RIGOROUS A-POSTERIORI ERROR BOUNDS FOR POLYNOMIAL APPROXIMATIONS TO STABLE/UNSTABLE MANIFOLDS OF FIXED POINTS AND ANALYTIC SHADOWING OF HOMOCLINIC ORBITS FOR DISCRETE TIME DYNAMICAL SYSTEMS

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Abstract. We present a new computer assisted proof scheme for rigorously establishing the existence of transverse homoclinic orbits in discrete time dynamical system. The novelty of the scheme is that it requires only a Newton-Kantorovich argument *in finite dimensions*. In order to apply the Newton-Kantorivich Theorem in this setting it is necessary to obtain explicit rigorous bounds on the truncation errors in the numerical approximation of the stable/unstable manifolds, as well as bounds on the first and second derivatives of the truncation errors. These bounds are the main technical contribution of the present work. We note that the bounds are obtained in the analytic category, so that the scheme provides a kind of "analytic shadowing" theorem for homoclinic orbits. Since the estimates of the truncation errors and their derivatives are interesting and potentially useful outside the context of computer assisted proof for connecting orbits, we give a complete and self-contained exposition. The main tool in these arguments is the so called *Parameterization Method* for invariant manifolds. We present applications of our numerical scheme for example systems in dimensions three and six.

1. Introduction. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a real analytic mapping with real analytic inverse and that p is a hyperbolic saddle point for f. Let $n_s, n_u \in \mathbb{N}$ denote respectively the dimension of the stable and unstable eigenspaces of $Df(p)$, and note that $n_s + n_u = n$. It follows from the stable manifold theorem [29] that there are $\nu_s, \nu_u, > 0$ and analytic chart maps

$$P : B_{\nu_u}(0) \subset \mathbb{R}^{n_u} \to \mathbb{R}^n \qquad \text{and} \qquad Q : B_{\nu_s}(0) \subset R^{n_s} \to \mathbb{R}^n$$

for the local unstable and stable manifolds at p, so that

$$P[B_{\nu_u}(0)] = W^u_{\text{loc}}(p) \qquad \text{and} \qquad Q[B_{\nu_s}(0)] = W^s_{\text{loc}}(p).$$

Define the *homoclinic operator equation* $F : \mathbb{R}^{nk} \to \mathbb{R}^{nk}$ by

$$F(\theta, x_1, x_2, \ldots, x_{k-2}, x_{k-1}, \phi) = \begin{bmatrix} f^{-1}(x_1) - P(\theta) \\ f^{-1}(x_2) - x_1 \\ f^{-1}(x_3) - x_2 \\ \vdots \\ f^{-1}(x_j) - x_{j-1} \\ f(x_j) - x_{j+1} \\ \vdots \\ f(x_{k-2}) - x_{k-1} \\ f(x_{k-1}) - Q(\phi) \end{bmatrix} \tag{1.1}$$

where $\theta \in \mathbb{R}^{n_u}$, $\phi \in \mathbb{R}^{n_s}$, $x_i \in \mathbb{R}^n$ for each $1 \leq i \leq k-1$, and $1 < j < k-1$ (so that x_j is a point whose inverse iterates lie on the local unstable manifold, and whose forward

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iterates lie on the local stable manifold). Let $\tilde{x} = (\tilde{\theta}, \tilde{x}_1, \ldots, \tilde{x}_{k-1}, \tilde{\phi})$ denote a zero of F, then $O = \{P(\tilde{\theta}), \tilde{x}_1, \ldots, \tilde{x}_{k-1}, Q(\tilde{\phi})\}$ is an orbit segment which begins on the local unstable manifold of p and ends, after k iterates, on the local stable manifold of p. It follows that orbit(q) is homoclinic to p for any $q \in O$.

Now, if P_N and Q_N are polynomial approximations of the chart maps P and Q, then one defines F_N in analogy with Equation 1.1 by replacing the the exact chart maps with their polynomial approximations. Numerically solving $F_N(x) = 0$ using a Newton Scheme leads to the *method of projected boundary conditions* of Beyn and Kleinkauf [7, 8], and enables fast and accurate numerical computation of homoclinic orbits.

Suppose that $\hat{x}$ is an approximate zero of F_N (perhaps, but not necassarily, computed numerically using the method of projected boundary conditions just described). Then it is natural to try to invoke the Newton-Kantorivich Theorem (thm 4.3) in order to prove that there is an exact zero $\tilde{x}$ of F (Equation 1.1) near $\hat{x}$. This approach to computer assisted proof of the existence of a connecting orbit presents the following difficulties; one must obtain

(i) rigorous bounds on the truncation errors in the approximations $P \approx P_N$ and $Q \approx Q_N$,
(ii) rigorous bounds on the derivative of the truncation errors at the approximate solution $\hat{x}$,
(iii) rigorous uniform bounds on the second derivative of the truncation errors in a neighborhood of the approximate solution $\hat{x}$.

To overcome difficulties (i), (ii), (iii), and implement a computer assisted version of the Newton-Kantorivich argument for the homoclinic operator equation (Equation 1.1) is the main goal of the present work. The tool which we use in order to obtain the necessary bounds is the so called *Parameterization Method* of Cabré, de la Llave, and Fontich [9, 10, 11]. By making a small modification to the arguments in [9, 11] we are able to prove the existence of an analytic function which represents the truncation error and obtain explicit bounds on this function. Then we obtain the necessary bounds on the first and second derivatives of the truncation function using the Cauchy Bounds of KAM theory.

The remainder of the paper is organized as follows. In Section 2 we discuss the background material used throughout the present work. We begin in Section 2.1 with a brief discussion of the computer assisted proof literature for existence of homoclinic orbits in discrete time dynamical systems. In Section 2.2 we establish the definitions and notation which will be used throughout the paper. In Section 2.3 we introduce the main class of examples, the Lomelí map, which we will work with later in the applications section.

In Section 3 we review the basic notions of the Parameterization Method for Stable and Unstable manifolds of fixed points of a diffeomorphism f. We illustrate in some detail the formal computation of the powerseries coefficients for the the one dimensional stable and unstable manifolds of the Lomelí map, explain how the computations generalize in higher dimensions, and discuss numerical aspects of the method.

Section 4 is devoted to the proof of Theorem 4.1, the main technical result of the present work. The section is organized as follows. In Section 4.1 we review the functional analytic and complex variables theory which is needed for the proof of Theorem 4.1, and in Section 4.2 we sketch the proof while introducing a series of Lemmas. In Seciton 4.3 we prove the lemmas in order to complete the proof of Theorem 4.1. Section 4.4 shows how to obtain one of the bounds in the hypothesis of

Theorem 4.1 if the case that f is polynomial.

In Section 5 we apply the a-posteriori estimates of Theorem 4.1 to the Newton-Kantorovich problem associated with zeros of Equation 1.1. The main result is Theorem 5.1; our analytic shadowing theorem. The proof of Theorem 5.1 is a straight forward application of the Newton-Kantorovich theorem and is given in Section 5.2

In Section 6 we present the results of several computer assisted proofs of the existence of transverse homoclinic orbits in the three dimensional Lomelí Map. Here the stable and unstable manifolds are one and two dimensional respectively. We provide examples of the use of high order approximations to the manifold (useful when proving the existence of many distinct homoclinic orbits at a single parameter set) and low order approximation of the manifold (useful when continuing a single orbit over a range of parameters). In order to demonstrate that the algorithms can be applied in dimensions higher than three, we also provide a six dimensional example computation for a pair of coupled Lomelí Maps. Here the proof involves establishing the existence of a transverse homoclinic orbit in the intersection of a four dimensional unstable manifold and a two dimensional stable manifold.

2. Background.

2.1. Relation to the Existing Literature. In 1965 Smale showed that non-degenerate connecting orbits give rise to complicated behavior in discrete time dynamical systems [47]. Since then substantial effort has been directed toward the the dual problems of using computers to (i) detect, and (ii) prove the existence of transverse connecting orbits and complicated/'chaotic' behavior in specific nonlinear dynamical systems. The present work focuses on (ii); using the computer to prove the existence of transverse homoclinic orbits, once a suitable numerical approximation has been found.

We mention only the work of [7, 8] on numerical computation of approximate homoclinic orbits (as this work is closely related to ours) and then take for granted the entire classical numerical literature. However we will attempt a brief survey of existing methods for computer assisted proof of connecting/horseshoe dynamics for discrete time dynamical systems. We focus on so called *a-posteriori* methods of proof. These are methods which allow one to conclude from the existence of a "good enough" numerical approximation of an orbit, that there exists a true orbit nearby. We also give a brief discussion of the parameterization method literature, as this is the main tool which we use in order to control the local stable and unstable manifolds in the remainder of the paper.

C^0 **A-Posteriori Techniques for Topological Horseshoes:** There exist several computer assisted proof schemes which make use of only topological information and pass directly from floating point or combinatorial approximations of connecting orbits to the existence of horseshoe dynamics. These methods bypass the question of whether or not connecting orbits between fixed/periodic points actually exist. The methods can be classified in terms of how the phase space near the approximate connecting orbit is represented. This choice of representation will in turn influence which topological tools are used to give the a-posteriori results.

For example, if the phase space is discretized by combinatorial complexes (simplicial or cubical) then it is natural to use theorems of combinatorial topology in the a-posteriori analysis. The Discrete Conley Index is a powerful tool in this setting, and was used for example in [39, 40] to prove the existence of horseshoe dynamics for a Poincare section of the Lorenz system. [49] shows how to obtain a-posteriori verification of the existence of a horseshoe from the existence of a combinatorial approximation to a connecting orbit in a quite general setting. The arguments make use of a Lefschitz fixed point theorem for topological index pairs. These methods were used recently in [19] to obtain entropy bounds in the Hénon map.

On the other hand it is sometimes desirable to discretize the phase space by parallelograms which are aligned with the expanding and contracting directions of the system. [23, 24] have developed an a-posteriori technique based on covering relations in order to prove the existence of horseshoe dynamics. The a-posteriori argument uses the notion of local Brower degree. This method is exploited for example in [2, 3] in order to establish chaotic dynamics in the Restricted Three Body Problem and the Hénon-Heiles Hamiltonian respectively. Similar windowing methods have been developed by [32, 33] and also by [46]. These methods have been used for example to validate numerical experiments for the standard map [25].

Lipschitz-C^2 A-Posteriori Techniques for Invariant Manifolds and Transverse Homoclinic Orbits: If one wants to prove statements about connecting orbits (orbits with prescribed asymptotic behavior at the fixed/periodic points) then it is necessary to exploit some regularity near the fixed/periodic point. On the other hand, even if one is only interested in establishing the existence of chaotic dynamics, some degree of regularity is needed in order to apply analytic rather than topological arguments. We note that while the methods described here make some assumptions on the differentiability of f, none of them require (or exploit) more than two derivatives.

There are many Lipschitz/low regularity methods for a-posteriori analysis of the local stable and unstable manifolds of fixed points. For example [54] develops an a-posteriori stable/unstable manifold theorem based on covering relations and cone conditions. This can be combined with the C^0 windowing methods mentioned above in order to obtain an a-posteriori scheme for heteroclinic and homoclinic connecting orbits. Such a strategy is used for example in [4] to study heteroclinic and homoclinic orbits in Hénon-Heiles, in [53] to study heteroclinic and homoclinic orbits and obtain entropy bounds for the Hénon map, and in [51] to study connecting dynamics on the Rossler attractor.

Covering-relation-plus-cone-condition methods have been extended in order to prove the existence of more general hyperbolic invariant sets in [14]. This generalization has been used recently by [15] to prove the existence of a center manifold in a celestial mechanics problem.

We also mention here the work of [41], where a rigorous box covering method for planar maps with real distinct eigenvalues is developed. The method is used to study homoclinic chaos in the standard map, by proving directly that the globalized stable and unstable manifolds intersect transversally.

Another method, which is similar to the methods developed in the present work in that it exploits high-order polynomial approximations of the stable and unstable manifold, is developed in [52]. Here the invariant manifolds are approximated by 'Taylor Models'. Existence of the manifolds and a-posteriori bounds on the Taylor Model errors are proved via a nonlinear-box covering argument, which is topological rather than analytical. The method has been used in [42] to study connecting orbits and obtain entropy bounds for the Hénon map.

Finally we mention another thread in the literature which is base on analytical a-posteriori (or shadowing) arguments rather than topological methods. The main tool here is the so called method of *exponential dichotomies*. In [48], an a-posteriori method is developed for proving the existence of a horseshoe given the existence of two numerically computed periodic orbits which pass near one another at a point. The a-posteriori argument is used to prove the existence of a horseshoe in the Hénon map. An extension of the method is given in [44] which allows exponential dichotomy arguments to be applied to homoclinic and heteroclinic orbits. The method is implemented in [17] and used to prove the existence of transverse hetero and homoclinic orbits in both the dissipative and area preserving Hénon map, as well as in the Cremona map.

