and farthest from the unit circle respectively. Equation (3.1) shows that the coefficient a_α is well defined as long as

$$\lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_k^{\alpha_k} \neq \lambda_i$$

for any $1 \leq i \leq k$. Then it is sufficient to directly check this condition for each multi-index α with

$$2 \leq |\alpha| \leq \frac{\ln(\mu_-)}{\ln(\mu_+)},$$

in order to rule out resonances at all orders. Since this gives a finite number of conditions we conclude that a generic set of eigenvalues will be non-resonant. Nevertheless, in practice we check this finite set of non-resonance conditions using rigorous interval arithmetic. The preceding discussion gives an algorithm for computing the coefficients of a generic chart map P to any desired finite order.

In the remainder of this section we study the computational costs associated with determining the coefficients for the Delayed Hénon map described in Section 2.2. All computations are carried out using the *IntLab* implementation of interval arithmetic in MatLab. The IntLab package is equipped with subroutines for computing rigorous interval enclosures of the usual elementary functions, eigenvalues and eigenvectors of $n \times n$ matrices, and solutions of linear systems of equations. See [40] for a more thorough discussion.

We compute the coefficients of the parameterization of the $n-1$ dimensional stable manifold at p_1 in dimensions $n = 3$ through $n = 11$ to various orders for $a = 1.6$ and $b = 0.1$. We obtain an interval enclosure of $x_1 \subset B(0.557857598881097, 2.221e-16)$ for the fixed point $p_1 = (x_1, \ldots, x_1)$.

First we consider the cost of computing parameterizations of the two dimensional stable manifold with varying polynomial order. We obtain interval enclosures of the stable eigenvalues

$$\lambda_s^1 \subset B(-0.25570156572582, 2.221e{-}16) \qquad \text{and} \qquad \lambda_s^2 \subset B(0.22314485443973, 1.388e{-}16).$$

and eigenvectors

$$\xi_s^1 \subset B\left(\begin{bmatrix} -0.06321850901795 \\ 0.24723551785264 \\ -0.96689090327176 \end{bmatrix}, 1.67e-16\right)$$

and

$$\xi_s^2 \subset B\left(\begin{bmatrix} 0.04854109103645 \\ 0.21753175155361 \\ 0.97484547470201 \end{bmatrix}, 1.67e-16\right)$$

Table 3.1 reports the performance data for computations with orders between $N = 2$ and $N = 60$. Each coefficient is a vector in $\mathbb{R}^3$ (the solution of the homological equation, which is a 3×3 linear system) so each non-zero coefficient consists of three intervals. Also given are the resulting computation times and the size of the largest containment interval of any coefficient.

We also compute the parameterization to third order for phase space dimensions 4 through 11 (manifold dimensions 3 through 10). The results are given in Table 3.1. Each increase in dimension leads to an increase in computation time of roughly a factor of four. The interval enclosure radii are related to the enclosure radii of the eigenvalues and eigenvectors, which get more difficult to enclose as the dimension increases. In every case all eigenvalues and eigenvectors are enclosed in balls with radii of no more than 5×10^{-16}.

Order	Number Non-Zero Coeff	Comp Time	Largest Interval Rad
2	6	0.064 sec	$3.61e-16$
3	10	0.143 sec	$3.61e-16$
4	15	0.257 sec	$3.61e-16$
5	21	0.396 sec	$3.61e-16$
10	66	1.41 sec	$3.61e-16$
15	136	3.80 sec	$3.61e-16$
20	231	8.35 sec	$3.61e-16$
30	496	32.01 sec	$3.61e-16$
60	1756	395.51 sec	$3.61e-16$

TABLE 3.1

Coefficient Computation Performance: Two-Dimensional Manifold; Three Dimensional Phase Space.

Phase Space Dim	Number Non-Zero Coeff	Comp Time	Largest Interval Rad
4 (3-D manifold)	20	0.478 sec	$4.11e-16$
5 (4-D manifold)	35	0.555 sec	$4.61e-16$
6 (5-D manifold)	56	0.820 sec	$5.46e-16$
7 (6-D manifold)	84	1.37 sec	$5.88e-16$
8 (7-D manifold)	120	3.43 sec	$6.98e-16$
9 (8-D manifold)	165	10.92 sec	$6.96e-16$
10 (9-D manifold)	220	52.72 sec	$9.11e-16$
11 (10-D manifold)	286	292.83 sec	$6.95e-16$

TABLE 3.2

Coefficient Computation Performance: Third order approximation of co-dimension one manifold in n-dimensional phase space.

3.2. Numerical Radius of Validity for Formal Solutions. Suppose that we have recursively solved the homological equations for the parameterization of a k dimensional (stable or unstable) manifold up to a fixed finite order N as discussed in the previous section. Then we have a polynomial approximation

$$P_N(\theta) = \sum_{0 \leq |\alpha| \leq N} a_\alpha \theta^\alpha$$

to the true parameterization P. While any truncated approximation P_N is entire (as P_N is a polynomial), we do not expect that P_N is a good approximation to P for all θ. Instead, we would like to determine a fixed domain on which the approximation is "good". The following definition makes this precise;

DEFINITION 3.1. Let $\epsilon > 0$ be a prescribed tolerance, $\nu > 0$, and $B = B(0, \nu) \subset \mathbb{R}^k$. We call the number ν *an ϵ-numerical radius of validity* for the approximation P_N if

$$\text{Error}_\nu(P_N) \equiv \sup_{\theta \in B} \| f[P_N(\theta)] - P_N(\Lambda\, \theta) \| \leq \epsilon. \tag{3.2}$$

REMARK 3.2.

- In practice, numerical experimentation is enough to select a good ν. Numerical examples and algorithm performance information for local manifold computations for the Lomelí map can be found in Section 5 and Appendix A of [35]
- We have the usefull bound

$$\mathrm{Error}_\nu(P_N) \leq \sum_{0\leq|\alpha|} |C_\alpha - D_\alpha|\nu^{|\alpha|} \tag{3.3}$$

  where C_α, D_α are the power series coefficients of $f[P_N]$ and $P_N(\Lambda\,\theta)$ respectively. (The inequality is due to the maximum modulus principle). When f is a polynomial, all but finitely many of A_α, and B_α are zero. Then the sum is finite and Equation (3.2) is easy to rigorously bound numerically using interval arithmetic.
- Theorem 4.1 shows that under certain conditions which are easy to validate numerically, we actually have $\|P - P_N\|_\nu \leq C\epsilon$ where C is an explicitly known constant. This provides a mathematically rigorous *a-posteriori* bound on the truncation error made in approximating P by P_N.

4. A-Posteriori Validation of the Formal Series. In this section we prove an *a-posteriori* validation theorem for parameterizations of stable and unstable manifolds for discrete time dynamical systems. From a theoretical view it is preferable to work with analytic functions defined on $\mathbb{C}^n$. For the sake of readability we re-state our assumptions.

A1 Let $p \in \mathbb{C}^n$, $\rho > 0$ and assume that that $f : B(p, \rho) \subset \mathbb{C}^n \to \mathbb{C}^n$ is a bounded analytic function, so that there is $K_0 > 0$ so that

$$\|f\|_\rho \leq K_0.$$

A2 Assume that $Df(p)$ is non-singular, diagonalizble, and hyperbolic. Let $\{\lambda_1^s, \ldots, \lambda_{n_s}^s\}$ and $\{\xi_1^2, \ldots, \xi_{n_s}^s\}$ denote the stable eigenvalues (which are distinct as $Df(p)$ is diagonalizable) and a choice of stable eigenvectors respectively. Let Λ denote the $n_s \times n_s$ diagonal matrix of stable eigenvalues and $Q_0 = [\xi_1^s|\ldots|\xi_{n_s}^s]$ denote the matrix whose columns are the stable eigenvectors.

A3 Assume that $P_N : B(0, \nu) \subset \mathbb{C}^{n_s} \to \mathbb{C}^n$ is an N-th order polynomial, with $N \geq 2$, which for each $\theta \in B(0, \nu)$ solves the equation

$$f[P_N(\theta)] = P_N(\Lambda\theta)$$

exactly to N-th order (in the sense that the power series coefficients of the function on the left are equal to the power series coefficients of the function on the right to N-th order).

Then we have the following definition.

Definition 4.1. [Validation values for discrete dynamical systems] The collection of positive constants ν, ϵ_{tol}, C_1, C_2, K_1, ρ, ρ', μ_* and μ^* are validation values for P_N if

1. $\|f \circ P_N - P_N \circ \Lambda\|_{\Sigma,\nu} \leq \epsilon_{\text{tol}}$;
2. $\|P_N\|_{\Sigma,\nu} \leq \rho' < \rho$;
3. $0 < \mu_* \leq \min_{1\leq i\leq n_s} |\lambda_i^s| \leq \max_{1\leq i\leq n_s} |\lambda_i^s| \leq \mu^* < 1$;
4. $$\|Df[P_n]^{-1}\|_{\Sigma,\nu} \leq C_1\mu_*^{-1} + C_2(\nu);$$

where, as we will see in the proof, we take C_1 to be any constant with

$$\|Q_0\|\|Q_0^{-1}\| \leq C_1,$$

and C_2 to be any bound on the theta dependent terms of $Df[P_N(\theta)]^{-1}$ on B_ν.

5.

$$\max_{\substack{\beta \in \mathbb{Z}^n \\ |\beta| = 2}} \max_{1 \leq j \leq n} \|\partial^\beta f_j\|_\rho \leq K_1(\rho).$$

The bounds in the validation theorem are improved if we take into account only the of non-zero second partials of f. Then we will define

$$N_f = \max_{1 \leq j \leq n} \#\{\beta \in \mathbb{Z}^n : |\beta| = 2 \text{ and } \partial^\beta f_j \neq 0\}, \tag{4.1}$$

and of course have that $N_f \leq n^2$. However for a given map N_f may be smaller than this.

THEOREM 4.1 (A-posteriori manifold validation). *Given validation values ν, ϵ_{tol} , K_1, C_1, C_2, ρ, ρ', μ_* and μ^*, assume that N and δ satisfy the three inequalities*

$$N + 1 > \frac{\ln(\mu_*) - \ln(C_1 + \mu_* C_2)}{\ln(\mu^*)}; \tag{4.2}$$

$$\delta < \min\left(\frac{[\mu_* - (C_1 + \mu_* C_2)(\mu^*)^N]}{2ne\pi N_f (C_1 + \mu_* C_2) K_1}, (\rho - \rho')e^{-1}\right) \tag{4.3}$$

$$\delta > \frac{2(C_1 + \mu_* C_2)\epsilon_{tol}}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^N} \tag{4.4}$$

Then there is a unique parameterization function $P : B(0, \nu) \subset \mathbb{C}^{n_s} \to \mathbb{C}^n$ solving Equation 2.5. Additionally, the truncation error is bounded by

$$\|P - P_N\|_\nu \leq \delta$$

and the parameterization coefficients $a_\alpha \in \mathbb{C}^n$ decay as

$$|a_\alpha| \leq \frac{\delta}{\nu^{|\alpha|}} \qquad \text{for } |\alpha| > N.$$

REMARK 4.2. [The Resonance Condition] While the meanings of the conditions given by Equations 4.2, 4.3, and 4.4 will become clear in the Sections 4.2 and 4.3, when we discuss the proof of Theorem 4.1, it is appropriate to make a small remark about Equation 4.2 presently. Note that the right hand side of Equation 4.2 is the natural logarithm of the ratio of the smallest to the largest eigenvalue of $Df(p)$ (the spectral gap) minus a correction term which reflects the nonlinearity of f at p. The condition given by Equation 4.2 guarantees that N is so large enough that there is no possibility of resonances in the coefficients of the remainder $P - P_N$.