Remarks 2.1.

a. While our method is closely related to the work of [44] we mention some differences. The methods of [44] require only C^2 assumptions, and as such they apply to a larger family of maps than the C^ω tools developed here.
On the other hand, exponential dichotomy arguments require a delicate local analysis near the fixed point (tail of the homoclinic orbit) in order to be able to apply a Newton-Kantorovich argument on an infinite dimensional sequence space. It is reasonable to think that if the dynamical systems of interest is in fact analytic, then the use of analytic tools could provide some simplification of the arguments.
We will show that this is the case; that when the dynamical system is C^ω we can replace the asymptotic segments of the orbit with a suitable approximation of the local stable and unstable manifolds, and then work with finite orbit segments which transition between the local manifolds. The resulting operator equation is finite dimensional, so that in the (often studied) case that the dynamics are analytic, we obtain a simplification of the a-posteriori argument.
In addition, using high order approximations of the local stable and unstable manifolds lets us work with orbit segments which begin and end farther from the fixed point. This allows us to avoid considering iterates of the map near the fixed point, where the dynamics are slow (and well understood). In principe this should allow for the study of orbits which spend a long time in transition from the local stable to the local unstable manifolds.

b. One could criticize our method on the grounds that it cannot be applied to differential equations, as Poincare and time-τ maps of analytic vector fields need not be analytic. We answer this criticism by pointing to the recent work of [6], which shows how the techniques developed here can applied also in the continuous time case.
The main idea in [6] is to work with an operator equation defined in the full phase space, rather than with a first return or time-τ map. The validated approximation to the connecting orbits obtained using this method are piecewise analytic arcs in phase space with rigorous error bounds along the entire arc (rather than only at the mesh or return points). We note also that the methods of [6] do not require rigorous integration of the system.

C^ω **A-Posteriori Techniques in KAM Theory and Celestial Mechanics:** Since the techniques developed in the present work are tailored for real analytic dynamical systems, our methods have much in common with the tools used by the numerical KAM and rigorous normal form communities, where working in the analytic category is common. For example a key component in the work of [50] is the use of a rigorous, high-order normal form about the equilibria at the origin of the Lorenz system. Since the normal form is used not simply as a computational tool, but rather as a critical ingredient in a computer assisted proof, it is necessary for the author to rigorously bound the truncation error associated with the coordinate change as well as its derivative and inverse (see [50] Proposition 3.1 as well as Lemmas 3.2 and 3.3).

Similar techniques are used in numerical KAM theory. See for example [16, 22] where stability of solar system astroids is studied by computing high-order normal forms about invariant tori in n-body problems. Again, since the authors are interested in computer assisted proofs they are required to rigorously bound the truncation errors in their expansions. In numerical KAM problems these bounds are usually obtained using analytical rather than topological methods. For a more thorough discussion of the numerical KAM literature see [35] and also [16].

Remark 2.1. Rigorous normal form and KAM computations typically involve the so called *small divisors* which arise due to resonant terms in the formal expansions. Overcoming the small divisors and proving the convergence of the formal series of KAM theory requires the use of powerful functional analytic tools (Nash-Moser quadratic convergence schemes, delicate majorant arguments, etc.) However small divisors do not arise in the formal expansions of stable and unstable manifolds, and this greatly simplifies the a-posteriori analysis.

The Parameterizatoin Method for Invariant Manifolds: The so called *Parameterization Method* of [9, 10, 11] provides a theoretical framework for studying the convergence of formal power series expansions of stable and unstable manifolds associated with fixed points of discrete and continuous time dynamical systems, under mild non-resonance conditions.

In [9] an existence theorem ([9] Theorem 1.1) is proved which gives, under quite general hypotheses, the existence of C^k chart maps for local stable and unstable manifolds of C^k diffeomorphisms on Banach spaces. The proof is constructive and, as noted by the authors in the beginning of [9] Section 3, lends itself to a-posteriori analysis and computer assisted proof.

[11] gives a number of applications of the parameterization method, including some elementary proofs of theorems about invariant manifolds in the analytic category, C^0 invariant manifold theorems, and a rigorous treatment of “slow invariant manifolds”. The proofs in [11] illustrate the use of the implicit function theorem. As a consequence they are not constructive (with the exception of [11] Theorem 5.4 on the existence of stable and unstable manifolds of hyperbolic periodic orbits of vector fields. This theorem is proven using the contraction mapping theorem, and explicit a-posteriori bounds are given).

[10] develops optimal regularity results for the parameterization method with respect to system parameters in the C^k category.

In recent years the parameterization method has been extend into a general method for studying a wide variety of invariant manifolds in dynamical systems theory. For example in [27, 28] a method is developed for computing invariant tori and their stable and unstable manifolds in quasiperiodic discrete time dynamical systems. In [34] the parameterization method is used to study KAM tori in symplectic maps without the use of the so called *action/angle coordinates*. Some extensions to invariant tori of infinite dynamical systems are given in [21], while in [36] the parameterization method is used to prove the existence of certain ‘mixed-stability’ invariant manifolds associated with hyperbolic fixed points of symplectic and volume preserving diffeomorphisms. That the parameterization method can be extended to the study of center manifolds (at least in the case of an eigenvalue of one) is shown in [5], while for example [26, 38, 12, 13] give numerical applications of the theory. All of the work mentioned in the present paragraph are based on constructive arguments and can in principle be adapted for use in computer assisted proof.

REMARKS 2.2.

(I) While the present work focuses on the development of tools for analytic systems, the ideas presented here could be modified to work with the C^k theory developed in [9].

(II) The main result of the present work is Theorem 4.1, which gives a-posteriori bounds on the truncation errors in the polynomial approximations to the local stable and unstable manifolds. We note that Theorem 4.1 is less general than Theorem 1.1 of [9] in the sense that we make both stronger non-resonance and regularity assumptions. On the other hand, Theorem 4.1 is specifically designed for a-posteriori application in computer assisted proof. For this reason we develop estimates which take into account explicitly the order of the polynomial approximation, and give explicit expressions for all the constants appearing in the proof. We also note that an analog of our Theorem 4.1 for local stable and unstable manifolds of equilibria of vector fields is given in [6].

2.2. Stable/Unstable Manifolds and Homoclinic Connections. In this section we establish some basic dynamical systems notation and terminology which we use throughout the remainder of the paper. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^k diffeomorphism with $k \in \mathbb{N}$, $k = \infty$ or $k = \omega$, and let $p \in \mathbb{R}^n$ be a hyperbolic fixed point of f. Then there are stable and unstable eigenvalues $\{\lambda_i^s\}_{i=1}^{n_s}$, $\{\lambda_i^u\}_{i=1}^{n_u}$, $\subset \mathbb{C}$ with $n_s + n_u = n$. Let $\{\xi_i^s\}_{i=1}^{n_s}$ and $\{xi_i^u\}_{i=1}^{n_u}$ denote some choice of associated eigenvectors and let $E_s = \text{span}(\{\lambda_i^s\})$ and $E_u = \text{span}(\{\lambda_i^u\})$ denote the stable and unstable eigenspaces. Let $W(p)^u$ and $W(p)^u$ denote the stable and unstable sets of p respectively, i.e.

$$W^s(p) = \{x \in \mathbb{R}^n \,:\, \lim_{n\to\infty} f^n(x) = p\} \qquad \text{and} \qquad W^u(p) = \{x \in \mathbb{R}^n \,:\, \lim_{n\to\infty} f^{-n}(x) = p\}.$$

The sets $W^s(p)$ and $W^u(p)$ are immersed C^k invariant $n_{s,u}$-disks tangent to $E_{s,u}$ respectively at p, by the celebrated Stable Manifold Theorem (see [29]).

Let $N \subset \mathbb{R}^n$ be an open neighborhood of about p. The local stable and unstable manifolds of p relative to N are

$$W^s_{\text{loc}}(p) = \{x \in \mathbb{R}^n \;:\; f^n(x) \in N \text{ for all } n \in \mathbb{N}\}$$

and

$$W^u_{\text{loc}}(p) = \{x \in \mathbb{R}^n \;:\; f^{-n} \in N \text{ for all } n \in \mathbb{N}\},$$

(we suppress the N dependance in this notation). $W^{s,u}_{\text{loc}}$ are embedded C^k $n_{s,u}$-disks and submanifolds of $W^{s,u}(p)$ respectively. Moreover one has that

$$W^s(p) = \bigcup_{n=0}^{\infty} f^{-n}[W^s_{\text{loc}}(p)] \qquad \text{and} \qquad W^u(p) = \bigcup_{n=0}^{\infty} f^{n}[W^u_{\text{loc}}(p)]$$

for any local stable and unstable manifolds.

Let $\nu_s, \nu_u > 0$ and $Q : B_{\nu_s}(0) \subset \mathbb{R}^{n_s} \to \mathbb{R}^n$ and $P : B_{\nu_u} \subset \subset R^{n_u} \to \mathbb{R}^n$ be chart maps for neighborhoods on $W^{s,u}(p)$ which map the origin in $\mathbb{R}^{n_{s,u}}$ to $p \in \mathbb{R}^n$. This means that image(P) and image(Q) are local unstable and stable manifolds respectively.

Let Λ_s denote the $n_s \times n_s$ matrix which has the stable eigenvalues $\{\lambda_i^s\}$ as diagonal entries and zero entries elsewhere, and similarly for Λ_u. Suppose that $S \subset B_{\nu_s}(0)$ is a topological $n_s - 1$-sphere having that $\Lambda_s S \cap S = \emptyset$, and that A is the annulus whose boundary is $S \cup \Lambda_s S$. Then $W^s_{\text{fd}}(p) = Q(A)$ is called a *fundamental domain* for $W^s(p)$. Note that $f[W^s_{\text{fd}}(p)] \cap W^s_{\text{fd}}(p) = Q[\Lambda_s S]$ and

$$W^s(p) = \bigcup_{i=0}^{\infty} f^{-n}[W^s_{\text{fd}}(p)],$$

so that the stable manifold is generated by iterates of a fundamental domain, and these iterates intersect only at their boundaries. In practice S is often take to be a sphere or an ellipsoid in parameter space whose principle axes are determined by the magnitudes of the stable eigenvalues. Similarly, a fundamental domain for $W^u(p)$ can be defined by taking a topological sphere in $B_{\nu_u}(0)$ which does not intersect its own image under Λ_u^{-1} and letting A be the annulus whose boundary is determined by these spheres.

A point $q \in \mathbb{R}^n$ is said to be *homoclinic* to the fixed point p if $q \neq p$ and

$$f^n(q) \to p \quad \text{as} \quad n \to \pm\infty.$$

Clearly q is homoclinic to p if and only if $q \neq p$ and

$$q \in W^u(p) \cap W^s(p),$$

so that locating homoclinic points is equivalent to locating intersections of the stable and unstable manifolds.

2.3. An Example System: The Lomelí Map. Consider the five parameter family of (quadratic) volume preserving diffeomorphisms $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x, y, z) = f_{\alpha,\tau,a,b,c}(x, y, z) = \left(\begin{array}{c} z + Q_{\alpha,\tau,a,b,c}(x, y) \\ x \\ y \end{array} \right), \tag{2.1}$$

where Q is the quadratic function

$$Q_{\alpha,\tau,a,b,c}(x, y) = \alpha + \tau x + a x^2 + bxy + cy^2, \qquad \text{with} \qquad a + b + c = 1. \tag{2.2}$$

The family of maps was introduced in [37], as an analog of the two dimensional area preserving Hénon map. We will refer to this as the *Lomelí Family*, or simply the *Lomelí Map* when it is understood that the parameters are fixed.

We note the following elementary facts;

1. When $\tau^2 - 4\alpha > 0$ the map has a pair of (real) distinct fixed points $p_\pm \in \mathbb{R}^3$

$$p_\pm = \left(\begin{array}{c} x_\pm \\ x_\pm \\ x_\pm \end{array} \right),$$

where

$$x_{\pm} = \frac{-\tau \pm \sqrt{\tau^2 - 4\alpha}}{2}.$$

These are the only possible fixed points of the family.

2. The map is volume preserving, i.e. $|\det(Df(p))| = 1$, for all $p \in \mathbb{R}^3$. Then, for example, at either of the fixed points $p_{\pm}$ the generic stability situation will be either two unstable and one stable eigenvalues, or vice verse (as the product of the three is required to be 1). The two eigenvalues with the same stability type (stable or unstable) are generically either real and distinct, or a complex conjugate pair.
3. At various points in the sequel it is useful to have an analytic expression for the inverse mapping. One can work out that f^{-1} is given explicitly by

$$f^{-1}(x,y,z) = f^{-1}_{\alpha,\tau,a,b,c}(x,y,z) = \left(\begin{array}{c} y \\ z \\ x - Q^{-1}_{\alpha,\tau,a,b,c}(y,z) \end{array} \right),$$

where $Q^{-1}_{\alpha,\tau,a,b,c}$ is the quadratic function

$$Q^{-1} = \alpha + \tau y + a y^2 + byz + cz^2.$$

4. The Jacobian differential of the Lomelí family is

$$Df(x,y,z) = \left(\begin{array}{ccc} 2ax + by + \tau & 2cy + bx & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right).$$

Moreover, the inverse function theorem states that $Df(x,y,z)^{-1} = Df^{-1}[f(x)]$ and allows us to write the inverse of the differential explicitly as

$$Df(x,y,z)^{-1} = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -\tau - 2ax - by & -bx - 2cy \end{array} \right).$$

An affine change of variables puts the Lomelí map in the form

$$g(x,y,z) = \left(\begin{array}{c} x + y \\ y + z - \epsilon + \mu y + P(x,y) \\ z - \epsilon + \mu y + P(x,y) \end{array} \right), \tag{2.3}$$

where $P(x,y) = \bar{a}x^2 + \bar{b}xy + \bar{c}z^2$. We call this the "Dullin-Meiss" form of the map [20], and it has the advantage that the two fixed points are located on the z-axis at $\pm\sqrt{\epsilon/\bar{a}}$. Nevertheless, we will use the standard form of the Lomelí map in our numerical applications, largely so that we can exploit the computational tools developed in [38].

Given a set of parameters in Dullin-Meiss form, it is possible to transform to a Lomelí Map with the parameters

$$
\begin{aligned}
a &= \bar{c} \\
c &= \bar{c} + \bar{a} - \bar{b} \\
b &= \bar{b} - 2\bar{c} \\
\tau &= \frac{2\bar{a}(3+\mu)}{2\bar{a} - \bar{b}} \\
\alpha &= \frac{(9 + 6\mu + \mu^2)\bar{a} - 4\epsilon\bar{a}^2 + 4\epsilon\bar{a}\bar{b} - \epsilon\bar{b}^2}{(2\bar{a} - \bar{b})^2}.
\end{aligned}
$$

These transformations allow us to relate the numerical studies carried out in this work, where we work with the Lomelií form of the map, with the numerical studies in [20].

In Section 6 we consider the six-dimensional dynamical system $G : \mathbb{R}^6 \to \mathbb{R}^6$ formed by coupling two Lomelí maps

$$
\begin{array}{ccc}
f_1(x_1, y_1, z_1) & \equiv & f_{\alpha_1, \tau_1, a_1, b_1, c_1}(x_1, y_1, z_1) \\
& \text{and} & \\
f_2(x_2, y_2, z_2) & \equiv & f_{\alpha_2, \tau_2, a_2, b_2, c_2}(x_2, y_2, z_2)
\end{array}
$$

in the following way;

$$
G(x_1, y_1, z_1, x_2, y_2, z_2) \equiv \left[\begin{array}{c} f_1(x_1, y_1, z_1) + \varepsilon g_2(y_2,) \\ f_2(x_2, y_2, z_2) + \varepsilon g_1(y_1) \end{array} \right], \tag{2.4}
$$

where

$$
g_1(y_1) \equiv (y_1 - x_1^+)(y_1 - x_1^-) \quad \text{and} \quad g_2(y_2) \equiv (y_2 - x_2^+)(y_2 - x_2^-).
$$

Here $x_{1,2}^{\pm}$ denotes a coordinate of the fixed points in the $f_{1,2}$ systems (recall that the fixed points are on the $x = y = z$ line so that it is enough to specify only the x coordinate of the fixed point). Note that this coupling does not move the fixed points in the $f_{1,2}$ systems, but does change the eigenvalues and eigenvectors. When ε is small we can approximate a connecting orbit for G by taking the product of connecting orbits for $f_{1,2}$.

3. Parameterization Method. In this section we review the Parameterization Method of [9, 10, 11]. We focus on the case where the map f is real analytic, the differential is diagonalizable, and there are no resonances between eigenvalues of like stability (these assumptions will be formalized below). For a more complete reference to the Parameterization method, the reader should consult [9, 10, 11].

Recalling the notation of Sectoin 1 we take $p \in \mathbb{R}^n$ to be a hyperbolic saddle for the real analytic diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. We assume that f is uniformly bounded on $B(0, \rho) \subset \mathbb{R}^n$ and that $Df(p)$ is diagnolizable over $\mathbb{C}$. Then $Df(p)$ has n_s distinct stable eigenvalues $\{\lambda_1^s, \ldots, \lambda_{n_s}^s\}$ with $|\lambda_i^s| < 1$, and n_u distinct unstable eigenvalues $\{\lambda_1^u, \ldots, \lambda_{n_u}^u\}$ with $|\lambda_i^u| > 1$, and $n_s + n_u = n$ as p is a saddle. We choose eigenvectors $\{\xi_1^s, \ldots, \xi_{n_s}^s\}$ and $\{\xi_1^u, \ldots, \xi_{n_u}^u\}$ associated with the stable and unstable eigenvalues respectively. For the moment we leave the lengths of the eigenvectors unspecified.

As mentioned in the introduction, the stable manifold theorem gives that $W^s(p)$ and $W^u(p)$ are n_s and n_u dimensional manifolds, respectively tangent to span$\{\xi_i^{n_s}\}$ and span$\{\xi_i^{n_u}\}$ at p. The goal of the parameterization method is to determine analytic mappings $Q : B(0, \nu_s) \subset \mathbb{R}^{n_s} \to \mathbb{R}^n$ and $P : B(0, \nu_u) \subset \mathbb{R}^{n_u} \to \mathbb{R}^n$ which parameterize the local stable and unstable manifolds $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(p)$ respectively at p. For the moment we focus our attention on the development of Q, and consider P at the end of the section.

We simplify our notation a little by letting $B_s \equiv B(0, \nu_s) \subset \mathbb{R}^{n_s}$, and Λ denote the $n_s \times n_s$ matrix with λ_i^s in the i-th diagonal entry and zeros elsewhere (this was called Λ_s above). The Parameterization Method is built on the fact that; $Q[B_s]$ is a local stable manifold for p if Q satisfies

1. $Q(0) = p$,
2. $DQ(0) = [\xi_1^s | \dots | \xi_{n_s}^s]$,
3. and
$$f[Q(\theta)] = Q(\Lambda\theta), \tag{3.1}$$
for all $\theta \in B_s$.

To see this note that for any Q satisfying these conditions, image(Q) is an immersed n_s-disk containing p and is tangent to span$\{\xi_i^{n_s}\}$ at p. Moreover Equation (3.1) implies that $(f \circ Q)(B_s) = Q[\Lambda B_s] \subset Q(B_s)$, so that the ω-limit set of image(Q) under f is p. Then

$$Q(B_s) = W^s_{\text{loc}}(p),$$

by definition.

In general it is impossible to compute Q in closed form. Instead, we note that Q satisfies a (functional) initial value problem with analytic data. Then it is natural to seek a power series expansion for Q of the form

$$Q(\theta) = \sum_{|\alpha| \geq 0} a_\alpha \theta^\alpha \qquad a_n \in \mathbb{R}^n, \quad \theta \in \mathbb{R}^{n_s}, \quad \alpha \in \mathbb{N}^{n_s} \tag{3.2}$$

convergent on B_s. Note that the first order constraints on Q demand that $a_{(0,\dots,0)} = p$ and $a_{e_i} = \xi_i^s$ (here e_i is the multi-index with one in the i-th component and zeros elsewhere). Then the problem is to try to determine the unknown coefficients a_α for $|\alpha| \geq 2$.

Remark 3.1. [Uniqueness] Note that the choice of the lengths of the eigenvectors ξ_i is free in the above formulation. This corresponds to the freedom in the choice of scaling of the parameterization of any manifold. Nevertheless, it is shown in [9] (and we will see again in Section 4) that the solution of Equation 3.1 is unique once the scale of the eigenvectors is fixed.

A formal solution of Equation (3.1) can be obtained by inserting the power series given by Equation (3.2) into Equation (3.1), expanding f as a power series, and computing recurrence relations for the coefficients of Q by matching like powers of θ. This approach is works especially when f is a polynomial map of low to moderate degree. Iterative approaches for solving Equation 3.1 are discussed in [9]. Numerical implementations of iterative algorithms for solving Equation 3.1 can be found in [38, 52, 42].

Finally, we note that the parameterization P of the local unstable manifold for f at p parameterizes the local stable manifold for f^{-1} at p, so that P must satisfy the functional equation

$$f^{-1} \circ P = P \circ \Omega^{-1}, \tag{3.3}$$

where Ω is the matrix of unstable eigenvaules of $Df(p)$. But if we right compose Equation (3.3) with Ω and left compose with f then we obtain

$$P \circ \Omega = f \circ P,$$

which is identical to Equatoin 3.1. Then P and Q solve the same functional equation, modulo the appropriate choose of linear map Λ or Ω.

3.1. Formal Computations for 1D Manifolds of the Lomelí Map. In order to illustrate the flavor of the formal computation of the power series coefficients for the stable/unstable manifold parameterizations we now consider the one-dimensional case for the Lomelí map

$$f(x, y, z) = \left(\begin{array}{c} \alpha + \tau x + z + a x^2 + bxy + cy^2 \\ x \\ y \end{array} \right).$$

Let p_0 denote either one of the fixed points of the map and recall that generically p_0 will have either a one dimensional stable eigenspace, or a one dimensional unstable eigenspace. Let

$$P(\theta) = \left(\begin{array}{c} P_1(\theta) \\ P_2(\theta) \\ P_3(\theta) \end{array} \right) = \left(\begin{array}{c} \sum_{n=0}^{\infty} v_n^1 \theta^n \\ \sum_{n=0}^{\infty} v_n^2 \theta^n \\ \sum_{n=0}^{\infty} v_n^3 \theta^n \end{array} \right), \tag{3.4}$$

be the unknown parameterization function for the one dimensional stable or unstable manifold, and let λ and ξ be the associated stable or unstable eigenvalue and eigenvector. Then $(v_0^1, v_0^2, v_0^3)^T = p_0$, and $(v_1^1, v_1^2, v_1^3)^T = \xi$ are the zero-th and first order power series coefficients.

Substituting the power series into Equation 3.1 gives

$$f \circ P = \left(\begin{array}{c} \alpha + \tau P_1 + P_3 + a[P_1]^2 + bP_1P_2 + c[P_2]^2 \\ P_1 \\ P_2 \end{array} \right)$$

on the left, and

$$P(\lambda\theta) = \left(\begin{array}{c} \sum_{n=0}^{\infty} v_n^1 (\lambda\theta)^n \\ \sum_{n=0}^{\infty} v_n^2 (\lambda\theta)^n \\ \sum_{n=0}^{\infty} v_n^3 (\lambda\theta)^n \end{array} \right) = \left(\begin{array}{c} \sum_{n=0}^{\infty} v_n^1 \lambda^n \theta^n \\ \sum_{n=0}^{\infty} v_n^2 \lambda^n \theta^n \\ \sum_{n=0}^{\infty} v_n^3 \lambda^n \theta^n \end{array} \right)$$

on the right. Equating the second and third components of the left and right hand sides gives

$$\sum_{n=0}^{\infty} v_n^1 \theta^n = \sum_{n=0}^{\infty} v_n^2 \lambda^n \theta^n,$$

and

$$\sum_{n=0}^{\infty} v_n^2 \theta^n = \sum_{n=0}^{\infty} v_n^3 \lambda^n \theta^n.$$

Upon matching like powers this is

$$v_n^1 - v_n^2 \lambda^n = 0 \qquad\qquad v_n^2 - v_n^3 \lambda^n = 0. \tag{3.5}$$

The first component equation is more involved. Expanding the left hand side of the first component and utilizing the Cauchy product formula gives

$$\alpha + \tau \sum_{n=0}^{\infty} v_n^1 \theta^n + \sum_{n=0}^{\infty} v_n^3 \theta^n$$

$$+a \left[\sum_{n=0}^{\infty} v_n^1 \theta^n\right]^2 + b \left[\sum_{n=0}^{\infty} v_n^1 \theta^n\right] \left[\sum_{n=0}^{\infty} v_n^2 \theta^n\right] + c \left[\sum_{n=0}^{\infty} v_n^2 \theta^n\right]^2$$

$$= \alpha + \sum_{n=0}^{\infty} \tau v_n^1 \theta^n + \sum_{n=0}^{\infty} v_n^3 \theta^n$$

$$+ \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a v_k^1 v_{n-k}^1\right) \theta^n + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b v_k^1 v_{n-k}^2\right) \theta^n + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c v_k^2 v_{n-k}^2\right) \theta^n$$

$$= \sum_{n=0}^{\infty} v_n^1 \lambda^n \theta^n.$$

Equating like powers gives that for $n \geq 2$ we have that

$$\tau v_n^1 + v_n^3 + 2a v_0^1 v_n^1 + b v_0^2 v_n^1 + b v_0^1 v_n^2 + 2c v_0^2 v_n^2$$

$$+ \sum_{k=1}^{n-1} \left[a v_k^1 v_{n-k}^1 + b v_k^1 v_{n-k}^2 + c v_k^2 v_{n-k}^2\right]$$

$$= \lambda^n v_n^1,$$

where we have removed from the sum any terms containing v_n^1, or v_n^2. We isolate the n-th order coefficients on the left hand side of the equality in order to obtain

$$(\tau + 2av_0^1 + bv_0^2 - \lambda^n)v_n^1 + (bv_0^1 + 2cv_0^2)v_n^2 + v_n^3$$

$$= -\sum_{k=1}^{n-1} \left[av_k^1 v_{n-k}^1 + bv_k^1 v_{n-k}^2 + cv_k^2 v_{n-k}^2 \right]$$

Combining the three component equations in matrix form gives

$$A_n \begin{bmatrix} v_n^1 \\ v_n^2 \\ v_n^3 \end{bmatrix} = \begin{bmatrix} s_n \\ 0 \\ 0 \end{bmatrix}$$

where

$$A_n = \begin{pmatrix} \tau + 2av_0^1 + bv_0^2 - \lambda^n & bv_0^1 + 2cv_0^2 & 1 \\ 1 & -\lambda^n & 0 \\ 0 & 1 & -\lambda^n \end{pmatrix} \tag{3.6}$$

and

$$s_n = -\sum_{k=1}^{n-1} \left[av_k^1 v_{n-k}^1 + bv_k^1 v_{n-k}^2 + cv_k^2 v_{n-k}^2 \right].$$

Note that if we let $y_n = (s_n, 0, 0)^T$, then the matrix equation has the form

$$\left[Df(p_\pm) - \lambda^n I \right] v_n = y_n. \tag{3.7}$$

This expression is seen to be correct by evaluating the formula for the Jacobian of the Lomelí Map at $p_0 = (v_0^1, v_0^2, v_0^3)$. Equation (3.7) is called the *homological equation* for the one dimensional stable manifold.