4.1. Analytic Preliminaries. If $x \in \mathbb{R}$, then we use $|x|$ to denote the usual absolute value. Similarly, for $z = a + ib \in \mathbb{C}$ we use the usual "Euclidian" norm $|z| = \sqrt{a^2 + b^2}$. We endow $\mathbb{R}^n$ and $\mathbb{C}^n$ with the so called sup or infinity norms generated by the real or complex absolute value functions, so that for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we have

$$|x| = \max_{1 \leq i \leq n} |x_i|, \qquad \text{and} \qquad |z| = \max_{1 \leq i \leq n} |z_i|$$

where in each case the $|\cdot|$ on the right is either the absolute value for $\mathbb{R}$ or $\mathbb{C}$, and the sup is taken over components. These norms are well suited for numerical work, as they are easy to evaluate and introduce no rounding errors.

For fixed $\hat{z} \in \mathbb{C}^m$ and $\nu > 0$ let $B_\nu(\hat{z}) \subset \mathbb{C}^m$ be the ball (or *poly-disk*) of radius ν about $\hat{z}$ generated by the sup-norm, so

$$B_\nu(\hat{z}) \equiv \{(h_1, \ldots, h_m) \in \mathbb{C}^m \ : \ |\hat{z}_i - h_i| < \nu \ \text{ for each } \ 1 \leq i \leq m\}.$$

A function $g : B_\nu(\hat{z}) \subset \mathbb{C}^m \to \mathbb{C}$ is analytic on the poly-disk $B_\nu(\hat{z})$ if g has a power series expansion

$$g(z) = \sum_{|\alpha| \geq 0} a_\alpha (\hat{z} - z)^\alpha \qquad \alpha \in \mathbb{N}^m \qquad a_\alpha \in \mathbb{C},$$

which converges for all $z \in B_\nu(\hat{z})$. Here we use the usual *multi-index* notation, so that if $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and $z \in \mathbb{C}^m$ then $|\alpha| = \alpha_1 + \ldots + \alpha_m$ and $z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_m^{\alpha_m}$.

We say that $f : B_\nu(\hat{z}) \subset \mathbb{C}^m \to \mathbb{C}^n$ is analytic on $B_\nu(\hat{z})$ if $f = (f_1, \ldots, f_n)$ and each $f_j : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}$, $1 \leq j \leq n$ is analytic in the sense just described. Such an f can also be expressed in power series form as

$$f(z) = \sum_{|\beta| > 0} b_\beta (\hat{z} - z)^\beta \qquad \beta \in \mathbb{N}^m \qquad b_\beta \in \mathbb{C}^n$$

which converges for all $z \in B_\nu(\hat{z})$. The space of bounded analytic functions on $B_\nu(\hat{z})$ forms a Banach space under the norm

$$\|f\|_{B_\nu(\hat{z}), \Sigma} \equiv \sum_{|\alpha| \geq 0} |b_\alpha| \nu^{|\alpha|}.$$

Of course the bounded analytic functions are also a Banach space under the usual C^0 norm, and that the two norms are related by

$$\|f\|_{B_\nu(\hat{z})} \equiv \max_{1 \leq j \leq n} \max_{1 \leq i \leq m} \sup_{|z_i - \hat{z}_i| \leq \nu} |f_j(z_1, \ldots, z_m)| \leq \|f\|_{B_\nu(\hat{z}), \Sigma}.$$

In theoretical arguments we often use the C^0 norm $\|\cdot\|_{B_\nu(\hat{z})}$, while in numerical applications it is often convenient to use the *sigma-norm* $\|\cdot\|_{B_\nu(\hat{z}), \Sigma}$ in conjunction with the above inequality. Also, by the maximum modulus principle we have that if f is uniformly bounded and analytic on (the open set) $B_\nu(\hat{z})$, then

$$\|f\|_{B_\nu(\hat{z})} = \max_{1 \leq j \leq n} \sup_{|z_i - \hat{z}_i| = \nu} |f_j(z_1, \ldots, z_m)|,$$

so that f is in fact bounded on the closed ball. It follows that f is continuous on $\partial B_\nu(\hat{z})$. If the ball in question is centered at the origin, i.e. is a ball of the form $B_\nu(0)$ then we sometimes use the notation $\|\cdot\|_{\nu,\Sigma}$ and $\|\cdot\|_\nu$ for $\|\cdot\|_{B_\nu(0),\Sigma}$ and $\|\cdot\|_{B_\nu(0)}$ respectively.

Suppose that A is an $n \times m$-matrix with entries $a_{ij} \in \mathbb{C}$. Then when we consider a A as a linear operator $A : \mathbb{C}^m \to \mathbb{C}^n$ we employ the usual operator norm

$$\|A\|_M = \sup_{|\eta|=1} |A \cdot \eta|,$$

where $\eta \in \mathbb{C}^m$ and $\cdot$ is matrix-vector multiplication. Since $|\cdot|$ is the sup-norm on components we have that

$$\|A\|_M \leq \sup_{1 \leq i \leq n} \sum_{j=1}^{m} |a_{ij}| \leq m \sup_{1 \leq i \leq n} \sup_{1 \leq j \leq m} |a_{ij}|. \tag{4.5}$$

Given a fixed $\hat{z} \in \mathbb{C}^k$ and $\nu > 0$, suppose that $g : B_\nu(\hat{z}) \subset \mathbb{C}^k \to \mathbb{C}^m$ is an analytic function and suppose that the entries of the $n \times m$ matrix A are themselves analytic functions $a_{ij} : B_\nu(\hat{z}) \subset \mathbb{C}^k \to \mathbb{C}$. We can define the norm of the non-constant matrix A to be

$$\|A\|_{M,B_\nu(\hat{z})} \equiv \max_{1 \leq i \leq n} \sum_{j=1}^{m} \|a_{ij}\|_{B_\nu(\hat{z})}$$

Then the non-constant matrix vector product $A \cdot g : B_\nu(\hat{z}) \subset \mathbb{C}^k \to \mathbb{C}^n$ is an analytic function and we have the bounds

$$\|A \cdot g\|_{B_\nu(\hat{z})} \leq \|A\|_{M,B_\nu(\hat{z})} \|g\|_{B_\nu(\hat{z})} \leq m \|g\|_{B_\nu(\hat{z}),\Sigma} \max_{1 \leq i \leq n} \max_{1 \leq j \leq m} \|a_{ij}\|_{B_\nu(\hat{z}),\Sigma},$$

the last bound being particularly useful for numerical applications.

The family of analytic functions which are zero to N-th order play an important in the arguments to follow. We say that $h : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}^n$ is an *analytic N-tail* if h is analytic on $B_\nu(0)$ and

$$h(0) = 0, \quad Dh(0) = 0, \quad \ldots \quad D^\alpha h(0) = 0, \qquad \text{for} \qquad |\alpha| \leq N.$$

Then an analytic N-tail h always has power series representation

$$h(z) = \sum_{|\beta| > N} b_\beta z^\beta \qquad \beta \in \mathbb{N}^m \qquad b_\beta \in \mathbb{C}^n$$

converging for each $|z| < \nu$. With m, n, and $\nu > 0$ fixed we define $\mathbb{H}_N$ to be the set of bounded analytic N-tials on $B_\nu(0) \subset \mathbb{C}^m$ taking values in $\mathbb{C}^n$ (n, m, and ν will always be clear from context).

We use freely the following well known facts about analytic functions and N-tails.

LEMMA 4.2.

1. *If $\hat{z} \in \mathbb{C}^m$, $\nu > 0$, $f : B_\nu(\hat{z}) \to \mathbb{C}^n$ is analytic and $\|f\|_\nu \leq M$, then one has for each $\beta \in \mathbb{N}^m$ the Cauchy Estimates*

$$|b_\beta| \leq \frac{M}{\nu^{|\beta|}}.$$

2. *Let h be a bounded analytic N-tail on $B_\nu(0) \subset \mathbb{C}^m$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$be non-zero complex numbers with $0 < |\lambda_j| < 1$, for $1 \leq j \leq m$. Suppose that Λ is the $m \times m$ matrix with λ_j in the j-th diagonal entry and zeros in the non-diagonal entries, and that $0 < \mu^* \equiv \sup_j |\lambda_j| < 1$. Then $h \circ \Lambda$ is a bounded analytic N-tail on $B_\nu(0)$ and*

$$\|h \circ \Lambda\|_\nu \leq (\mu^*)^{N+1} \|h\|_\nu \ .$$

3. *If $g : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}$ is analytic and $\hat{z} \in \mathbb{C}^m$ has $|\hat{z}| < \nu$, then g is analytic on the poly-disk $B_s(\hat{z})$, $s = \nu - |\hat{z}|$ and for any $\eta \in B_s(\hat{z})$, g can be expanded as*

$$g(\hat{z} + \eta) = g(\hat{z}) + Dg(\hat{z}) \cdot \eta + R_{\hat{z}}(\eta)$$

where

$$\|R_{\hat{z}}\|_s \leq N_g K s^2.$$

Here N_g is the number of non-zero second partial derivatives of of g at $\hat{z}$ (so $N_g \leq m^2$) and K is any constant having

$$\sup_{|\beta|=2} \|\partial_\beta g\|_s \leq K.$$

If f is analytic on $B_\nu(0) \subset \mathbb{C}^m$ with values in $\mathbb{C}^n$ then the result can be applied to f component by component.

4. *If $f : B_\nu(\hat{z}) \subset \mathbb{C}^m \to \mathbb{C}^n$ is analytic and $z_1, z_2 \in B_\nu(\hat{z})$ then*

$$|f(z_1) - f(z_2)| \leq \|Df\|_{M, B_\nu(\hat{z})} |z_1 - z_2|.$$

For (1) see any standard text on complex analysis (for example [1]). The elementary proof of (2) is in [6]. (3) is the Lagrange form of the Taylor remainder theorem (also for example in [1]), while (4) is the mean value theorem combined with our norm definitions.

In the following let X be a Banach space, $\mathbb{L}(X)$ be the Banach space of all bounded linear operators on X, and $A \in \mathbb{L}(X)$. Then

$$\|A\|_{\mathbb{L}(X)} \equiv \sup_{x \in X, \|x\|_X = 1} \|Ax\|_X = M < \infty.$$

We make use of the following standard theorems from non-linear analysis.