Then the coefficient v_n is well defined whenever A_n is invertible. But Equation 3.7 shows that A_n has the form of the characteristic matrix $Df(p_0) - \tau I$ of $Df(p_0)$, and the characteristic matrix is invertible precisely when τ is not an eigenvalue of $Df(p_0)$. Now, if λ is stable and the rest of the eigenvalues are unstable then $|\lambda| < 1 < |\lambda_i|$ so that $|\lambda^n| < |\lambda| < |\lambda_i|$ for all $n > 1$ (a similar arguments holds if $|\lambda| > 1$ and the remaining eigenvalues are stable). Then λ^n is never an eigenvalue of $Df(p_0)$ and the series solution $\sum v_n \theta^n = P(\theta)$ is formally well defined to all orders.

Remark 3.2.

- The computation above provides a numerical scheme for computing approximations to the stable manifold. Namely, we can compute a polynomial P_N which approximates P to any desired finite order by recursively solving the homological Equation (3.7) for $2 \leq n \leq N$.
- The magnitude of $\xi = v_1$ is free in the preceding discussion. This can be used to control the growth of the coefficients of P in numerical computations.
- We treat the convergence of the formal series defined by Equations 3.4 and 3.7 in Theorem 4.1.

3.2. Formal Computation of Two Dimensional Manifolds for the Lomelí Map. In order to parameterize a two dimensional (stable or unstable) manifold associated with a pair of real, distinct (stable or unstable) eigenvalues λ_1, λ_2, of $Df(p_0)$, having $|\lambda_1|, |\lambda_2| < 1$, we choose associated eigenvectors ξ_1 and ξ_2 and assume that the parameterization $P : \mathbb{R}^2 \to \mathbb{R}^3$ has power series expansion

$$P(\theta_1, \theta_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} v_{mn} \theta_1^m \theta_2^n,$$

where $v_{mn} \in \mathbb{R}^3$ are coefficients having

$$v_{00} = p_{\pm}, \qquad v_{10} = \xi_1 \qquad \text{and} \qquad v_{01} = \xi_2.$$

The remaining v_{mn}, $m + n \geq 2$, are determined by requiring that P satisfy the functional equation $f \circ P = P \circ \Lambda$ which in this case is

$$f[P(\theta_1, \theta_2)] = P(\lambda_1 \theta_1, \lambda_2 \theta_2).$$

If we let $v_{mn} = (v_{mn}^1, v_{mn}^2, v_{mn}^3)^T$, then a formal computation similar to the one given in Section 3.1 shows that the coefficients for a two dimensional (stable or unstable) manifold solve the homological equation

$$\begin{pmatrix} \tau + 2av_{00}^1 + bv_{00}^2 - \lambda_1^m \lambda_2^n & bv_{00}^1 + 2cv_{00}^2 & 1 \\ 1 & -\lambda_1^m \lambda_2^n & 0 \\ 0 & 1 & -\lambda_1^m \lambda_2^n \end{pmatrix} \begin{pmatrix} v_{mn}^1 \\ v_{mn}^2 \\ v_{mn}^3 \end{pmatrix} = \begin{pmatrix} -s_{mn} \\ 0 \\ 0 \end{pmatrix}, \quad (3.8)$$

where

$$s_{mn} = \sum_{j=0}^{n} \sum_{i=0}^{m} a\, \bar{v}_{(m-i)(n-j)}^1 \bar{v}_{ij}^1 + b\, \bar{v}_{(m-i)(n-j)}^1 \bar{v}_{ij}^2 + c\, \bar{v}_{(m-i)(n-j)}^2 \bar{v}_{ij}^2$$

and

$$\bar{v}_{k\ell}^s = \begin{cases} 0 & \text{if} \quad k = m \text{ and } \ell = n \\ v_{k\ell}^s & \text{otherwise} \end{cases}$$

for $s = 1, 2, 3$.

Remark 3.3.

- If a fixed point of the Lomelí map has a complex conjugate pair of eigenvalues λ and $\bar{\lambda}$, then we complexify and proceed exactly as in the distinct real case. More precisely we take $\bar{P}$ to have the form

$$\bar{P}(x + iy, x - iy) = \sum_{n=0} \sum_{m=0} v_{mn} (x + iy)^m (x - iy)^n$$

with $v_{mn} \in \mathbb{C}^3$, and impose that $\bar{P}$ solves the invariance equation

$$f[\bar{P}(z_1, z_2)] = \bar{P}(\lambda z_1, \bar{\lambda} z_2).$$

Proceeding as in the case of two distinct real eigenvalues we see that in this case the coefficients still solve the homological equation given by Equation (3.8) with $\lambda_1 = \lambda$

and $\lambda_2 = \bar{\lambda}$. The resulting complex coefficients have that $v_{(m,n)} = \overline{v_{(n,m)}}$, so that $P(x, y) = \bar{P}(x+iy, x-iy)$ is a real valued function. Then image(P) is again a (real) local stable manifold of p. A more thorough discussion of the complex conjugate case is found in [38].

- Note that the homological equation for the power series coefficients of the two dimensional stable/unstable parameterization has the form

$$[Df(p_0) - \lambda_1^m \lambda_2^n I]v_{mn} = \begin{pmatrix} -s_{mn} \\ 0 \\ 0 \end{pmatrix},$$

which is analogous to the one dimensional result. Then the coefficients of the formal series exist uniquely for all m, n with $m + n \geq 2$, so long as the following *non-resonance* conditions is satisfied;

$$\lambda_1^m \lambda_2^n \neq \lambda_i \tag{3.9}$$

for $i = 1, 2$.

Let $\mu_- = \min(|\lambda_1|, |\lambda_2|)$ and $\mu_+ = \max(|\lambda_1|, |\lambda_2|)$. Then it is sufficient to check the non-resonance condition given by Equation (3.9) for each pair $(m, n) \in \mathbb{N}^2$ having

$$2 \leq m + n \leq \frac{\ln(\mu_-)}{\ln(\mu_+)}, \tag{3.10}$$

as $m + n > \ln(\mu_-)/\ln(\mu_+)$ implies that

$$|\lambda_1|^m |\lambda_2|^n \leq (\mu_+)^{m+n} \leq \mu_- \tag{3.11}$$

Then the non-resonance conditions given by Equation (3.9) reduce to a finite number of conditions. In practice we check the non-resonance conditions using rigorous interval arithmetic for each m, n given by Inequality (3.10). If we can confirm all of these conditions, then Inequality (3.11) implies there are no resonances at higher order.

- In fact the situation just describe is quite general. If $P : B \subset \mathbb{R}^k \to \mathbb{R}^n$ parameterizes a k dimensional (stable or unstable) invariant manifold of $f : \mathbb{R}^n \to \mathbb{R}^n$ and, using the notation of Section 3, we suppose that

$$P(\theta) = \sum_{|\alpha|>0} a_\alpha \theta^\alpha,$$

then a formal computation shows that the $|\alpha| \geq 2$ coefficients of the parameterization P satisfy the homological equation

$$[Df(p_0) - \Lambda^\alpha I]a_\alpha = s(\alpha')$$

where

$$\Lambda^\alpha = \lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_k^{\alpha_k} \in \mathbb{C},$$

s depends only on coefficients α' with $|\alpha'| < |\alpha|$, and the form of the function s depends only on the form of the nonlinearity of the function f (this formal computation is discussed in general in [9]). The coefficients a_α are then formally well defined as long as there are no resonances of the form

$$\lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_k^{\alpha_k} = \lambda_i$$

for any $1 \leq i \leq k$ and any $|\alpha| \geq 2$. In precise analogy with Inequalities 3.10 and 3.11 of the previous remark it is sufficient to check that there are no resonances for each

$$2 \leq |\alpha| \leq \frac{\ln(\mu_-)}{\mu_+}.$$

Again, this gives a finite number of conditions which can be checked rigorously using interval arithmetic.

3.3. Numerical Radius of Validity for Formal Solutions. Suppose that we have recursively solved the homological equations for the parameterization of a k dimensional (stable or unstable) manifold up to a fixed finite order N. Then we have a polynomial approximation

$$P_N(\theta) = \sum_{0\leq|\alpha|\leq N} a_\alpha \theta^\alpha$$

to the true parameterization P. While any truncated approximation P_N is entire (as P_N is a polynomial), we do not expect that P_N is a good approximation to P for all θ. Instead, we would like to determine a fixed domain on which the approximation is "good". The following definition makes this precise;

Definition 3.4. Let $\epsilon > 0$ be a prescribed tolerance, $\nu > 0$, and $B = B(0, \nu) \subset \mathbb{R}^k$. We call the number ν *an ϵ-numerical radius of validity* for the approximation P_N if

$$\text{Error}_\nu(P_N) \equiv \sup_{\theta\in B} \|f[P_N(\theta)] - P_N(\Lambda\,\theta)\| \leq \epsilon. \tag{3.12}$$

Remark 3.5.

- In practice, numerical experimentation is enough to select a good ν. Numerical examples and algorithm performance information for local manifolds computations for the Lomelí map can be found in Section 5 and Appendix A of [38]
- We have the usefull bound

$$\text{Error}_\nu(P_N) \leq \sum_{0\leq|\alpha|} |C_\alpha - D_\alpha|\nu^{|\alpha|} \tag{3.13}$$

  where C_α, D_α are the power series coefficients of $f[P_N]$ and $P_N(\Lambda\,\theta)$ respectively. (The inequality is due to the maximum modulus principle). When f is a polynomial, all but finitely many of A_α, and B_α are zero. Then the sum is finite and Equation (3.5) is easy to rigorously bound numerically using interval arithmetic.
- Theorem 4.1 shows that under certain conditions which are easy to validate numerically, we actually have $\|P - P_N\|_\nu \leq C\epsilon$ where C is an explicitly known constant. This provides a mathematically rigorous *a-posteriori* bound on the truncation error made in approximating P by P_N.

4. A-Posteriori Validation of the Formal Series. In this section we prove an *a-posteriori* validation theorem for parameterizations of stable and unstable manifolds for discrete time dynamical systems. From a theoretical view it is preferable to work with analytic functions defined on $\mathbb{C}^n$. For the sake of readability we re-state our assumptions.

A1 Let $p \in \mathbb{C}^n$, $\rho > 0$ and assume that that $f : B(p, \rho) \subset \mathbb{C}^n \to \mathbb{C}^n$ is a bounded analytic function, so that there is $K_0 > 0$ so that

$$\|f\|_\rho \leq K_0.$$

A2 Assume that $Df(p)$ is non-singular, diagonalizble, and hyperbolic. Let $\{\lambda_1^s, \ldots, \lambda_{n_s}^s\}$ and $\{\xi_1^2, \ldots, \xi_{n_s}^s\}$ denote the stable eigenvalues (which are distinct as $Df(p)$ is diagonalizable) and a choice of stable eigenvectors respectively. Let Λ denote the $n_s \times n_s$ diagonal matrix of stable eigenvalues and $Q_0 = [\xi_1^s| \ldots |\xi_{n_s}^s]$ denote the matrix whose columns are the stable eigenvectors.

A3 Assume that $P_N : B(0, \nu) \subset \mathbb{C}^{n_s} \to \mathbb{C}^n$ is an N-th order polynomial, with $N \geq 2$, which for each $\theta \in B(0, \nu)$ solves the equation

$$f[P_N(\theta)] = P_N(\Lambda\theta)$$

exactly to N-th order (in the sense that the power series coefficients of the function on the left are equal to the power series coefficients of the function on the right to N-th order).

Then we have the following definition.

DEFINITION 4.1. [Validation values for discrete dynamical systems] The collection of positive constants ν, ϵ_{tol}, C_1, C_2, K_1, ρ, ρ', μ_* and μ^* are validation values for P_N if

1. $\|f \circ P_N - P_N \circ \Lambda\|_{\Sigma,\nu} \leq \epsilon_{\text{tol}}$;
2. $\|P_N\|_{\Sigma,\nu} \leq \rho' < \rho$;
3. $0 < \mu_* \leq \min_{1 \leq i \leq n_s} |\lambda_i^s| \leq \max_{1 \leq i \leq n_s} |\lambda_i^s| \leq \mu^* < 1$;
4. $$\|Df[P_n]^{-1}\|_{\Sigma,\nu} \leq C_1 \mu_*^{-1} + C_2(\nu);$$
   where, as we will see in the proof, we take C_1 to be any constant with
   $$\|Q_0\|\|Q_0^{-1}\| \leq C_1,$$
   and C_2 to be any bound on the theta dependent terms of $Df[P_N(\theta)]^{-1}$ on B_ν.
5. $$\max_{\substack{\beta \in \mathbb{Z}^n \\ |\beta| = 2}} \max_{1 \leq j \leq n} \|\partial^\beta f_j\|_\rho \leq K_1(\rho).$$

The bounds in the validation theorem are improved if we take into account only the of non-zero second partials of f. Then we will define

$$N_f = \max_{1 \leq j \leq n} \#\{\beta \in \mathbb{Z}^n : |\beta| = 2 \;\text{ and }\; \partial^\beta f_j \neq 0\}, \tag{4.1}$$

and of course have that $N_f \leq n^2$. However for a given map N_f may be smaller than this.

THEOREM 4.1 (A-posteriori manifold validation). *Given validation values ν, ϵ_{tol}, K_1, C_1, C_2, ρ, ρ', μ_* and μ^*, assume that N and δ satisfy the three inequalities*

$$N + 1 > \frac{\ln(\mu_*) - \ln(C_1 + \mu_* C_2)}{\ln(\mu^*)}; \tag{4.2}$$

$$\delta < \min\left(\frac{[\mu_* - (C_1 + \mu_* C_2)(\mu^*)^N]}{2ne\pi N_f (C_1 + \mu_* C_2) K_1}, (\rho - \rho')e^{-1}\right) \tag{4.3}$$

$$\delta > \frac{2(C_1 + \mu_* C_2)\epsilon_{tol}}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^N} \tag{4.4}$$

Then there is a unique parameterization function $P : B(0,\nu) \subset \mathbb{C}^{n_s} \to \mathbb{C}^n$ solving Equation 3.1. Additionally, the truncation error is bounded by

$$\|P - P_N\|_\nu \leq \delta$$

and the parameterization coefficients $a_\alpha \in \mathbb{C}^n$ decay as

$$|a_\alpha| \leq \frac{\delta}{\nu^{|\alpha|}} \qquad \text{for } \; |\alpha| > N.$$

REMARK 4.2. [The Resonance Condition] While the meanings of the conditions given by Equations 4.2, 4.3, and 4.4 will become clear in the Sections 4.2 and 4.3, when we discuss the proof of Theorem 4.1, it is appropriate to make a small remark about Equation 4.2 presently. Note that the right hand side of Equation 4.2 is the natural logarithm of the ratio of the smallest to the largest eigenvalue of $Df(p)$ (the spectral gap) minus a correction term which reflects the nonlinearity of f at p. The condition given by Equation 4.2 guarantees that N is so large enough that there is no possibility of resonances in the coefficients of the remainder $P - P_N$.

4.1. Analytic Preliminaries. If $x \in \mathbb{R}$, then we use $|x|$ to denote the usual absolute value. Similarly, for $z = a + ib \in \mathbb{C}$ we use the usual "Euclidian" norm $|z| = \sqrt{a^2 + b^2}$. We endow $\mathbb{R}^n$ and $\mathbb{C}^n$ with the so called sup or infinity norms generated by the real or complex absolute value functions, so that for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we have

$$|x| = \max_{1 \le i \le n} |x_i|, \qquad \text{and} \qquad |z| = \max_{1 \le i \le n} |z_i|$$

where in each case the $| \cdot |$ on the right is either the absolute value for $\mathbb{R}$ or $\mathbb{C}$, and the sup is taken over components. These norms are well suited for numerical work, as they are easy to evaluate and introduce no rounding errors.

For fixed $\hat{z} \in \mathbb{C}^m$ and $\nu > 0$ let $B_\nu(\hat{z}) \subset \mathbb{C}^m$ be the ball (or *poly-disk*) of radius ν about $\hat{z}$ generated by the sup-norm, so

$$B_\nu(\hat{z}) \equiv \{(h_1, \ldots, h_m) \in \mathbb{C}^m \ : \ |\hat{z}_i - h_i| < \nu \ \text{ for each } \ 1 \le i \le m\}.$$

A function $g : B_\nu(\hat{z}) \subset \mathbb{C}^m \to \mathbb{C}$ is analytic on the poly-disk $B_\nu(\hat{z})$ if g has a power series expansion

$$g(z) = \sum_{|\alpha| \ge 0} a_\alpha (\hat{z} - z)^\alpha \qquad \alpha \in \mathbb{N}^m \qquad a_\alpha \in \mathbb{C},$$

which converges for all $z \in B_\nu(\hat{z})$. Here we use the usual *multi-index* notation, so that if $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and $z \in \mathbb{C}^m$ then $|\alpha| = \alpha_1 + \ldots + \alpha_m$ and $z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_m^{\alpha_m}$.

We say that $f : B_\nu(\hat{z}) \subset \mathbb{C}^m \to \mathbb{C}^n$ is analytic on $B_\nu(\hat{z})$ if $f = (f_1, \ldots, f_n)$ and each $f_j : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}$, $1 \le j \le n$ is analytic in the sense just described. Such an f can also be expressed in power series form as

$$f(z) = \sum_{|\beta| > 0} b_\beta (\hat{z} - z)^\beta \qquad \beta \in \mathbb{N}^m \qquad b_\beta \in \mathbb{C}^n$$

which converges for all $z \in B_\nu(\hat{z})$. The space of bounded analytic functions on $B_\nu(\hat{z})$ forms a Banach space under the norm

$$\|f\|_{B_\nu(\hat{z}), \Sigma} \equiv \sum_{|\alpha| \ge 0} |b_\alpha| \nu^{|\alpha|}.$$

Of course the bounded analytic functions are also a Banach space under the usual C^0 norm, and that the two norms are related by

$$\|f\|_{B_\nu(\hat{z})} \equiv \max_{1 \le j \le n} \ \max_{1 \le i \le m} \ \sup_{|z_i - \hat{z}_i| \le \nu} |f_j(z_1, \ldots, z_m)| \le \|f\|_{B_\nu(\hat{z}), \Sigma}.$$

In theoretical arguments we often use the C^0 norm $\| \cdot \|_{B_\nu(\hat{z})}$, while in numerical applications it is often convenient to use the *sigma-norm* $\| \cdot \|_{B_\nu(\hat{z}), \Sigma}$ in conjunction with the above inequality. Also, by the maximum modulus principle we have that if f is uniformly bounded and analytic on (the open set) $B_\nu(\hat{z})$, then

$$\|f\|_{B_\nu(\hat{z})} = \max_{1 \le j \le n} \ \sup_{|z_i - \hat{z}_i| = \nu} |f_j(z_1, \ldots, z_m)|,$$

so that f is in fact bounded on the closed ball. It follows that f is continuous on $\partial B_\nu(\hat{z})$. If the ball in question is centered at the origin, i.e. is a ball of the form $B_\nu(0)$ then we sometimes use the notation $\| \cdot \|_{\nu, \Sigma}$ and $\| \cdot \|_\nu$ for $\| \cdot \|_{B_\nu(0), \Sigma}$ and $\| \cdot \|_{B_\nu(0)}$ respectively.

Suppose that A is an $n \times m$-matrix with entries $a_{ij} \in \mathbb{C}$. Then when we consider a A as a linear operator $A : \mathbb{C}^m \to \mathbb{C}^n$ we employ the usual operator norm

$$\|A\|_M = \sup_{|\eta| = 1} |A \cdot \eta|,$$

where $\eta \in \mathbb{C}^m$ and $\cdot$ is matrix-vector multiplication. Since $|\cdot|$ is the sup-norm on components we have that

$$\|A\|_M \leq \sup_{1\leq i\leq n} \sum_{j=1}^{m} |a_{ij}| \leq m \sup_{1\leq i\leq n} \sup_{1\leq j\leq m} |a_{ij}|. \tag{4.5}$$

Given a fixed $\hat{z} \in \mathbb{C}^k$ and $\nu > 0$, suppose that $g : B_\nu(\hat{z}) \subset \mathbb{C}^k \to \mathbb{C}^m$ is an analytic function and suppose that the entries of the $n \times m$ matrix A are themselves analytic functions $a_{ij} : B_\nu(\hat{z}) \subset \mathbb{C}^k \to \mathbb{C}$. We can define the norm of the non-constant matrix A to be

$$\|A\|_{M,B_\nu(\hat{z})} \equiv \max_{1\leq i\leq n} \sum_{j=1}^{m} \|a_{ij}\|_{B_\nu(\hat{z})}$$

Then the non-constant matrix vector product $A \cdot g : B_\nu(\hat{z}) \subset \mathbb{C}^k \to \mathbb{C}^n$ is an analytic function and we have the bounds

$$\|A \cdot g\|_{B_\nu(\hat{z})} \leq \|A\|_{M,B_\nu(\hat{z})} \|g\|_{B_\nu(\hat{z})} \leq m \|g\|_{B_\nu(\hat{z}),\Sigma} \max_{1\leq i\leq n} \max_{1\leq j\leq m} \|a_{ij}\|_{B_\nu(\hat{z}),\Sigma},$$

the last bound being particularly useful for numerical applications.

The family of analytic functions which are zero to N-th order play an important in the arguments to follow. We say that $h : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}^n$ is an *analytic N-tail* if h is analytic on $B_\nu(0)$ and

$$h(0) = 0, \quad Dh(0) = 0, \quad \ldots \quad D^\alpha h(0) = 0, \qquad \text{for} \qquad |\alpha| \leq N.$$

Then an analytic N-tail h always has power series representation

$$h(z) = \sum_{|\beta|>N} b_\beta z^\beta \qquad \beta \in \mathbb{N}^m \qquad b_\beta \in \mathbb{C}^n$$

converging for each $|z| < \nu$. With m, n, and $\nu > 0$ fixed we define $\mathbb{H}_N$ to be the set of bounded analytic N-tials on $B_\nu(0) \subset \mathbb{C}^m$ taking values in $\mathbb{C}^n$ (n, m, and ν will always be clear from context).

We use freely the following well known facts about analytic functions and N-tails.

Lemma 4.2.

1. *If $\hat{z} \in \mathbb{C}^m$, $\nu > 0$, $f : B_\nu(\hat{z}) \to \mathbb{C}^n$ is analytic and $\|f\|_\nu \leq M$, then one has for each $\beta \in \mathbb{N}^m$ the Cauchy Estimates*
$$|b_\beta| \leq \frac{M}{\nu^{|\beta|}}.$$
2. *Let h be a bounded analytic N-tail on $B_\nu(0) \subset \mathbb{C}^m$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$be non-zero complex numbers with $0 < |\lambda_j| < 1$, for $1 \leq j \leq m$. Suppose that Λ is the $m \times m$ matrix with λ_j in the j-th diagonal entry and zeros in the non-diagonal entries, and that $0 < \mu^* \equiv \sup_j |\lambda_j| < 1$. Then $h \circ \Lambda$ is a bounded analytic N-tail on $B_\nu(0)$ and*
$$\|h \circ \Lambda\|_\nu \leq (\mu^*)^{N+1} \|h\|_\nu \ .$$
3. *If $g : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}$ is analytic and $\hat{z} \in \mathbb{C}^m$ has $|\hat{z}| < \nu$, then g is analytic on the poly-disk $B_s(\hat{z})$, $s = \nu - |\hat{z}|$ and for any $\eta \in B_s(\hat{z})$, g can be expanded as*
$$g(\hat{z} + \eta) = g(\hat{z}) + Dg(\hat{z}) \cdot \eta + R_{\hat{z}}(\eta)$$
where
$$\|R_{\hat{z}}\|_s \leq N_g K s^2.$$

Here N_g is the number of non-zero second partial derivatives of of g at $\hat{z}$ (so $N_g \leq m^2$) and K is any constant having

$$\sup_{|\beta|=2} \|\partial_\beta g\|_s \leq K.$$

If f is analytic on $B_\nu(0) \subset \mathbb{C}^m$ with values in $\mathbb{C}^n$ then the result can be applied to f component by component.