- **Contraction Mapping Theorem** Let $x \in X$,

$$B_r(x) = \{y \in X \,:\, \|x - y\|_X \leq r\},$$

and suppose that $\Phi : B_r(x) \to B_r(x)$ is continuous. If in addition there is a $0 < \kappa < 1$ so that for any $x_1, x_2 \in B_r(x)$ we have

$$\|\Phi(x_1) - \Phi(x_2)\|_X \leq \kappa \|x_1 - x_2\|_X$$

then there is a unique $\hat{x} \in B_r(x)$ so that $\Phi(\hat{x}) = \hat{x}$.
- **Neumann Series** If $I : X \to X$ is the identity map and $A : X \to X$ is a bounded linear operator with $\|A\|_{\mathbb{L}(X)} \leq 1$ then $I - A$ is boundedly invertible and

$$[I - A]^{-1} = \sum_{k=0}^{\infty} A^k,$$

from which it follows that

$$\|(I - A)^{-1}\|_{\mathbb{L}(X)} \leq \sum_{k=0}^{\infty} \|A\|^k_{\mathbb{L}(X)} \leq \frac{1}{1 - M}.$$

Our "analytic homoclinic shadowing theorem" (Theorem 5.1) is based on the Newton-Kantorovich Theorem [26, 27].

Theorem 4.3 (Newton-Kantorovich Method). *Let X, Y be Bancah spaces and $F : X \to Y$ be a differentiable mapping. Assume that there as an $\hat{x} \in X$ and an $r > 0$ such that*

(i) $DF(\hat{x})$ has bounded inverse, and

(ii) $\|DF(x) - DF(y)\|_{B(X,Y)} \leq \kappa \|x - y\|$ for all $x, y \in B_r(\hat{x})$.

If

(I)

$$\epsilon_{NK} \geq \|DF(\hat{x})^{-1} F(\hat{x})\|_X,$$

(II)

$$\epsilon_{NK} \leq \frac{r}{2},$$

and

(III)

$$4\epsilon_{NK} \, \kappa \, \|DF(\hat{x})^{-1}\|_{B(X,Y)} \leq 1,$$

then the equation

$$F(x) = 0$$

has a unique solution in $B(r, \hat{x})$.

For an english language exposition of the proof, see also [38]

Finally we require the following bounds for derivatives of analytic functions. The Lemma 4.3 tells us how to bound the derivatives of an analytic function in terms of a bound on the function itself, *so long as we are willing to give up some portion of the domain of analyticity.* The estimates are considered "standard" in KAM theory. (For example they are left as an exercise in [29], and are similar to the bounds for Fourier series found in Section 2.5.7 of [**?**]. Similar, but less optimal, estimates are in [**?**, 6]) We include a brief proof in order to obtain explicitly the constants, as we must apply the bounds in the context of computer assisted arguments. Our aim is to give an elementary and brief computation and we note that our constants are obviously not sharp. On the other hand we do take care to obtain the optimal order in the *loss of domain parameter* σ.

LEMMA 4.3 (Cauchy Bounds). *Suppose that* $f : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}^n$ *is bounded and analytic. Then for any* $0 < \sigma \leq 1$ *we have that*

$$\|\partial_i f\|_{\nu e^{-\sigma}} \leq \frac{2\pi}{\nu\sigma}\|f\|_\nu \quad \text{so that} \quad \|Df\|_{\nu e^{-\sigma}} \leq \frac{2\pi m}{\nu\sigma}\|f\|_\nu, \tag{4.6}$$

as well as

$$\|\partial_i\partial_j f\|_{\nu e^{-\sigma}} \leq \frac{4\pi^2}{\nu^2\sigma^2}\|f\|_\nu \quad \text{and} \quad \|D^2 f\|_{\nu e^{-\sigma}} \leq \frac{4\pi^2 m^2}{\nu^2\sigma^2}\|f\|_\nu. \tag{4.7}$$

Proof: Consider first the one dimensional case, where $\nu > 0$ and $f : B_\nu(0) \subset \mathbb{C} \to \mathbb{C}$ is analytic. Let $0 < \sigma \leq 1$. Then using Cauchy's formula [1] we have that for any $z \in B_{\nu e^{-\sigma}}(0)$

$$f'(z) = \frac{1}{2\pi i}\int_{|\xi|=\nu} \frac{f(\xi)}{(\xi - z)^2}\, d\xi.$$

Note that the denominator is bounded precisely because $|z| \leq \nu e^{-\sigma}$, i.e. because we are taking z in a reduced domain. (Choosing to reduce the domain by an amount exponential in σ gives the optimal $1/\sigma$ dependance in the final estimate, as will be seen in the proof). We parameterize the path $|\xi| = \nu$ by $\xi(\theta) = \nu e^{i\theta}$ and take norms to obtain

$$\begin{aligned} |f'(z)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f[\nu e^{i\theta}] i\nu e^{i\theta}}{(\nu e^{i\theta} - z)^2}\, d\theta\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\nu\|f\|_\nu}{|\nu e^{i\theta} - z|^2}\, d\theta \\ &\leq \frac{\|f\|_\nu}{2\pi\nu}\int_0^{2\pi} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta, \end{aligned} \tag{4.8}$$

where the last inequality is due to the fact that $|z| \leq \nu e^{-\sigma}$, so that the denominator is minimized when $|z| = \nu e^{-\sigma}$. Since the integrand is radially symmetric once we take the norm of f, we are free to take $z = \nu e^{-\sigma}$, and then factor a ν^2 out of the denominator of the integrand.

Noting that $e^\sigma \geq 1 + \sigma$ for all real σ, we have that $\sigma/(1 + \sigma) \leq 1 - e^{-\sigma}$ for all $\sigma > -1$. Then for $0 < \sigma \leq 1$ we have

$$\sigma/2 \leq \frac{\sigma}{1+\sigma} \leq 1 - e^{-\sigma} \leq |e^{i\theta} - e^{-\sigma}|, \tag{4.9}$$

for all $0 \leq \theta \leq 2\pi$. Naive application of Eq (4.9) to Eq (4.8) would yield $|f'(z)| \leq 4\|f\|_\nu/\sigma^2$. However a slightly more subtle argument yields an estimate which is only inverse proportional to σ. Eq (4.8) can be re-written as

$$\frac{\|f\|_\nu}{2\pi\nu} \int_0^{2\pi} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta$$

$$= \frac{\|f\|_\nu}{2\pi\nu} \left(\int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta + \int_{\frac{\sigma}{2}}^{2\pi - \frac{\sigma}{2}} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta \right) \tag{4.10}$$

For the first of the integrals on the right in Eq (4.10) we exploit Eq (4.9) to obtian

$$\int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta \leq \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{|\frac{\sigma}{2}|^2}\, d\theta \leq \frac{4}{\sigma}. \tag{4.11}$$

On the other hand, since $|e^{i\theta} - e^{-\sigma}| \geq \sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/4$, the second integral on the right in Eq (4.10) satisfies the bound

$$\int_{\frac{\sigma}{2}}^{2\pi - \frac{\sigma}{2}} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta \leq 4 \int_{\frac{\sigma}{2}}^{\frac{\pi}{2}} \frac{\pi^2}{4\theta^2} \leq \frac{2\pi^2}{\sigma} \tag{4.12}$$

Racalling that $z \in B_{\nu e^{-\sigma}}(0)$ we note that Eq (4.11) and Eq (4.12) are uniform in z and combine them with Eq (4.10) to obtain

$$\|f'\|_{\nu e^{-\sigma}} \leq \frac{1}{2\pi\nu} \left(\frac{4}{\sigma} + \frac{2\pi^2}{\sigma} \right) \|f\|_\nu \leq \frac{2\pi}{\nu\sigma} \|f\|_\nu. \tag{4.13}$$

If $f : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}^n$ then each $f_k(z_1, \ldots, z_i, \ldots, z_m)$, $1 \leq i \leq m$, $1 \leq k \leq n$ is analytic in the i-th variable (with the other variables held fixed), so that we obtain

$$\left| \frac{\partial}{\partial z_i} f_k(z) \right| \leq \frac{2\pi}{\nu\sigma} \|f\|_\nu,$$

for any $|z| \leq \nu e^{-\sigma}$ by applying the same argument to the Cauchy integral of $\partial/\partial z_i f_k(z)$. Since this is uniform in i, k and z we apply the estimate given by Equation (4.5) in order to obtian

$$\|Df\|_{\nu e^{-\sigma}} \leq \frac{2\pi m}{\nu\sigma} \|f\|_\nu,$$

as desired. The same estimates can be applied to the Cauchy type integral

$$\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} f(z) = \frac{1}{(2\pi i)^2}\int_{|\xi_i|=\nu}\int_{|\xi_j|=\nu} \frac{f(z_1,\ldots,\xi_i,\ldots,\xi_j,\ldots,z_m)}{(\xi_i - z_i)^2\,(\xi_j - z_j)^2}\, d\xi_i\, d\xi_j$$

to obtain in a similar fashion that

$$\|D^2 f\|_{\nu e^{-\sigma}} \leq \frac{4\pi^2 m^2}{\nu^2\sigma^2}\|f\|_{\nu},$$

as desired.

$\square$

4.2. Proof of the Validation Theorem. We seek an analytic N-tail $h : B_\nu \to \mathbb{R}^n$ so that $P = P_N + h$ and having $\|h\|_\nu \leq \delta$ as small as possible (note that δ bounds the truncation error in the approximation P_N). The key observation is that h itself solves a certain functional equation. To see this let $P = P_N + h$ so that Equation 2.5 becomes

$$f[P_N + h] = [P_N + h](\Lambda).$$

Since f is analytic in $B_\rho \subset \mathbb{R}^n$, and since $\|P_N\|_\nu \leq \rho' \leq \rho$, f has a Taylor expansion about $P_N(\theta)$ for each $\theta \in B_s$. Then let $\theta \in B_s$ so that

$$f[P_N(\theta) + h(\theta)] = f[P_N(\theta)] + Df[P_N(\theta)]\, h(\theta) + R_{P_N(\theta)}(h(\theta)), \tag{4.14}$$

where for any $|z| \leq \rho'$, R_z is the Taylor remainder of f expanded at z. Again, since f is analytic on $\rho > \rho'$ we have that $R_z(\eta)$ is analytic on a disk of radius $s = \rho - \rho'$. Let

$$E(\theta) = f[P_N(\theta)] - P_N(\Lambda\theta) \tag{4.15}$$

and note that E is an analytic N-tail by the assumption that P_N solves Equation 2.5 exactly to N-th order. Then using Equations 4.14 and 4.15 in Equation 2.5 we have a new operator equation in terms of h

$$h[\Lambda\theta] - Df[P_N(\theta)]h(\theta) = E(\theta) + R_{P_N}(h). \tag{4.16}$$

In order to re-write Equation 4.16 as a fixed point equaiton on $\mathbb{H}_N$, the Banach Space of all analytic N-tails from B into $\mathbb{C}^n$, consider the linear operator $\mathfrak{L} : \mathbb{H}_N \to \mathbb{H}_N$ defined by the left hand side of Equation 4.16. So for any $p, q \in \mathbb{H}_N$ we define $\mathfrak{L}[q]$ to be

$$\mathfrak{L}[q](\theta) = q[\Lambda\theta] - Df[P_N(\theta)]h(\theta),$$

and our first task is to study the equation $\mathfrak{L}[q] = p$. He have that

LEMMA 4.4. *Let C_1, C_2, μ_* and μ^* be validation values as in Definition 4.1. Suppose that N satisfies the assumption given by Equation 4.2 of Theorem 4.1. Then the linear operator $\mathfrak{L}$ is boundedly invertible on H_N, so that for any $p \in H$ there exists a unique solution to the equation*

$$\mathfrak{L}[q] = p.$$

Moreover we have the bound

$$\|\mathfrak{L}^{-1}\| \leq \frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^N}.$$

Using Lemma 4.4 we apply $\mathfrak{L}^{-1}$ to both sides of Equation 4.16 to see that if $P = P_N + h$ then

$$h = \mathfrak{L}^{-1}\left[E(\theta) + R_{P_N}[h(\theta)].\right]$$

Define the non-linear operator $\Phi : \mathbb{H}_N \to \mathbb{H}_N$ to be

$$\Phi(h) = \mathfrak{L}^{-1}\left[E(\theta) + R_{P_N}[h(\theta)]\right]. \tag{4.17}$$

The preceding discussion makes it clear that $P = P_N + h$ is an exact solution of Equation 2.5 if and only if h is a fixed point of Equation 4.17. What remains is to show that if the assumptions given by Equations 4.2, 4.3 and 4.4 are satisfied, then Φ admits a unique fixed point h. A natural strategy is to employ the Banach Contraction Mapping Theorem. In fact, as we will see in the next section, the assumptions given by Equations 4.3 and 4.4 are exactly the conditions which make Φ a local contraction near P_N.

LEMMA 4.5. *Under the hypotheses of Theorem 4.1 Φ is a contraction on the ball $U_\delta = \{h \in \mathbb{H}_N : \|h\|_\nu \leq \delta\}$. Hence there is a unique fixed point h of Φ on U_δ so that $P_N + h$ is an exact solution of Equation 2.5.*

Then Theorem 4.1 is true as soon as the lemmas are proved. Note that on an heuristic level, it is natural to expect that Φ is a contraction as E is a small constant (with respect to h), and R_{P_N} should depend "quadratically" on h.

4.3. Proofs of the Lemmas. Now we complete the proof of Theorem 4.1 by providing the proofs of the lemmas.

Proof of Lemma 4.4: Let p and q be bounded analytic N-tails on B_ν and consider the equation

$$\mathfrak{L}[q](\theta) \equiv q[\Lambda\theta] - Df[P_N(\theta)]q(\theta) = p(\theta). \tag{4.18}$$

If we let $\bar{p}(\theta) \equiv -Df[P_N(\theta)]^{-1}p(\theta)$ then this is equivalent to

$$q(\theta) - Df[P_N(\theta)]^{-1}q(\Lambda\theta) = \bar{p}(\theta),$$

which upon defining the linear operator

$$A[q](\theta) \equiv Df[P_N(\theta)]^{-1} q(\Lambda\theta)$$

becomes

$$(I - A)[q](\theta) = \bar{p}(\theta).$$

Now consider the norm

$$\|A\|_{\mathbb{H}_N} \equiv \sup_{\|\eta\|_\nu = 1} \|A[\eta](\theta)\|_\nu$$

$$= \sup_{\|\eta\|_\nu = 1} \|Df[P_N](\eta \circ \Lambda)\|_\nu$$

$$\leq \sup_{\|\eta\|_\nu = 1} (C_1 \mu_*^{-1} + C_2)|\Lambda|^{N+1} \|\eta\|_\nu$$

$$\leq \mu_*^{-1}(C_1 + \mu_* C_2)(\mu^*)^{N+1},$$

where we have used the bound from Equation 4.19 and Estimate 2 of Lemma 4.2. Then we apply the assumption given by Equation (4.2) of Theorem 4.1 and see that

$$\|A\|_{\mathbb{H}_N} \leq \frac{(C_1 + \mu_* C_2)(\mu^*)^{N+1}}{\mu_*} < 1.$$

It follows from the Neumann Theorem that $(I - A)$ is boundedly invertible, and that we have the bound

$$\|(I - A)^{-1}\|_{\mathbb{H}_N} \leq \sum_{i=0}^{\infty} \|A\|_{\mathbb{H}_N}^i = \frac{1}{1 - \frac{C_1(\mu^*)^{N+1}}{\mu_*}}.$$

From the bounded invertability of $(I - A)$ we obtain a unique solution to Equaiton 4.18 in the form

$$q(\theta) = (I - A)^{-1}[\bar{p}](\theta) = -(I - A)^{-1} Df[P_N(\theta)]^{-1} p(\theta).$$

Since p and q were arbitrary we have

$$\|\mathfrak{L}^{-1}\|_{\mathbb{H}_N} \leq \|(I - A)^{-1}\|_{\mathbb{H}_N} \|Df[P_N]^{-1}\|_{\Sigma,\nu}$$

$$\leq \frac{1}{1 - \frac{(C_1 + \mu_* C_2)(\mu^*)^{N+1}}{\mu_*}} (\mu_*^{-1} C_1 + C_2)$$

$$\leq \frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}},$$

as desired.

$\square$

Proof of Lemma 4.5: Since we hypothesized Equation 4.2, we can apply Lemma 4.4 and have that $\mathfrak{L}^{-1}$ is a well defined bounded linear operator. Then the operator

$$\Phi[h](\theta) \equiv \mathfrak{L}^{-1}\left[E(\theta) + R_{P_N(\theta)}[h](\theta)\right]$$

is well defined. To employ the Banach Fixed Point Theorem we must establish that when $U_\delta = \{h \in H_N : \|h\|_\nu \leq \delta\}$ is a δ-neighborhood in the space of analytic N-tails and δ satsfies the hypotheses of Theorem 4.1 and then

(i) Φ maps U_δ into itself.

(ii) there is a $0 < \kappa < 1$ so that for any $h_1, h_2 \in U_\delta$ one has $\|\Phi(h_1) - \Phi(h_2)\|_\nu \leq \kappa\|h_1 - h_2\|_\nu$.

In order to establish (i) we first note that for any $z, \eta \in \mathbb{C}^n$ with $|z| \leq \rho'$ and $|\eta| \leq s \equiv \rho - \rho'$ we have that

$$|R_z^j(\eta)| \leq N_f K_1 s^2$$

by straightforward application of the Lagrange Form of the Taylor Remainder to each of the $1 \leq j \leq n$ components of $R_z(\eta)$ (this estimate is carried out explicitly in [6] see Equaiton (75)). Then since $\|P_N\|_\nu \leq \|P_N\|_{\Sigma,\nu} \leq \rho'$ by by the definition of validation values (def 4.1) and $\delta < se^{-1} < s$ we have for each $\theta \in B_\nu$

$$|R_{P_N(\theta)}(h(\theta))|] \leq |R_z(h(\theta))| \leq \|R_z\|_\delta \leq \frac{\delta^2}{s^2}\|R_z\|_s \leq N_f K_1 \delta^2.$$

Then

$$\|\Phi(h)\|_\nu \leq \|\mathfrak{L}^{-1}\| \left(\|E\|_\nu + \|R_{P_N}(h)\|_\nu\right)$$

$$\leq \frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}} \left(\epsilon_{\text{tol}} + N_f K_1 \delta^2\right)$$

But

$$\frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}} \epsilon_{\text{tol}} \leq \frac{\delta}{2}$$

and

$$\frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}} N_f K_1 \delta^2 \leq \frac{\delta}{2},$$

as we see by applying the hypotheses given by Equations 4.3 and 4.4 respectively. Then Φ does in fact map into U_δ, as desired.

To establish (ii) we begin by considering the differential of the remainder term. Then let $\theta \in B_\nu$ and $z = P_N(\theta)$ and note that $|z| \leq \rho'$ (due to the definition of validation values, see Def (4.1)). Since $\delta < se^{-1} < s$ we choose a $0 < \sigma \leq 1$ and let $\omega = \delta/se^{-\sigma}$ so that for any and $h \in U_\delta$ we have the bound

$$\begin{aligned}
\|DR_z(h(\theta))\|_\delta &= \|DR_z \circ \omega\|_{se^{-\sigma}} \\
&\leq \omega \|DR_z\|_{se^{-\sigma}} \\
&\leq \frac{\delta}{se^{-\sigma}} \frac{2\pi n \sigma^{-1}}{s} \|R_z\|_s \\
&\leq \frac{2n\pi e^{\sigma} N_f K_1}{\sigma} \delta, \\
&\leq 2ne\pi N_f K_1 \delta.
\end{aligned}$$

Here we have used the Taylor Estimate of Lemma 4.2, the Cauchy Bounds of Estimate 4.3, the N-tail scaling estimate of Lemma 4.2, the fact that $\sigma^{-1}e^{\sigma}$ is minimized at $\sigma = 1$, and the assumption that that $\delta < e^{-1}s$.

Then for any $h_1, h_2 \in U_\delta$ we have

$$|R_z^j(h_1(\theta)) - R_z^j(h_2(\theta))| \leq 2ne\pi N_f K_1 \delta \|h_1 - h_2\|_\nu$$

by the mean value theorem. So

$$\begin{aligned}
\|\Phi(h_1) - \Phi(h_2)\|_\nu &= \left\|\mathfrak{L}^{-1}[E - R_{P_N}(h_1)] - \mathfrak{L}^{-1}[E - R_{P_N}(h_2)]\right\|_\nu \\
&= \|\mathfrak{L}^{-1}[R_{P_N}(h_1) - R_{P_N}(h_2)\|_\nu \\
&\leq \frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}} 2ne\pi N_f K_1 \delta \|h_1 - h_2\|_\nu \\
&\leq \kappa \|h_1 - h_2\|_\nu,
\end{aligned}$$

where

$$\kappa \equiv \frac{2ne\pi N_f (C_1 + \mu_* C_2) K_1}{[\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}]} \delta < 1,$$

as δ satisfies the hypothesis given by Equation (4.3) of Theorem (4.1).

$\square$

4.4. The Bounds C_1 and C_2 when f is polynomial. In this section we describe how to obtain the bounds on the non-constant matrix $Df[P_N(\theta)]^{-1}$ required in the definition of validation values. We focus on the case where f is a polynomial. This is the only part of the validation argument that makes the polynomial assumption. We note that if f is a general analytic function then we can use the Taylor expansion of f to obtain that f is polynomial plus a remainder as small as we wish. The argument given here can be modified to work in this more general case as well. We do not pursue the details here.