4. *If $f : B_\nu(\hat{z}) \subset \mathbb{C}^m \to \mathbb{C}^n$ is analytic and $z_1, z_2 \in B_\nu(\hat{z})$ then*

$$|f(z_1) - f(z_2)| \leq \|Df\|_{M, B_\nu(\hat{z})} |z_1 - z_2|.$$

For (1) see any standard text on complex analysis (for example [1]). The elementary proof of (2) is in [6]. (3) is the Lagrange form of the Taylor remainder theorem (also for example in [1]), while (4) is the mean value theorem combined with our norm definitions.

In the following let X be a Banach space, $\mathbb{L}(X)$ be the Banach space of all bounded linear operators on X, and $A \in \mathbb{L}(X)$. Then

$$\|A\|_{\mathbb{L}(X)} \equiv \sup_{x \in X, \|x\|_X = 1} \|Ax\|_X = M < \infty.$$

We make use of the following standard theorems from non-linear analysis.

- **Contraction Mapping Theorem** Let $x \in X$,

$$B_r(x) = \{y \in X \, : \, \|x - y\|_X \leq r\},$$

  and suppose that $\Phi : B_r(x) \to B_r(x)$ is continuous. If in addition there is a $0 < \kappa < 1$ so that for any $x_1, x_2 \in B_r(x)$ we have

$$\|\Phi(x_1) - \Phi(x_2)\|_X \leq \kappa \|x_1 - x_2\|_X$$

  then there is a unique $\hat{x} \in B_r(x)$ so that $\Phi(\hat{x}) = \hat{x}$.

- **Neumann Series** If $I : X \to X$ is the identity map and $A : X \to X$ is a bounded linear operator with $\|A\|_{\mathbb{L}(X)} \leq 1$ then $I - A$ is boundedly invertible and

$$[I - A]^{-1} = \sum_{k=0}^{\infty} A^k,$$

  from which it follows that

$$\|(I - A)^{-1}\|_{\mathbb{L}(X)} \leq \sum_{k=0}^{\infty} \|A\|_{\mathbb{L}(X)}^k \leq \frac{1}{1 - M}.$$

Our "analytic homoclinic shadowing theorem" (Theorem 5.1) is based on the Newton-Kantorovich Theorem [30, 31].

THEOREM 4.3 (Newton-Kantorovich Method). *Let X, Y be Bancah spaces and $F : X \to Y$ be a differentiable mapping. Assume that there as an $\hat{x} \in X$ and an $r > 0$ such that*

(i) $DF(\hat{x})$ has bounded inverse, and

(ii) $\|DF(x) - DF(y)\|_{B(X,Y)} \leq \kappa \|x - y\|$ for all $x, y \in B_r(\hat{x})$.

If

(I)

$$\epsilon_{NK} \geq \|DF(\hat{x})^{-1} F(\hat{x})\|_X,$$

(II)

$$\epsilon_{NK} \leq \frac{r}{2},$$

and

(III)

$$4\epsilon_{NK}\,\kappa\,\|DF(\hat{x})^{-1}\|_{B(X,Y)} \leq 1,$$

then the equation

$$F(x) = 0$$

has a unique solution in $B(r, \hat{x})$.

For an english language exposition of the proof, see also [43]

Finally we require the following bounds for derivatives of analytic functions. The Lemma 4.3 tells us how to bound the derivatives of an analytic function in terms of a bound on the function itself, *so long as we are willing to give up some portion of the domain of analyticity.* The estimates are considered "standard" in KAM theory. (For example they are left as an exercise in [35], and are similar to the bounds for Fourier series found in Section 2.5.7 of [16]. Similar, but less optimal, estimates are in [50, 6]) We include a brief proof in order to obtain explicitly the constants, as we must apply the bounds in the context of computer assisted arguments. Our aim is to give an elementary and brief computation and we note that our constants are obviously not sharp. On the other hand we do take care to obtain the optimal order in the *loss of domain parameter* σ.

Lemma 4.3 (Cauchy Bounds). *Suppose that* $f : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}^n$ *is bounded and analytic. Then for any* $0 < \sigma \leq 1$ *we have that*

$$\|\partial_i f\|_{\nu e^{-\sigma}} \leq \frac{2\pi}{\nu\sigma}\|f\|_\nu \quad \textit{so that} \quad \|Df\|_{\nu e^{-\sigma}} \leq \frac{2\pi m}{\nu\sigma}\|f\|_\nu, \tag{4.6}$$

as well as

$$\|\partial_i\partial_j f\|_{\nu e^{-\sigma}} \leq \frac{4\pi^2}{\nu^2\sigma^2}\|f\|_\nu \quad \textit{and} \quad \|D^2 f\|_{\nu e^{-\sigma}} \leq \frac{4\pi^2 m^2}{\nu^2\sigma^2}\|f\|_\nu. \tag{4.7}$$

Proof: Consider first the one dimensional case, where $\nu > 0$ and $f : B_\nu(0) \subset \mathbb{C} \to \mathbb{C}$ is analytic. Let $0 < \sigma \leq 1$. Then using Cauchy's formula [1] we have that for any $z \in B_{\nu e^{-\sigma}}(0)$

$$f'(z) = \frac{1}{2\pi i}\int_{|\xi|=\nu} \frac{f(\xi)}{(\xi - z)^2}\, d\xi.$$

Note that the denominator is bounded precisely because $|z| \leq \nu e^{-\sigma}$, i.e. because we are taking z in a reduced domain. (Choosing to reduce the domain by an amount exponential in σ gives the optimal $1/\sigma$ dependance in the final estimate, as will be seen in the proof). We parameterize the path $|\xi| = \nu$ by $\xi(\theta) = \nu e^{i\theta}$ and take norms to obtain

$$\begin{aligned} |f'(z)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f[\nu e^{i\theta}] i\nu e^{i\theta}}{(\nu e^{i\theta} - z)^2}\, d\theta\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\nu\|f\|_\nu}{|\nu e^{i\theta} - z|^2}\, d\theta \\ &\leq \frac{\|f\|_\nu}{2\pi\nu}\int_0^{2\pi} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta, \end{aligned} \tag{4.8}$$

where the last inequality is due to the fact that $|z| \leq \nu e^{-\sigma}$, so that the denominator is minimized when $|z| = \nu e^{-\sigma}$. Since the integrand is radially symmetric once we take the norm of f, we are free to take $z = \nu e^{-\sigma}$, and then factor a ν^2 out of the denominator of the integrand.

Noting that $e^\sigma \geq 1+\sigma$ for all real σ, we have that $\sigma/(1+\sigma) \leq 1-e^{-\sigma}$ for all $\sigma > -1$. Then for $0 < \sigma \leq 1$ we have

$$\sigma/2 \leq \frac{\sigma}{1+\sigma} \leq 1 - e^{-\sigma} \leq |e^{i\theta} - e^{-\sigma}|, \tag{4.9}$$

for all $0 \leq \theta \leq 2\pi$. Naive application of Eq (4.9) to Eq (4.8) would yield $|f'(z)| \leq 4\|f\|_\nu/\sigma^2$. However a slightly more subtle argument yields an estimate which is only inverse proportional to σ. Eq (4.8) can be re-written as

$$\begin{aligned} &\frac{\|f\|_\nu}{2\pi\nu} \int_0^{2\pi} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta \\ &= \frac{\|f\|_\nu}{2\pi\nu} \left(\int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta + \int_{\frac{\sigma}{2}}^{2\pi - \frac{\sigma}{2}} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta \right) \end{aligned} \tag{4.10}$$

For the first of the integrals on the right in Eq (4.10) we exploit Eq (4.9) to obtian

$$\int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta \leq \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{|\frac{\sigma}{2}|^2}\, d\theta \leq \frac{4}{\sigma}. \tag{4.11}$$

On the other hand, since $|e^{i\theta} - e^{-\sigma}| \geq \sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/4$, the second integral on the right in Eq (4.10) satisfies the bound

$$\int_{\frac{\sigma}{2}}^{2\pi - \frac{\sigma}{2}} \frac{1}{|e^{i\theta} - e^{-\sigma}|^2}\, d\theta \leq 4 \int_{\frac{\sigma}{2}}^{\frac{\pi}{2}} \frac{\pi^2}{4\theta^2} \leq \frac{2\pi^2}{\sigma} \tag{4.12}$$

Racalling that $z \in B_{\nu e^{-\sigma}}(0)$ we note that Eq (4.11) and Eq (4.12) are uniform in z and combine them with Eq (4.10) to obtain

$$\|f'\|_{\nu e^{-\sigma}} \leq \frac{1}{2\pi\nu} \left(\frac{4}{\sigma} + \frac{2\pi^2}{\sigma} \right) \|f\|_\nu \leq \frac{2\pi}{\nu\sigma} \|f\|_\nu. \tag{4.13}$$

If $f : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}^n$ then each $f_k(z_1, \ldots, z_i, \ldots, z_m)$, $1 \leq i \leq m$, $1 \leq k \leq n$ is analytic in the i-th variable (with the other variables held fixed), so that we obtain

$$\left| \frac{\partial}{\partial z_i} f_k(z) \right| \leq \frac{2\pi}{\nu\sigma} \|f\|_\nu,$$

for any $|z| \leq \nu e^{-\sigma}$ by applying the same argument to the Cauchy integral of $\partial/\partial z_i f_k(z)$. Since this is uniform in i, k and z we apply the estimate given by Equation (4.5) in order to obtian

$$\|Df\|_{\nu e^{-\sigma}} \leq \frac{2\pi m}{\nu\sigma} \|f\|_\nu,$$

as desired. The same estimates can be applied to the Cauchy type integral

$$\frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} f(z) = \frac{1}{(2\pi i)^2} \int_{|\xi_i| = \nu} \int_{|\xi_j| = \nu} \frac{f(z_1, \ldots, \xi_i, \ldots, \xi_j, \ldots, z_m)}{(\xi_i - z_i)^2 (\xi_j - z_j)^2}\, d\xi_i\, d\xi_j$$

to obtain in a similar fashion that

$$\|D^2 f\|_{\nu e^{-\sigma}} \leq \frac{4\pi^2 m^2}{\nu^2 \sigma^2} \|f\|_\nu,$$

as desired.

□

4.2. Proof of the Validation Theorem. We seek an analytic N-tail $h : B_\nu \to \mathbb{R}^n$ so that $P = P_N + h$ and having $\|h\|_\nu \leq \delta$ as small as possible (note that δ bounds the truncation error in the approximation P_N). The key observation is that h itself solves a certain functional equation. To see this let $P = P_N + h$ so that Equation 3.1 becomes

$$f[P_N + h] = [P_N + h](\Lambda).$$

Since f is analytic in $B_\rho \subset \mathbb{R}^n$, and since $\|P_N\|_\nu \leq \rho' \leq \rho$, f has a Taylor expansion about $P_N(\theta)$ for each $\theta \in B_s$. Then let $\theta \in B_s$ so that

$$f[P_N(\theta) + h(\theta)] = f[P_N(\theta)] + Df[P_N(\theta)]\,h(\theta) + R_{P_N(\theta)}(h(\theta)), \tag{4.14}$$

where for any $|z| \leq \rho'$, R_z is the Taylor remainder of f expanded at z. Again, since f is analytic on $\rho > \rho'$ we have that $R_z(\eta)$ is analytic on a disk of radius $s = \rho - \rho'$. Let

$$E(\theta) = f[P_N(\theta)] - P_N(\Lambda\theta) \tag{4.15}$$

and note that E is an analytic N-tail by the assumption that P_N solves Equation 3.1 exactly to N-th order. Then using Equations 4.14 and 4.15 in Equation 3.1 we have a new operator equation in terms of h

$$h[\Lambda\theta] - Df[P_N(\theta)]h(\theta) = E(\theta) + R_{P_N}(h). \tag{4.16}$$

In order to re-write Equation 4.16 as a fixed point equaiton on $\mathbb{H}_N$, the Banach Space of all analytic N-tails from B into $\mathbb{C}^n$, consider the linear operator $\mathfrak{L} : \mathbb{H}_N \to \mathbb{H}_N$ defined by the left hand side of Equation 4.16. So for any $p, q \in \mathbb{H}_N$ we define $\mathfrak{L}[q]$ to be

$$\mathfrak{L}[q](\theta) = q[\Lambda\theta] - Df[P_N(\theta)]h(\theta),$$

and our first task is to study the equation $\mathfrak{L}[q] = p$. He have that

Lemma 4.4. *Let C_1, C_2, μ_* and μ^* be validation values as in Definition 4.1. Suppose that N satisfies the assumption given by Equation 4.2 of Theorem 4.1. Then the linear operator $\mathfrak{L}$ is boundedly invertible on H_N, so that for any $p \in H$ there exists a unique solution to the equation*

$$\mathfrak{L}[q] = p.$$

Moreover we have the bound

$$\|\mathfrak{L}^{-1}\| \leq \frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^N}.$$

Using Lemma 4.4 we apply $\mathfrak{L}^{-1}$ to both sides of Equation 4.16 to see that if $P = P_N + h$ then

$$h = \mathfrak{L}^{-1}\left[E(\theta) + R_{P_N}[h(\theta)].\right]$$

Define the non-linear operator $\Phi : \mathbb{H}_N \to \mathbb{H}_N$ to be

$$\Phi(h) = \mathfrak{L}^{-1}\left[E(\theta) + R_{P_N}[h(\theta)]\right]. \tag{4.17}$$

The preceding discussion makes it clear that $P = P_N + h$ is an exact solution of Equation 3.1 if and only if h is a fixed point of Equation 4.17. What remains is to show that if the assumptions given by Equations 4.2, 4.3 and 4.4 are satisfied, then Φ admits a unique fixed point h. A natural strategy is to employ the Banach Contraction Mapping Theorem. In fact, as we will see in the next section, the assumptions given by Equations 4.3 and 4.4 are exactly the conditions which make Φ a local contraction near P_N.