By the inverse function theorem we have

$$Df[P_N(\theta)]^{-1} = Df^{-1}[f \circ P_N(\theta)],$$

which can be used to compute an analytic expression for $Df[P_N]^{-1}$ as long as f^{-1} is known explicitly. Then we let

$$Df(x)^{-1} = \sum_{|\beta| \geq 0}^{M-1} B_\beta x^\beta$$

where each B_β is an $n \times n$ matrix, and M is the order of f. Recall also that

$$P_N(\theta) = \sum_{0 \leq |\alpha| \leq N} a_\alpha \theta^\alpha.$$

Then if $\bar{N} = N(M-1)$ we have that $Df[P_N(\theta)]^{-1}$ is an $\bar{N}$-th order polynomial with matrix coefficients. Then we let

$$Df[P_N(\theta)]^{-1} = \sum_{0 \leq |\alpha| \leq \bar{M}} C_\alpha \theta^\alpha$$

where the coefficients C_α, depend on the B_β and c_α, can be worked out via Cauchy Products.

Let $Q_0 \Sigma Q_0^{-1} = Df(p)$ be the eigenvector/eigenvalue decomposition of the differential and note that

$$C_0 = Df[P_N(0)]^{-1} = Df(p)^{-1} = Q_0^{-1} \Sigma^{-1} Q_0.$$

Then

$$\|Df[P_N]^{-1}\|_{\Sigma,\nu} \leq \left\| Q_0^{-1} \Sigma^{-1} Q_0 + \sum_{1 \leq |\alpha| \bar{M}} C_\alpha \theta^\alpha \right\|_{\Sigma,\nu}$$

$$\leq \|Q_0\| \|Q_0^{-1}\| \mu_*^{-1} + \sum_{|\alpha|=1}^{\bar{N}} \|C_\alpha\| \nu^{|\alpha|}.$$

Then we define C_1 and C_2 to be any bounds of the form

$$\|Q_0\|\|Q_0^{-1}\| \leq C_1,$$

and

$$\sum_{|\alpha|=1}^{\bar{N}} \|C_\alpha\|\nu^{|\alpha|} \leq C_2.$$

Note that since these expressions involve bounding finite sums of known quantities, both C_1 and C_2 are easily found using interval arethmetic. Finally we have that

$$\|Df[P_N]\|_{\Sigma,\nu} \leq C_1\mu_*^{-1} + C_2. \tag{4.19}$$

as needed in the definition of the validation values.

Of course if f is not a polynomial map it is possible to make a similar argument using at M-th order Taylor expansion by including a remainder term. This is a technicality not needed in the present work but which could be easily added to the scheme. In this case C_2 would simply have to incorporate as well the truncation error on the ball of radius ρ'.

5. Rigorous Computation of Transverse Homoclinic Orbits. Throughout this section we make the following definitions and assumptions.

- P1: Let $p \in \mathbb{R}^n$ be a hyperbolic fixed point of the analyticomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. Assume that $Df(p)$ is diagonalizable, and that n_s, $n_u > 0$, the number of stable and unstable eigenvalues respectively, have $n_u + n_s = n$.
- P2: Let P_N be the N-th order polynomial approximate parameterization of $W^u(p)$. In addition let ν_u, ϵ_u, C_1^u, C_2^u, ρ, ρ', and μ_*, μ^* be validation values for P_N. Assume that these validation values satisfy the hypotheses of Theorem (4.1) applied to f^{-1}, so that the is a unique analytic N-tail h with $\|h\|_{\nu_u} \leq \delta_u$ so that $P = P_N + h$ is a parameterization of $W^u_{\text{loc}}(p)$.
- P3: Similarly, let Q_N be the N-th order polynomial approximate parameterization of $W^s(p)$ and ν_s, ϵ_s, C_1^s, C_2^s, ρ, ρ', and μ_-, μ^+ be validation values for Q_N and assume that these validation values satisfy the hypotheses of Theorem 4.1 so that the is a unique analytic N-tail g with $\|g\|_{\nu_u} \leq \delta_s$ so that $Q = Q_N + g$ is a parameterization of $W^s_{\text{loc}}(p)$.

Then we can write the homoclinic functional equation (Equation 1.3) in the form

$$F(\theta, x_1, x_2, \ldots, x_{k-2}, x_{k-1}, \phi) =$$

$$\begin{bmatrix} f^{-1}(x_1) - P_N(\theta) - h(\theta) \\ f^{-1}(x_2) - x_1 \\ f^{-1}(x_3) - x_2 \\ \vdots \\ f^{-1}(x_j) - x_{j-1} \\ f(x_j) - x_{j+1} \\ \vdots \\ f(x_{k-2}) - x_{k-1} \\ f(x_{k-1}) - Q_N(\phi) - g(\phi) \end{bmatrix} \equiv F_N(\theta, x_1, \ldots, x_{k-1}, \phi) + H(\theta, \phi), \tag{5.1}$$

where again we stress that P_N and Q_N are explicitly know polynomials and h, and g are unknown analytic N-tails for which we have the mathematically rigorous bounds given in $P3$. We call F_N the *discretized homoclinic functional equation*.

Heuristically our validation scheme is as follows. Assume that there is $\hat{x} = (\hat{\theta}, \hat{x}_1, \ldots, \hat{x}_{k-1}, \hat{\phi}) \in \mathbb{R}^{nk}$ with $\hat{\theta} \in B_u^\circ$ and $\hat{\phi} \in B_s^\circ$ having that $\hat{x}$ is an approximate zero of the discretized homoclinic equation, i.e. assume that

$$\|F_N(\hat{x})\| \approx 0.$$

If in addition δ_s and δ_u are small, then we have that $\hat{x}$ is also an approximate zero of F, so that orbit$(\hat{x}_j)$ is approximately homoclinic to p for each $1 \leq j \leq k-1$. Our goal is to apply the Newton-Kantorovich Theorem (Thm 4.3) in order to conclude that there exists a true solution x_* of the full homoclinic functional equation near $\hat{x}$. These notions are formalized in the next section.

5.1. Validation of Homoclinic Connections. We now formalize the heuristic scheme just described. Assume, in addition to $P1$, $P2$ and $P3$, that we have computed, or are otherwise given, the following "quasi-local" data, which we refer to as *homoclinic validation values.*

Definition 5.1 (Homoclinic validation values). We say that the vector $\hat{x} = (\hat{\theta}, \hat{x}_1, \ldots, \hat{x}_{k-1}, \hat{\phi}) \in \mathbb{R}^{nk}$, and positive constants A_N, M_N, C_β, C_P, κ, $\hat{\delta}$, $\hat{\epsilon}$, and r are *validation values* for the homoclinic functional equation if the following conditions are met:

1. Define the point $x_0 \in \mathbb{R}^{nk}$ to be given by $x_0 = (0_{n_u}, p, \ldots, p, 0_{n_s})$ where p is the fixed point of f described in $P1-P3$ and 0_{n_u} and 0_{n_s} are the zero vectors in $\mathbb{R}^{n_u}$ and $\mathbb{R}^{n_s}$. Assume that x_0 is not in the poly-disk $B_r(\hat{x}) \subset \mathbb{R}^{nk}$.
2. $\hat{x} = (\hat{\theta}, \hat{x}_1, \ldots, \hat{x}_{k-1}, \hat{\phi}) \in \mathbb{R}^{nk}$ is an $\hat{\epsilon}$-approximate solution of $F = 0$, in the sense that
$$|DF_N(\hat{x})^{-1} F_N(\hat{x})| \leq \hat{\epsilon}.$$
3. $DF_N(\hat{x})$ is non-singular and the positive constant A_M has that $\|DF_N(\hat{x})^{-1}\|_M \leq A_N$.
4. $|\hat{\theta}| < \nu_u$ and $|\hat{\phi}| < \nu_s$ so that we can define what we will call *the first order loss of domain parameters*
$$\hat{\sigma}_s = -\ln\left(\frac{|\hat{\theta}|}{\nu_s}\right), \qquad \textit{and} \qquad \hat{\sigma}_u = -\ln\left(\frac{|\hat{\phi}|}{\nu_u}\right).$$

5. The positive constant M_N has that

$$\left(\max_{1\le i\le nk}\sum_{j=1}^{n}[DF_N^{-1}(\hat{x})]_{ij}\right)\frac{2\pi n_u}{\nu_u\hat{\sigma}_u}\delta_u+\left(\max_{1\le i\le nk}\sum_{j=nk-n+1}^{nk}[DF_N^{-1}(\hat{x})]_{ij}\right)\frac{2\pi n_s}{\nu_s\hat{\sigma}_s}\delta_s\le M_N.$$

6. The positive constant $\hat{\delta}$ has that

$$\left(\max_{1\le i\le nk}\sum_{j=1}^{n}[DF_N^{-1}(\hat{x})]_{ij}\right)\delta_u+\left(\max_{1\le i\le nk}\sum_{j=nk-n+1}^{nk}[DF_N^{-1}(\hat{x})]_{ij}\right)\delta_s.\le\hat{\delta}.$$

7. The parameters $\hat{\theta}$, $\hat{\phi}$ and the positive constant r also satisfy $|\hat{\theta}|+r<\nu_u$ and $|\hat{\phi}|+r<\nu_s$ so that we can define the *second order loss of domain parameters*

$$\sigma_s=-\ln\left(\frac{|\hat{\theta}|+r}{\nu_s}\right),\qquad \textit{and}\qquad \sigma_u=-\ln\left(\frac{|\hat{\phi}|+r}{\nu_u}\right).$$

8. The positive constant C_β has that

$$\max_{1\le j\le k-1}\ \max_{1\le i\le n}\ \max_{|\beta|=2}\left\{\|\partial^\beta f_i\|_{B_r(\hat{x}_j)},\|\partial^\beta f_i^{-1}\|_{B_r(\hat{x}_j)}\right\}\le C_\beta.$$

9. The positive constant C_P has

$$\max\left(\|D^2P_N\|_{B_r(\hat{\theta})}+\frac{2\pi^2n^2}{\nu_u^2\sigma_u^2}\delta_u,\quad \|D^2Q_N\|_{B_r(\hat{\phi})}+\frac{2\pi^2n^2}{\nu_s^2\sigma_s^2}\delta_s\right)\le C_P.$$

10. Finally, κ is positive constant having

$$N_fC_\beta+C_P\le\kappa,$$

where N_f is the max of the number of non-zero second partials of f and f^{-1}.

We sometimes write $C_\beta(r)$, $C_P(r)$ and $\kappa(r)$ to emphasize that these constants should be thought of as depending on the radius r of the $\mathbb{R}^{nk}$ poly-disk about $\hat{x}$. In other words they are the members of a validation values set which carry global information about the ball $B_r(\hat{x})\subset\mathbb{R}^{nk}$. In the next section we will prove the following *a-posteriori* result for F, which is based on a standard Newton-Kantorivich argument combined with the rigorous a-posteriori bounds on the parameterizations.

THEOREM 5.1 (A-posteriori validation of a homoclinic connection). *Given assumptions $[P1]-[P3]$ let $\hat{x}$, A_N, M_N, C_{beta}, C_P, κ, $\hat{\delta}$, $\hat{\epsilon}$, and r be a set of homoclinic validation values as in Def 5.1. We call ϵ_{NK} a "Newton-Kantorovich Epsilon" if*

$$\frac{1}{1-M_N}\left(\hat{\epsilon}+\hat{\delta}\right)\le\epsilon_{NK}. \tag{5.2}$$

With ϵ_{NK} fixed suppose that
A. $0<M_N<1$,
B. $2\epsilon_{NK}\le r$,
C. $\frac{A_N}{1-M_N}4\kappa\epsilon_{NK}\ \le\ 1.$

Then there is a unique $x_* \in B_r(\hat{x}) \subset \mathbb{R}^{nk}$ which is a non-trivial solution of the equation $F(x_*) = 0$. Such an x_* clearly has that

$$|x_* - \hat{x}| \leq r.$$

Moreover, if for all $x \in B_r(\hat{x}) \subset \mathbb{R}^{nk}$ we have both that $DF_N(x)^{-1}$ exists, and that

$$\|DF_N(x)^{-1} DH(x)\|_{M, B_r(\hat{x})} < 1, \tag{5.3}$$

then it follows that $W^s(p) \cap W^u(p)$, which is non-empty due to the existence of x_*, is also transverse.

5.2. Proof of Theorem 5.1. The proof consists of two parts. First we use Theorem 4.3 to show that the hypotheses of Theorem 5.1 combined with the definition of homoclinic validation values imply the existence of a non-trivial zero of F in $B_r(\hat{x})$. Then we study the form of the differential in order to establish the transversality. The subtly throughout is that while $F_N(\hat{x})$ and $DF_N^{-1}(\hat{x})$ are known, it is F and DF which must be explicitly bound.

In order to apply the Newton-Kantorovic Theorem (thm 4.3) we must show that
(i) $DF(\hat{x})$ has bounded inverse,
(ii) DF is Lipschitz on $B_r(\hat{x})$ with Lipschitz constant κ,
(I) $|DF(\hat{x})^{-1} F(\hat{x})| \leq \varepsilon_{NK}$,
(II) $\varepsilon_{NK} \leq r/2$,
(III) $4\varepsilon_{NK}\kappa \|DF(\hat{x})^{-1}\|_M \leq 1$.
Here the roman numerals refer to the nomenclature established in the statement of Theorem 4.3.

Let $[DF_N^{-1}(\hat{x})]_{(a:b)}$, with $a < b \in \mathbb{N}$, denote the submatrix of $DF_N^{-1}(\hat{x})$ composed of columns a through b. We begin by noting that

$$\begin{aligned} DF_N^{-1}(\hat{x}) DH(\hat{x}) &= DF_N^{-1}(\hat{x}) \begin{bmatrix} D_\theta h(\hat{\theta}) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & D_\phi g(\hat{\phi}) \end{bmatrix} \\ &= \left[[DF_N^{-1}(\hat{x})]_{(1:n)} Dh(\hat{\theta}) \mid 0 \mid \ \dots \ \mid 0 \mid [DF_N^{-1}(\hat{x})]_{(nk-n+1:nk)} Dg(\hat{\phi}) \right], \end{aligned}$$

so that

$$\begin{aligned} \|DF_N^{-1}(\hat{x}) DH(\hat{x})\|_M &\leq \left(n_u \max_{1 \leq i \leq nk} \sum_{j=1}^{n} [DF_N^{-1}(\hat{x})]_{ij} \right) \|Dh\|_{\nu_u e^{-\hat{\sigma}_u}} \\ &\quad + \left(n_s \max_{1 \leq i \leq nk} \sum_{j=nk-n+1}^{nk} [DF_N^{-1}(\hat{x})]_{ij} \right) \|Dg\|_{\nu_s e^{-\hat{\sigma}_s}} \\ &\leq M_N \\ &< 1, \end{aligned}$$

by part 5 of Definition 5.1, The Cauchy bounds of Lemma (4.3), and Assumption A of the present Theorem. It follows from the Neumann Series Theorem that the matrix $I + DF_N^{-1}(\hat{x})DH(\hat{x})$ is invertible and that

$$\left\| [I + DF_N^{-1}(\hat{x})DH(\hat{x})]^{-1} \right\|_M \leq \frac{1}{1 - M_N}. \tag{5.4}$$

Then we have that

$$\begin{aligned} DF(\hat{x})^{-1} &= [DF_N(\hat{x}) + DH(\hat{x})]^{-1} \\ &= \left[DF_N(\hat{x}) \ \left(I + DF_N(\hat{x})^{-1} DH(\hat{x}) \right) \ \right]^{-1} \\ &= [I + DF_N(\hat{x})^{-1} DH(\hat{x})]^{-1} \ DF_N(\hat{x})^{-1} \end{aligned} \tag{5.5}$$

exists, and obtain the bound

$$\|DF(\hat{x})^{-1}\|_M \leq \frac{A_N}{1 - M_N}. \tag{5.6}$$

This establishes (i) of Theorem 4.3.

In order to investigate the Lipschitz condition on the differential DF we define the real valued functions $g_{ij} : B_r(\hat{x}) \subset \mathbb{R}^{nk} \to \mathbb{R}$ where $1 \leq i, j \leq nk$ by the expressions

$$g_{ij}(z) = \partial_j F_i(z).$$

Then for $x, y \in B_r(\hat{x})$ we have that

$$\begin{aligned} |g_{ij}(x) - g_{ij}(y)| &\leq \|\nabla g_{ij}\|_{M, B_r(\hat{x})} |x - y| \\ &\leq \sum_{\ell=1}^{nk} \|\partial_\ell g_{ij}\|_{B_r(\hat{x})} |x - y| \\ &\leq \left(\sum_{\ell=1}^{nk} \|\partial_\ell \partial_j F_i\|_{B_r(\hat{x})} \right) |x - y|, \end{aligned} \tag{5.7}$$

by the Mean Value Theorem. Then

$$\begin{aligned} \|DF(x) - DF(y)\|_M \equiv & \sup_{\substack{v \in \mathbb{R}^{nk} \\ |v| = 1}} |[DF(x) - DF(y)]v| \\ &\leq \max_{1 \leq i \leq nk} \sum_{1 \leq j \leq nk} |[DF(x) - DF(y)]_{ij}| \\ &= \max_{1 \leq i \leq nk} \sum_{1 \leq j \leq nk} |\partial_j F_i(x) - \partial_j F_i(y)| \\ &\leq \left(\max_{1 \leq i \leq nk} \sum_{j=1}^{nk} \sum_{\ell=1}^{nk} \|\partial_\ell \partial_j F_i\|_{B_r(\hat{x})} \right) |x - y|, \end{aligned} \tag{5.8}$$

where we have used the estimate of Inequality 5.7.

Note that from 7 of Definition 5.1 and the Cauchy Bounds of Lemma 4.3 we have that for any $1 \le i \le n$

$$\begin{aligned}\|\partial_\ell \partial_j h_i\|_{B_r(\hat{x})} &= \|\partial_\ell \partial_j h_i\|_{B_r(\hat{\theta})} \\ &\le \|\partial_\ell \partial_j h_i\|_{\nu_u e^{-\sigma_u}} \\ &\le \frac{2\pi^2}{\nu_u^2 \sigma_u^2} \delta_u,\end{aligned}$$

and similarly

$$\|\partial_\ell \partial_j g_i\|_{B_r(\hat{x})} \le \frac{2\pi^2}{\nu_s^2 \sigma_s^2} \delta_s.$$

Using these estimates and considering the second partial derivatives of F one component at a time we recall 8, 9, and 10 of Definition 5.1 and obtain that

$$\max_{1 \le i \le nk} \sum_{j=1}^{nk} \sum_{\ell=1}^{nk} \|\partial_\ell \partial_j F_i\|_{B_r(\hat{x})} \le N_f C_\beta + C_P = \kappa.$$

Combining this with Inequality (5.8) gives (ii) of Theorem 4.3.

For (I) of Theorem 4.3 we use the notation $[DF_N^{-1}(\hat{x})]_{(a:b)}$ as above and have that

$$\begin{aligned}\left|DF_N^{-1}(\hat{x}) H(\hat{x})\right| &= \left| DF_N^{-1}(\hat{x}) \begin{bmatrix} h(\hat{\theta}) \\ 0 \\ \vdots \\ 0 \\ g(\hat{\phi}) \end{bmatrix} \right| \\ &= \left| [DF_N^{-1}(\hat{x})]_{(1:n)}\ h(\hat{\theta}) + [DF_N^{-1}(\hat{x})]_{(nk-n+1:nk)}\ g(\hat{\phi}) \right| \\ &\le \left(\max_{1 \le i \le nk} \sum_{j=1}^{n} \left| [DF_N^{-1}(\hat{x})]_{ij} \right| \right) \|h\|_{\nu_u} \\ &+ \left(\max_{1 \le i \le nk} \sum_{j=nk-n+1}^{nk} \left| [DF_N^{-1}(\hat{x})]_{ij} \right| \right) \|g\|_{\nu_s} \\ &\le \hat{\delta}, \end{aligned} \tag{5.9}$$

where we have used 6 of Definition 5.1. Then, recalling Equation 5.5 and Inequality 5.6 we have

$$\begin{aligned}
\left|DF(\hat{x})^{-1}F(\hat{x})\right| &\leq \left|[I + DF_N(\hat{x})^{-1}DH(\hat{x})]^{-1}\ DF_N(\hat{x})^{-1}F(\hat{x})\right| \\
&= \left|[I + DF_N(\hat{x})^{-1}DH(\hat{x})]^{-1}\ DF_N(\hat{x})^{-1}\left(F_N(\hat{x}) + H(\hat{x})\right)\right| \\
&\leq \frac{1}{1-M_N}\left(\left|DF_N^{-1}(\hat{x})F_N(\hat{x})\right| + \left|DF_N^{-1}(\hat{x})H(\hat{x})\right|\right) \\
&\leq \frac{1}{1-M_N}\left(\hat{\epsilon} + \hat{\delta}\right) \\
&\leq \epsilon_{NK}, \qquad (5.10)
\end{aligned}$$

where we have used 2 of Definition 5.1, the Estimate given by Inequality 5.9, and the the defintion of ϵ_{NK} given by Equation 5.2. This establishes condition (I) of Theorem 4.3. Finally note that (III) of Theorem 4.3 follows directly from assumption C of the present theorem and Inequality 5.6, while (II) of Theorem 4.3 is assumption B of the present Theorem.

Then the conditions of Theorem 4.3 are satisfied and we obtain the existence of a unique $x_* \in B_r(\hat{x})$ so that $F(x_*) = 0$. Note that since $x_* \neq x_0$ by 1 of Definition 5.1, we obtain a non-trivial homoclinic orbit.