Lemma 4.5. *Under the hypotheses of Theorem 4.1 Φ is a contraction on the ball $U_\delta = \{h \in \mathbb{H}_N : \|h\|_\nu \leq \delta\}$. Hence there is a unique fixed point h of Φ on U_δ so that $P_N + h$ is an exact solution of Equation 3.1.*

Then Theorem 4.1 is true as soon as the lemmas are proved. Note that on an heuristic level, it is natural to expect that Φ is a contraction as E is a small constant (with respect to h), and R_{P_N} should depend "quadratically" on h.

4.3. Proofs of the Lemmas. Now we complete the proof of Theorem 4.1 by providing the proofs of the lemmas.

Proof of Lemma 4.4: Let p and q be bounded analytic N-tails on B_ν and consider the equation

$$\mathfrak{L}[q](\theta) \equiv q[\Lambda\theta] - Df[P_N(\theta)]q(\theta) = p(\theta). \tag{4.18}$$

If we let $\bar{p}(\theta) \equiv -Df[P_N(\theta)]^{-1}p(\theta)$ then this is equivalent to

$$q(\theta) - Df[P_N(\theta)]^{-1}q(\Lambda\theta) = \bar{p}(\theta),$$

which upon defining the linear operator

$$A[q](\theta) \equiv Df[P_N(\theta)]^{-1}q(\Lambda\theta)$$

becomes

$$(I - A)[q](\theta) = \bar{p}(\theta).$$

Now consider the norm

$$\begin{aligned}
\|A\|_{\mathbb{H}_N} &\equiv \sup_{\|\eta\|_\nu=1} \|A[\eta](\theta)\|_\nu \\
&= \sup_{\|\eta\|_\nu=1} \|Df[P_N](\eta\circ\Lambda)\|_\nu \\
&\leq \sup_{\|\eta\|_\nu=1} (C_1\mu_*^{-1} + C_2)|\Lambda|^{N+1}\|\eta\|_\nu \\
&\leq \mu_*^{-1}(C_1 + \mu_* C_2)(\mu^*)^{N+1},
\end{aligned}$$

where we have used the bound from Equation 4.19 and Estimate 2 of Lemma 4.2. Then we apply the assumption given by Equation (4.2) of Theorem 4.1 and see that

$$\|A\|_{\mathbb{H}_N} \leq \frac{(C_1 + \mu_* C_2)(\mu^*)^{N+1}}{\mu_*} < 1.$$

It follows from the Neumann Theorem that $(I - A)$ is boundedly invertible, and that we have the bound

$$\|(I - A)^{-1}\|_{\mathbb{H}_N} \leq \sum_{i=0}^{\infty} \|A\|_{\mathbb{H}_N}^i = \frac{1}{1 - \frac{C_1(\mu^*)^{N+1}}{\mu_*}}.$$

From the bounded invertability of $(I - A)$ we obtain a unique solution to Equaiton 4.18 in the form

$$q(\theta) = (I - A)^{-1}[\bar{p}](\theta) = -(I - A)^{-1}Df[P_N(\theta)]^{-1}p(\theta).$$

Since p and q were arbitrary we have

$$\begin{aligned}
\|\mathfrak{L}^{-1}\|_{\mathbb{H}_N} &\leq \|(I - A)^{-1}\|_{\mathbb{H}_N}\|Df[P_N]^{-1}\|_{\Sigma,\nu} \\
&\leq \frac{1}{1 - \frac{(C_1+\mu_* C_2)(\mu^*)^{N+1}}{\mu_*}}(\mu_*^{-1}C_1 + C_2) \\
&\leq \frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}},
\end{aligned}$$

as desired.

$\square$

Proof of Lemma 4.5: Since we hypothesized Equation 4.2, we can apply Lemma 4.4 and have that $\mathfrak{L}^{-1}$ is a well defined bounded linear operator. Then the operator

$$\Phi[h](\theta) \equiv \mathfrak{L}^{-1}\left[E(\theta) + R_{P_N(\theta)}[h](\theta)\right]$$

is well defined. To employ the Banach Fixed Point Theorem we must establish that when $U_\delta = \{h \in H_N : \|h\|_\nu \leq \delta\}$ is a δ-neighborhood in the space of analytic N-tails and δ satsfies the hypotheses of Theorem 4.1 and then

(i) Φ maps U_δ into itself.

(ii) there is a $0 < \kappa < 1$ so that for any $h_1, h_2 \in U_\delta$ one has $\|\Phi(h_1) - \Phi(h_2)\|_\nu \leq \kappa\|h_1 - h_2\|_\nu$.

In order to establish (i) we first note that for any $z, \eta \in \mathbb{C}^n$ with $|z| \leq \rho'$ and $|\eta| \leq s \equiv \rho - \rho'$ we have that

$$|R_z^j(\eta)| \leq N_f K_1 s^2$$

by straightforward application of the Lagrange Form of the Taylor Remainder to each of the $1 \leq j \leq n$ components of $R_z(\eta)$ (this estimate is carried out explicitly in [6] see Equaiton (75)). Then since $\|P_N\|_\nu \leq \|P_N\|_{\Sigma,\nu} \leq \rho'$ by by the definition of validation values (def 4.1) and $\delta < se^{-1} < s$ we have for each $\theta \in B_\nu$

$$|R_{P_N(\theta)}(h(\theta))|] \leq |R_z(h(\theta))| \leq \|R_z\|_\delta \leq \frac{\delta^2}{s^2}\|R_z\|_s \leq N_f K_1 \delta^2.$$

Then

$$\begin{aligned}
\|\Phi(h)\|_\nu &\leq \|\mathfrak{L}^{-1}\|\left(\|E\|_\nu + \|R_{P_N}(h)\|_\nu\right) \\
&\leq \frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}}\left(\epsilon_{\text{tol}} + N_f K_1 \delta^2\right)
\end{aligned}$$

But

$$\frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}}\epsilon_{\text{tol}} \leq \frac{\delta}{2}$$

and

$$\frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}} N_f K_1 \delta^2 \leq \frac{\delta}{2},$$

as we see by applying the hypotheses given by Equations 4.3 and 4.4 respectively. Then Φ does in fact map into U_δ, as desired.

To establish (ii) we begin by considering the differential of the remainder term. Then let $\theta \in B_\nu$ and $z = P_N(\theta)$ and note that $|z| \leq \rho'$ (due to the definition of validation values, see Def (4.1)). Since $\delta < se^{-1} < s$ we choose a $0 < \sigma \leq 1$ and let $\omega = \delta/se^{-\sigma}$ so that for any and $h \in U_\delta$ we have the bound

$$\begin{aligned}
\|DR_z(h(\theta))\|_\delta &= \|DR_z \circ \omega\|_{se^{-\sigma}} \\
&\leq \omega\|DR_z\|_{se^{-\sigma}} \\
&\leq \frac{\delta}{se^{-\sigma}} \frac{2\pi n\sigma^{-1}}{s}\|R_z\|_s \\
&\leq \frac{2n\pi e^\sigma N_f K_1}{\sigma}\delta, \\
&\leq 2ne\pi N_f K_1 \delta.
\end{aligned}$$

Here we have used the Taylor Estimate of Lemma 4.2, the Cauchy Bounds of Estimate 4.3, the N-tail scaling estimate of Lemma 4.2, the fact that $\sigma^{-1}e^{\sigma}$ is minimized at $\sigma = 1$, and the assumption that that $\delta < e^{-1}s$.

Then for any $h_1, h_2 \in U_\delta$ we have

$$|R_z^j(h_1(\theta)) - R_z^j(h_2(\theta))| \le 2ne\pi N_f K_1 \delta \|h_1 - h_2\|_\nu$$

by the mean value theorem. So

$$\begin{aligned}
\|\Phi(h_1) - \Phi(h_2)\|_\nu &= \left\|\mathfrak{L}^{-1}[E - R_{P_N}(h_1)] - \mathfrak{L}^{-1}[E - R_{P_N}(h_2)]\right\|_\nu \\
&= \|\mathfrak{L}^{-1}[R_{P_N}(h_1) - R_{P_N}(h_2)\|_\nu \\
&\le \frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}} 2ne\pi N_f K_1 \delta \|h_1 - h_2\|_\nu \\
&\le \kappa \|h_1 - h_2\|_\nu,
\end{aligned}$$

where

$$\kappa \equiv \frac{2ne\pi N_f (C_1 + \mu_* C_2) K_1}{[\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}]} \delta < 1,$$

as δ satisfies the hypothesis given by Equation (4.3) of Theorem (4.1).

□

4.4. The Bounds C_1 and C_2 when f is polynomial. In this section we describe how to obtain the bounds on the non-constant matrix $Df[P_N(\theta)]^{-1}$ required in the definition of validation values. We focus on the case where f is a polynomial. This is the only part of the validation argument that makes the polynomial assumption. We note that if f is a general analytic function then we can use the Taylor expansion of f to obtain that f is polynomial plus a remainder as small as we wish. The argument given here can be modified to work in this more general case as well. We do not pursue the details here.

By the inverse function theorem we have

$$Df[P_N(\theta)]^{-1} = Df^{-1}[f \circ P_N(\theta)],$$

which can be used to compute an analytic expression for $Df[P_N]^{-1}$ as long as f^{-1} is known explicitly. Then we let

$$Df(x)^{-1} = \sum_{|\beta| \ge 0}^{M-1} B_\beta x^\beta$$

where each B_β is an $n \times n$ matrix, and M is the order of f. Recall also that

$$P_N(\theta) = \sum_{0 \le |\alpha| \le N} a_\alpha \theta^\alpha.$$

Then if $\bar{N} = N(M-1)$ we have that $Df[P_N(\theta)]^{-1}$ is an $\bar{N}$-th order polynomial with matrix coefficients. Then we let

$$Df[P_N(\theta)]^{-1} = \sum_{0 \le |\alpha| \le \bar{M}} C_\alpha \theta^\alpha$$

where the coefficients C_α, depend on the B_β and c_α, can be worked out via Cauchy Products.

Let $Q_0 \Sigma Q_0^{-1} = Df(p)$ be the eigenvector/eigenvalue decomposition of the differential and note that

$$C_0 = Df[P_N(0)]^{-1} = Df(p)^{-1} = Q_0^{-1} \Sigma^{-1} Q_0.$$

Then

$$\|Df[P_N]^{-1}\|_{\Sigma,\nu} \leq \left\| Q_0^{-1}\Sigma^{-1}Q_0 + \sum_{1\leq|\alpha|\bar{M}} C_\alpha \theta^\alpha \right\|_{\Sigma,\nu}$$

$$\leq \|Q_0\|\|Q_0^{-1}\|\mu_*^{-1} + \sum_{|\alpha|=1}^{\bar{N}} \|C_\alpha\|\nu^{|\alpha|}.$$

Then we define C_1 and C_2 to be any bounds of the form

$$\|Q_0\|\|Q_0^{-1}\| \leq C_1,$$

and

$$\sum_{|\alpha|=1}^{\bar{N}} \|C_\alpha\|\nu^{|\alpha|} \leq C_2.$$

Note that since these expressions involve bounding finite sums of known quantities, both C_1 and C_2 are easily found using interval arethmetic. Finally we have that

$$\|Df[P_N]\|_{\Sigma,\nu} \leq C_1\mu_*^{-1} + C_2. \tag{4.19}$$

as needed in the definition of the validation values.

Of course if f is not a polynomial map it is possible to make a similar argument using at M-th order Taylor expansion by including a remainder term. This is a technicality not needed in the present work but which could be easily added to the scheme. In this case C_2 would simply have to incorporate as well the truncation error on the ball of radius ρ'.

5. Rigorous Computation of Transverse Homoclinic Orbits. Throughout this section we make the following definitions and assumptions.

- P1: Let $p \in \mathbb{R}^n$ be a hyperbolic fixed point of the analyticomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. Assume that $Df(p)$ is diagonalizable, and that n_s, $n_u > 0$, the number of stable and unstable eigenvalues respectively, have $n_u + n_s = n$.
- P2: Let P_N be the N-th order polynomial approximate parameterization of $W^u(p)$. In addition let ν_u, ϵ_u, C_1^u, C_2^u, ρ, ρ', and μ_*, μ^* be validation values for P_N. Assume that these validation values satisfy the hypotheses of Theorem (4.1) applied to f^{-1}, so that the is a unique analytic N-tail h with $\|h\|_{\nu_u} \leq \delta_u$ so that $P = P_N + h$ is a parameterization of $W^u_{\text{loc}}(p)$.
- P3: Similarly, let Q_N be the N-th order polynomial approximate parameterization of $W^s(p)$ and ν_s, ϵ_s, C_1^s, C_2^s, ρ, ρ', and μ_-, μ^+ be validation values for Q_N and assume that these validation values satisfy the hypotheses of Theorem 4.1 so that the is a unique analytic N-tail g with $\|g\|_{\nu_u} \leq \delta_s$ so that $Q = Q_N + g$ is a parameterization of $W^s_{\text{loc}}(p)$.