Now we turn to the question of transversality of the intersection at x_*. An argument similar to the one used to derive Equation 5.5, except with $\hat{x}$ replaced by a variable $x \in B_r(\hat{x})$ shows that $DF(x)$ is invertible for all $x \in B_r(\hat{x})$ as long as $DF_N(x)$ is invertible for all $x \in B_r(\hat{x})$ and the condition given by Equation 5.3 is met. Since we have assumed that both of these conditions are met, it follows that $DF(x_*)$ is non-singular.

What remains is to show is that the non-singularity of $DF(x_*)$ implies that the homoclinic orbit is transverse. Assume for the moment that $k = 1$, so that the local manifolds $W^u_{\text{loc}}(p) = P[B_{\nu_u}(0)]$ and $W^s_{\text{loc}}(p) = Q[B_{\nu_s}(0)]$ intersect at x_*. In this case the operator F reduces to

$$F(\theta, \phi) = P(\theta) - Q(\phi).$$

and we have a solution $x_* = (\theta_*, \phi_*) \in B_r(\hat{x})$. Since $DF(x_*)$ is non-singular, the columns of

$$DF(x_*) = [D_\theta P(\theta_*)| - D_\phi Q(\phi_*)]$$

span $\mathbb{R}^n$. But the columns of $D_\theta P(\theta_*)$ and $D_\phi Q_{\phi_*}$ span $T_{P(\theta_*)}W^u(p)$ and $T_{Q(\phi_*)}W^s(p)$ respectively. It follows that $T_{P(\theta_*)}W^u(p)$ and $T_{Q(\phi_*)}W^s(p)$ span $\mathbb{R}^n$, which is to say that x_* is a point of transverse intersection.

Now suppose $K > 1$, and $x_* \in \mathbb{R}^{nk}$ is the solution of $F = 0$. Since any f-iterate of a local unstable manifold is again a local unstable manifold, and any f-iterate of a homoclinic point is another homoclinic point, we have that the local unstable manifold $f^k[W^u_{\text{loc}}(p)] = f^k[P(B_{n_u}(0))]$ intersects $W^s_{\text{loc}}(p) = Q[B_{\nu_s}(0)]$ at the phase space point $Q(\phi_*) = f^k[P(\theta_*)]$. Then we are in exactly the same situation as above, and the intersection is transverse if and only if the matrix

$$[-D_\theta f^k[P(\theta_*)]\,|D_\phi Q(\phi_*)] = [-D_x f^k[P(\theta_*)]D_\theta P(\theta_*)|D_\phi Q(\phi_*)]$$

is non-singular. Note that $D_x f^k(x)$ is non-singular for any $x \in \mathbb{R}^n$ as f is a diffeomorphism.

Now, by hypothesis the matrix

$$DF(x_*) = \begin{pmatrix} -D_\theta P(\theta_*) & & 0 \\ \vdots & \mathbf{A} & \vdots \\ 0 & & -D_\phi Q(\phi_*) \end{pmatrix},$$

is non-singular, so that if we construct the non-singular matrix

$$\mathbf{B} = \begin{pmatrix} D_x f^k[P(\theta_*)] & \mathbf{0} \\ \mathbf{0} & Id_{n(k-1)\times n(k-1)} \end{pmatrix}$$

and multiply, we have that the product

$$\mathbf{B}DF(x_*) = \begin{pmatrix} -D_x f^k[P(\theta_*)]D_\theta P(\theta_*) & & 0 \\ \vdots & \mathbf{C} & \vdots \\ 0 & & -D_\phi Q(\phi_*) \end{pmatrix}$$

is the product of non-singular matrices hence is itself non-singular (here the actual form of $\mathbf{C}$ is unimportant to us). Since $\mathbf{B}DF(x_*)$ is non-singular, it has linearly independent columns. Exploiting this linear independence gives that the columns of

$$[-D_\theta f^k[P(\theta_*)]\,|D_\phi Q(\phi_*)] = [-D_x f^k[P(\theta_*)]D_\theta P(\theta_*)|D_\phi Q(\phi_*)],$$

span $\mathbb{R}^n$, which is to say that the local manifolds $W^s_{\text{loc}}(p) = Q[B_{\nu_s}(0)]$ and $W^u_{\text{loc}}(p) = f^k[P(B_{\nu_u}(0)]$ intersect transversally, as desired.

$\square$

6. Computer Assisted Proofs of Transverse Homoclinic Orbits and Chaos. We begin by considering a Lomelí Map with parameters $a = 0.5$, $b = -0.5$, $c = 1$, $\alpha = -5.34$, and $\tau = 0.8$. These correspond to Dullin-Meiss parameters of $\bar{a} = 1$, $\bar{b} = 0.5$, $\bar{c} = 0.5$, $\mu = -2.4$ and $\epsilon = 5.5$. For these parameters values there is a hyperbolic fixed point at $p = (x_-, x_-, x_-)$ with $x_- = -2.745207879911715$. Then $Df(p)$ has unstable complex conjugate eigenvalues $-0.402451645443971 \pm i2.035392592347574$ and stable eigenvalue 0.232299350932085. Table 6 illustrates the results of the parameterization computations, which are carried out using the rigorous interval arithmetic library IntLab (which runs under Matlab).

The table records the dimension of the manifolds, the approximation order N used in each case, the time taken to compute the coefficients of the polynomial approximations P_N and Q_N, the time taken to a-posteriori validated the approximations, the magnitudes of the resulting bounds on the truncation errors $\|h\|_{\nu_u} = \delta_u$ and $\|g\|_{\nu_s} = \delta_s$, the size of the parameter domain radii ν_u and ν_s, the size of the eigenvector scaling, and finally a rigorous bound on the size of the local manifolds in the sigma-norm.

Dim	Order	Approx Time	Valid Time	Validated Error	Radius	$\lvert\xi\rvert$	$\lVert \cdot \rVert_{\nu,\Sigma}$
1	50	5.16 sec	0.40 sec	8.71×10^{-13}	0.9	2	1.96
2	25	1.68 min	2.84 sec	5.67×10^{-12}	0.4	1.5	1.21

TABLE 6.1
Manifold Validation Performance: Example 1 ($\epsilon = 5.5, \mu = -2.4$)

K	$\hat{x}_1$	r
6	$(-1.648314148155201, -3.605864990373435, -2.750773367689280)$	1.1×10^{-11}
6	$(-1.692334813290302, -3.652591337627915, -2.718741184627647)$	1.06×10^{-11}

TABLE 6.2
Primary Intersection Validation ($\epsilon = 5.5, \mu = -2.4$): 3.21 sec for proof of both orbits. Chaos confirmed in both cases.

We then use a classical, numerical Newton scheme to find an approximate numerical solution to the discretized homoclinic functional equaiton $F_N(x) = 0$ with $k = 6$ and of course $n = 3$. This leads to an approximate zero

$$\hat{x} = \begin{bmatrix} \hat{\theta} \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{x}_5 \\ \phi \end{bmatrix} = \begin{bmatrix} (-0.337379322019076, 0.088431234641040) \\ (-1.648314148155201, -3.605864990373435, -2.750773367689280) \\ (1.979508268106647, -1.648314148155201, -3.605864990373435) \\ (-1.054666610773029, 1.979508268106647, -1.648314148155201) \\ (-2.313572985270695, -1.054666610773029, 1.979508268106647) \\ (-2.642742570718999, -2.313572985270695, -1.054666610773029) \\ 0.228218016117584 \end{bmatrix}$$

Using Theorem 5.1 we can validate that there is a true solution of the homoclinic functional equation in a polydisk $B_r(\hat{x})$ with $r = 1.1 \times 10^{-11}$. Table 6 gives computation data for the proof just described, and also for the proof of a second distinct solution of the homoclinic operator equation for $k = 6$. In each case only the $\hat{x}_1$ data is recorded. Figure 6.1 shows the time series data for the x component of the first of these two orbits. Black dots represent points in $\hat{x}$. Red points represent iterates on the local manifolds.

We note that in these first two proofs is that the time taken to compute the rigorous interval enclosures of the coefficients for the two variable polynomial P_N is 1 minute 68 seconds, while the validation of the two homoclinic orbits takes only 3.21 seconds. Since we can use the same polynomial approximations P_N and Q_N in any homoclinic functional equation, regardless of the size of k, we compute 32 more distinct homoclinic orbits with k varying. The results are tabulated in Table 6, and again only $\hat{x}_1$ components are recorded. Note that the time required to validate all 34 of orbits is a little less than the time needed to compute the rigorous approximation of the stable manifold. This suggests that high order approximation of the manifolds is most useful when computing many distinct homoclinic orbits at a given parameter set. Figures 6.2 and 6.3 show time series data for the x-component of the shortest and longest homoclinic orbits validated.

We note that in the previous example the dynamics is “fast” in the sense that as few as 6 iterates are needed in order to transition from the local unstable to the local stable manifold. In order to compute orbits with longer ‘time of flight’ (higher k) we consider a Lomelí map with parameters a, b, c, and τ as before, but with $\alpha = -0.04$.

K	$\hat{x}_1$	r	time
8	$(-1.878269557294666 - 3.704360821688669 - 2.644177124855255)$	1.0×10^{-11}	3.13 sec
·	$(-1.598486534326447 - 3.712394711133192 - 2.715338895232408)$	1.1×10^{-11}	·
9	$(-1.693365888596068 - 3.516449414154529 - 2.776271298390562)$	1.05×10^{-11}	4.95 sec
·	$(-2.033965491062911 - 3.691036738831221 - 2.607784382423848)$	1.05×10^{-11}	·
·	$(-3.649752275192224 - 2.876479215542708 - 2.487231377447373)$	1.00×10^{-11}	·
11	$(-1.724921906236488 - 3.503098391735548 - 2.773685700840596)$	1.06×10^{-11}	10.1 sec
·	$(-2.089900084565888 - 3.686144425839955 - 2.594568562106802)$	1.0×10^{-11}	·
·	$(-3.634873256589227 - 2.906134859387798 - 2.482573305537549)$	1.0×10^{-11}	·
·	$(-3.620917995724487 - 2.915222901577827 - 2.483676082433866)$	1×10^{-11}	·
·	$(-2.114585182128023 - 3.679401143907701 - 2.591096188024756)$	1.03×10^{-11}	·
·	$(-1.768297176557754 - 3.496683655844906 - 2.765421447288818)$	1.05×10^{-11}	·
12	$(-1.613946132963925 - 3.601054205346514 - 2.761528716808955)$	1.1×10^{-11}	6.8 sec
·	$(-1.672093712060165 - 3.527103879468739 - 2.777334962197874)$	1.06×10^{-11}	·
·	$(-2.122097145983667 - 3.674130503709248 - 2.591708802528494)$	1.04×10^{-11}	·
·	$(-1.822510500455057 - 3.571768208555173 - 2.720873332899369)$	1.1×10^{-11}	·
13	$(-3.644121861531430 - 2.872709464400592 - 2.489856336907984)$	1×10^{-11}	10.35 sec
·	$(-1.720320939862523 - 3.656391590596805 - 2.709687450800172)$	1.0×10^{-11}	·
·	$(-1.972320520664557 - 3.693712699582179 - 2.623618282117915)$	1.0×10^{-11}	·
·	$(-3.647170226591191 - 2.867372482172479 - 2.490553201321108)$	1×10^{-11}	·
·	$(-1.582489566947040 - 3.527839146851471 - 2.799122415227350)$	1.07×10^{-11}	·
·	$(-1.574224069064366 - 3.529792848481951 - 2.800384605320380)$	1.1×10^{-11}	·
20	$(-1.931148725862011 - 3.707646666557216 - 2.627909579872393)$	1.0×10^{-11}	4.01 sec
·	$(-3.638326627639060 - 2.901176380906034 - 2.483107270172258)$	1.0×10^{-11}	
21	$(-3.690719490424216 - 2.823690393936636 - 2.490880594199791)$	1×10^{-11}	16.44 sec
·	$(-1.957194765763665 - 3.705297800511473 - 2.621878341459779)$	1.05×10^{-11}	·
·	$(-1.729640666364290 - 3.510087223951199 - 2.769773329564600)$	1.06×10^{-11}	·
·	$(-1.690639165363386 - 3.669844178437995 - 2.711227287037635)$	1.1×10^{-11}	·
·	$(-1.950380561442004 - 3.705777860019821 - 2.623527619587280)$	1.1×10^{-11}	·
·	$(-3.702924845715120 - 2.791265552326865 - 2.496202529149488)$	1.0×10^{-11}	·
·	$(-3.708117158393551 - 2.774277602263187 - 2.499173703371658)$	0.98×10^{-11}	·
·	$(-1.932029291989042 - 3.707691973193318 - 2.627641932935536)$	1.04×10^{-11}	·
·	$(-3.616786394029812 - 2.918973341522530 - 2.483676503055390)$	1×10^{-11}	·

TABLE 6.3

Secondary Homoclinic Orbits ($\epsilon = 5.5, \mu = -2.4$): Total Time for Proofs: 55.0 sec. Transversality confirmed in all cases.

This corresponds to a Dullin-Meiss value of $\epsilon = 0.2$ with all other parameters as above. At these parameter values we study the fixed point $p = (x_-, x_-, x_-)$ with $x_- = -0.847213595499957$. The differential $Df(p)$ has unstable complex conjugate eigenvalues of $-0.150742620101308 \pm i1.205183554810613$ and a stable eigenvalue of 0.677878442452638. The data for the parameterization computations is given in Table 6, with format identical to before. Table 6 gives data for the results of the homoclinic validation computations for five different orbits with values of k varying between 75 and 121. Figures 6.3 and 6.4 show time series data for the shortest and longest of these homoclinic orbits (x-component in both cases). Note that for the orbit with $k = 121$ the discretized homoclinic functional equation $F_N : \mathbb{R}^{nk} \to \mathbb{R}^{nk}$ has $nk = 121 \times 3 = 363$.

Finally we cary out a similar computation for the map $G : \mathbb{R}^6 \to \mathbb{R}^6$ obtained

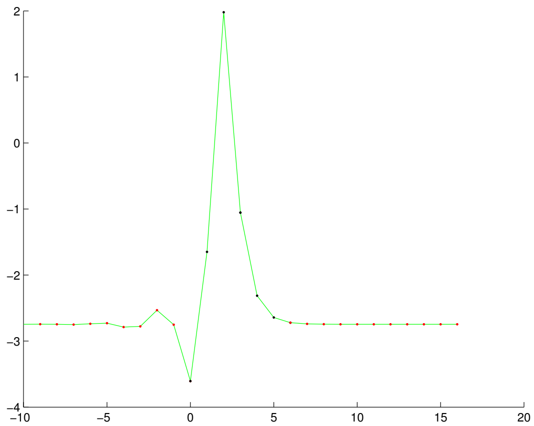

FIG. 6.1. *x-axis projection of the validated homoclinic; $k = 6$, $\epsilon = 5.5$.*

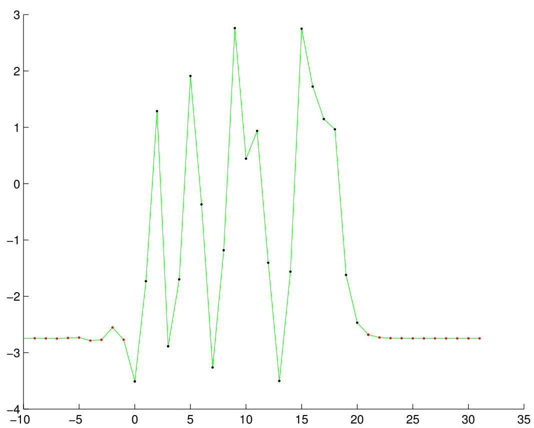

FIG. 6.2. *x-axis projection of the validated homoclinic; $k = 21$, $\epsilon = 5.5$.*

by a coupling a pair of Lomelí maps as discussed in Section 2.2. We take parameters $a_1 = a_2 = 0.5$, $b_1 = b_2 = -0.5$, $c_1 = c_2 = 1$, $\tau_1 = \tau_2 = 0.8$, $\alpha_1 = -5.339999999999998$ and $\alpha_2 = -5.939999999999998$ (corresponding to Dullin-Meiss parameters of $\epsilon_1 = 5.5$ and $\epsilon_2 = 6.1$). The maps are coupled with a strength of $\varepsilon = 5 \times 10^{-7}$. The reason for the small coupling strength is that we obtain a numerical guess by continuing away from the product system having $\varepsilon = 0$. The coupled system is quite sensitive to this parameter, and a tangency develops for coupling strengths much larger than this. However our proof does not in any way depend on the use of the small parameter, other than that it is helpful for locating an initial guess for a homoclinic in coupled system. We have made no attempt at an exhaustive study of the six dimensional system. The coupled system only serves to illustrate that the computations go through in higher dimensions.

We study the fixed point $p = (x_-^1, x_-^1, x_-^1, x_-^2, x_-^2, x_-^2)$ with $x_-^1 = -2.74507879911714$

Dim	Order	Approx Time	Proof Time	Validated Error	Radius	$\lvert\xi\rvert$	$\lVert\cdot\rVert_\nu$
1	50	4.95 sec	0.45 sec	2.71×10^{-11}	0.9	1.5	5.63
2	25	1.66 min	2.94 sec	4.30×10^{-13}	0.4	0.5	0.32

TABLE 6.4
Manifold Validation Performance: Example 2 ($\epsilon = 0.2, \mu = -2.4$)

K	$\hat{x}_1$	r	time
75	$(-0.717248519714197 - 1.043252947479510 - 0.860812112677259)$	1.04×10^{-7}	6.32 sec
76	$(-1.107394504655081 - 0.745731963636135 - 0.642025567084575)$	1.4×10^{-7}	6.15 sec
111	$(-1.104148108665029 - 0.729631044649217 - 0.648872760710501)$	1.05×10^{-7}	15.04 sec
118	$(-1.087535686140795 - 0.715568561563514 - 0.669111490970251)$	1.3×10^{-7}	16.11 sec
121	$(-0.995810895350469 - 0.972045779061998 - 0.671276957464922)$	1.04×10^{-7}	18.6 sec

TABLE 6.5
Homoclinic Orbits ($\epsilon = 0.2, \mu = -2.4$): Transversality confirmed in all cases.

and $x_-^2 = -2.869817807045693$. The differential $DG(p)$ has two pair of unstable complex conjugate eigenvalues $-0.428678184042694 \pm i2.076458156435394$ and $-0.402451645448668 \pm i2.035392592342751$, and a pair of real distinct stable eigenvalues 0.232299350933555 and 0.222447464570467. Then fixed point has a four dimensional unstable manifold and a two dimensional stable manifold. We show that these manifolds intersect transversally using the arguments developed above. The results of the computer assisted proofs are recorded in Tables 6.6 and 6.7. Note that since we are only doing one proof, we use lower order approximations and smaller parameter domains. In fact we choose the lowest order for the four dimensional manifolds allowed by the non-resonance condition. This helps minimize the cost, in seconds, of computing the higher dimensional manifold.

7. Conclusions.

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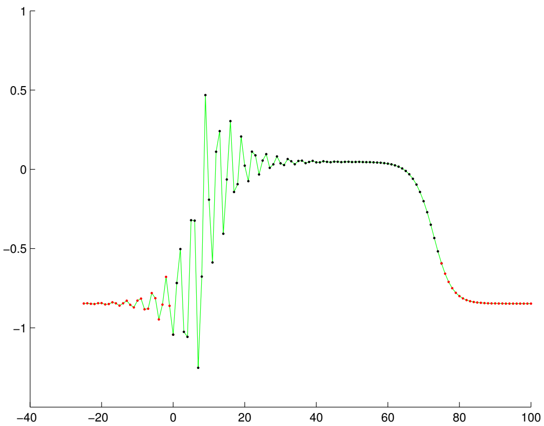

FIG. 6.3. *x-axis projection of the validated homoclinic;* $k = 75$*,* $\epsilon = 0.2$

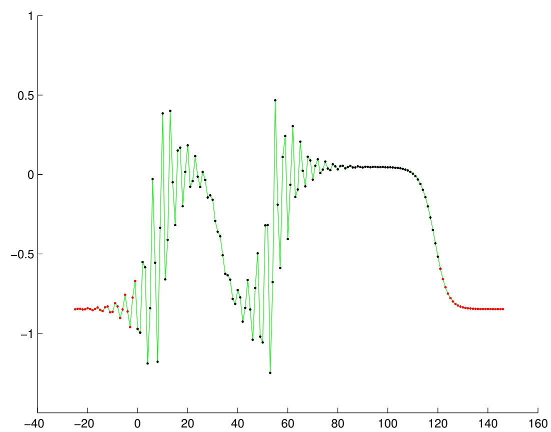

FIG. 6.4. *x-axis projection of the validated homoclinic;* $k = 121$*,* $\epsilon = 0.2$

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Dim	Order	Approx Time	Proof Time	Validated Error	Radius	$\lvert\xi\rvert$	$\Vert \cdot \Vert_\nu$
2	9	5.1 sec	0.43 sec	4.95×10^{-13}	0.001	1.4	1.46×10^{-3}
4	5	11.3 sec	1.5 sec	2.96×10^{-12}	0.001	0.4	7.1×10^{-4}

Table 6.6

Manifold Validation Performance: Six Dimensional Example.

K	$\hat{x}_1$	r	time
20	$\begin{pmatrix} -2.743916182731272 \\ -2.745285611304841 \\ -2.745493386071762 \\ -2.868750115142552 \\ -2.869921042268439 \\ -2.870035612699910 \end{pmatrix}$	1.49×10^{-11}	4.4 sec

Table 6.7

Primary Intersection Validation: Six Dimensional Example.Chaos confirmed. The similarity in the coordinates of x_1 is due to the fact that the orbit begins close to the fixed point.

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