Then we can write the homoclinic functional equation (Equation 1.1) in the form

$$F(\theta, x_1, x_2, \ldots, x_{k-2}, x_{k-1}, \phi) =$$

$$\begin{bmatrix} f^{-1}(x_1) - P_N(\theta) - h(\theta) \\ f^{-1}(x_2) - x_1 \\ f^{-1}(x_3) - x_2 \\ \vdots \\ f^{-1}(x_j) - x_{j-1} \\ f(x_j) - x_{j+1} \\ \vdots \\ f(x_{k-2}) - x_{k-1} \\ f(x_{k-1}) - Q_N(\phi) - g(\phi) \end{bmatrix} \equiv F_N(\theta, x_1, \ldots, x_{k-1}, \phi) + H(\theta, \phi), \tag{5.1}$$

where again we stress that P_N and Q_N are explicitly know polynomials and h, and g are unknown analytic N-tails for which we have the mathematically rigorous bounds given in $P3$. We call F_N the *discretized homoclinic functional equation.*

Heuristically our validation scheme is as follows. Assume that there is $\hat{x} = (\hat{\theta},\ \hat{x}_1,\ \ldots, \hat{x}_{k-1},\ \hat{\phi}) \in \mathbb{R}^{nk}$ with $\hat{\theta} \in B_u^\circ$ and $\hat{\phi} \in B_s^\circ$ having that $\hat{x}$ is an approximate zero of the discretized homoclinic equation, i.e. assume that

$$\|F_N(\hat{x})\| \approx 0.$$

If in addition δ_s and δ_u are small, then we have that $\hat{x}$ is also an approximate zero of F, so that orbit$(\hat{x}_j)$ is approximately homoclinic to p for each $1 \leq j \leq k-1$. Our goal is to apply the Newton-Kantorovich Theorem (Thm 4.3) in order to conclude that there exists a true solution x_* of the full homoclinic functional equation near $\hat{x}$. These notions are formalized in the next section.

5.1. Validation of Homoclinic Connections. We now formalize the heuristic scheme just described. Assume, in addition to $P1$, $P2$ and $P3$, that we have computed, or are otherwise given, the following "quasi-local" data, which we refer to as *homoclinic validation values.*

Definition 5.1 (Homoclinic validation values). We say that the vector $\hat{x} = (\hat{\theta}, \hat{x}_1,\ \ldots, \hat{x}_{k-1},\ \hat{\phi}) \in \mathbb{R}^{nk}$, and positive constants A_N, M_N, C_β, C_P, κ, $\hat{\delta}$, $\hat{\epsilon}$, and r are *validation values* for the homoclinic functional equation if the following conditions are met:

1. Define the point $x_0 \in \mathbb{R}^{nk}$ to be given by $x_0 = (0_{n_u}, p, \ldots, p, 0_{n_s})$ where p is the fixed point of f described in $P1 - P3$ and 0_{n_u} and 0_{n_s} are the zero vectors in $\mathbb{R}^{n_u}$ and $\mathbb{R}^{n_s}$. Assume that x_0 is not in the poly-disk $B_r(\hat{x}) \subset \mathbb{R}^{nk}$.
2. $\hat{x} = (\hat{\theta}, \hat{x}_1, \ldots, \hat{x}_{k-1}, \hat{\phi}) \in \mathbb{R}^{nk}$ is an $\hat{\epsilon}$-approximate solution of $F = 0$, in the sense that
$$|DF_N(\hat{x})^{-1}\, F_N(\hat{x})| \leq \hat{\epsilon}.$$
3. $DF_N(\hat{x})$ is non-singular and the positive constant A_M has that $\|DF_N(\hat{x})^{-1}\|_M \leq A_N$.
4. $|\hat{\theta}| < \nu_u$ and $|\hat{\phi}| < \nu_s$ so that we can define what we will call *the first order loss of domain parameters*
$$\hat{\sigma}_s = -\ln\left(\frac{|\hat{\theta}|}{\nu_s}\right), \qquad \textit{and} \qquad \hat{\sigma}_u = -\ln\left(\frac{|\hat{\phi}|}{\nu_u}\right).$$
5. The positive constant M_N has that
$$\left(\max_{1\leq i\leq nk} \sum_{j=1}^{n} [DF_N^{-1}(\hat{x})]_{ij}\right) \frac{2\pi n_u}{\nu_u \hat{\sigma}_u}\delta_u + \left(\max_{1\leq i\leq nk} \sum_{j=nk-n+1}^{nk} [DF_N^{-1}(\hat{x})]_{ij}\right) \frac{2\pi n_s}{\nu_s \hat{\sigma}_s}\delta_s \leq M_N.$$

6. The positive constant $\hat{\delta}$ has that

$$\left(\max_{1\le i\le nk}\sum_{j=1}^{n}[DF_N^{-1}(\hat{x})]_{ij}\right)\delta_u+\left(\max_{1\le i\le nk}\sum_{j=nk-n+1}^{nk}[DF_N^{-1}(\hat{x})]_{ij}\right)\delta_s.\le\hat{\delta}.$$

7. The parameters $\hat{\theta}$, $\hat{\phi}$ and the positive constant r also satisfy $|\hat{\theta}|+r<\nu_u$ and $|\hat{\phi}|+r<\nu_s$ so that we can define the *second order loss of domain parameters*

$$\sigma_s=-\ln\left(\frac{|\hat{\theta}|+r}{\nu_s}\right),\qquad \textit{and}\qquad \sigma_u=-\ln\left(\frac{|\hat{\phi}|+r}{\nu_u}\right).$$

8. The positive constant C_β has that

$$\max_{1\le j\le k-1}\ \max_{1\le i\le n}\ \max_{|\beta|=2}\left\{\|\partial^\beta f_i\|_{B_r(\hat{x}_j)},\|\partial^\beta f_i^{-1}\|_{B_r(\hat{x}_j)}\right\}\le C_\beta.$$

9. The positive constant C_P has

$$\max\left(\|D^2P_N\|_{B_r(\hat{\theta})}+\frac{2\pi^2n^2}{\nu_u^2\sigma_u^2}\delta_u,\ \ \|D^2Q_N\|_{B_r(\hat{\phi})}+\frac{2\pi^2n^2}{\nu_s^2\sigma_s^2}\delta_s\right)\le C_P.$$

10. Finally, κ is positive constant having

$$N_fC_\beta+C_P\le\kappa,$$

where N_f is the max of the number of non-zero second partials of f and f^{-1}.

We sometimes write $C_\beta(r)$, $C_P(r)$ and $\kappa(r)$ to emphasize that these constants should be thought of as depending on the radius r of the $\mathbb{R}^{nk}$ poly-disk about $\hat{x}$. In other words they are the members of a validation values set which carry global information about the ball $B_r(\hat{x})\subset\mathbb{R}^{nk}$. In the next section we will prove the following *a-posteriori* result for F, which is based on a standard Newton-Kantorivich argument combined with the rigorous a-posteriori bounds on the parameterizations.

THEOREM 5.1 (A-posteriori validation of a homoclinic connection). *Given assumptions $[P1]-[P3]$ let $\hat{x}$, A_N, M_N, C_{beta}, C_P, κ, $\hat{\delta}$, $\hat{\epsilon}$, and r be a set of homoclinic validation values as in Def 5.1. We call ϵ_{NK} a "Newton-Kantorovich Epsilon" if*

$$\frac{1}{1-M_N}\left(\hat{\epsilon}+\hat{\delta}\right)\le\epsilon_{NK}. \tag{5.2}$$

With ϵ_{NK} fixed suppose that

A. $0<M_N<1$,

B. $2\epsilon_{NK}\le r$,

C. $\frac{A_N}{1-M_N}4\kappa\epsilon_{NK}\ \le\ 1$.

Then there is a unique $x_*\in B_r(\hat{x})\subset\mathbb{R}^{nk}$ which is a non-trivial solution of the equation $F(x_*)=0$. Such an x_* clearly has that

$$|x_*-\hat{x}|\le r.$$

Moreover, if for all $x\in B_r(\hat{x})\subset\mathbb{R}^{nk}$ we have both that $DF_N(x)^{-1}$ exists, and that

$$\|DF_N(x)^{-1}DH(x)\|_{M,B_r(\hat{x})}<1, \tag{5.3}$$

then it follows that $W^s(p)\cap W^u(p)$, which is non-empty due to the existence of x_*, is also transverse.

5.2. Proof of Theorem 5.1. The proof consists of two parts. First we use Theorem 4.3 to show that the hypotheses of Theorem 5.1 combined with the definition of homoclinic validation values imply the existence of a non-trivial zero of F in $B_r(\hat{x})$. Then we study the form of the differential in order to establish the transversality. The subtly throughout is that while $F_N(\hat{x})$ and $DF_N^{-1}(\hat{x})$ are known, it is F and DF which must be explicitly bound.

In order to apply the Newton-Kantorovic Theorem (thm 4.3) we must show that

(i) $DF(\hat{x})$ has bounded inverse,

(ii) DF is Lipschitz on $B_r(\hat{x})$ with Lipschitz constant κ,

(I) $|DF(\hat{x})^{-1}F(\hat{x})| \leq \varepsilon_{NK}$,

(II) $\varepsilon_{NK} \leq r/2$,

(III) $4\varepsilon_{NK}\kappa\|DF(\hat{x})^{-1}\|_M \leq 1$.

Here the roman numerals refer to the nomenclature established in the statement of Theorem 4.3.

Let $[DF_N^{-1}(\hat{x})]_{(a:b)}$, with $a < b \in \mathbb{N}$, denote the submatrix of $DF_N^{-1}(\hat{x})$ composed of columns a through b. We begin by noting that

$$\begin{aligned}
DF_N^{-1}(\hat{x})DH(\hat{x}) &= DF_N^{-1}(\hat{x}) \begin{bmatrix} D_\theta h(\hat{\theta}) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & D_\phi g(\hat{\phi}) \end{bmatrix} \\
&= \left[[DF_N^{-1}(\hat{x})]_{(1:n)} Dh(\hat{\theta}) \mid 0 \mid \dots \mid 0 \mid [DF_N^{-1}(\hat{x})]_{(nk-n+1:nk)} Dg(\hat{\phi}) \right],
\end{aligned}$$

so that

$$\begin{aligned}
\|DF_N^{-1}(\hat{x})DH(\hat{x})\|_M &\leq \left(n_u \max_{1\leq i\leq nk} \sum_{j=1}^{n} [DF_N^{-1}(\hat{x})]_{ij} \right) \|Dh\|_{\nu_u e^{-\hat{\sigma}_u}} \\
&\quad + \left(n_s \max_{1\leq i\leq nk} \sum_{j=nk-n+1}^{nk} [DF_N^{-1}(\hat{x})]_{ij} \right) \|Dg\|_{\nu_s e^{-\hat{\sigma}_s}} \\
&\leq M_N \\
&< 1,
\end{aligned}$$

by part 5 of Definition 5.1, The Cauchy bounds of Lemma (4.3), and Assumption A of the present Theorem. It follows from the Neumann Series Theorem that the matrix $I + DF_N^{-1}(\hat{x})DH(\hat{x})$ is invertible and that

$$\left\| [I + DF_N^{-1}(\hat{x})DH(\hat{x})]^{-1} \right\|_M \leq \frac{1}{1 - M_N}. \tag{5.4}$$

Then we have that

$$\begin{aligned}
DF(\hat{x})^{-1} &= [DF_N(\hat{x}) + DH(\hat{x})]^{-1} \\
&= \left[DF_N(\hat{x}) \ \left(I + DF_N(\hat{x})^{-1} DH(\hat{x}) \right) \right]^{-1} \\
&= [I + DF_N(\hat{x})^{-1} DH(\hat{x})]^{-1} \ DF_N(\hat{x})^{-1}
\end{aligned} \tag{5.5}$$

exists, and obtain the bound

$$\|DF(\hat{x})^{-1}\|_M \leq \frac{A_N}{1 - M_N}. \tag{5.6}$$

This establishes (i) of Theorem 4.3.

In order to investigate the Lipschitz condition on the differential DF we define the real valued functions $g_{ij} : B_r(\hat{x}) \subset \mathbb{R}^{nk} \to \mathbb{R}$ where $1 \leq i, j \leq nk$ by the expressions

$$g_{ij}(z) = \partial_j F_i(z).$$

Then for $x, y \in B_r(\hat{x})$ we have that

$$\begin{aligned} |g_{ij}(x) - g_{ij}(y)| &\leq \|\nabla g_{ij}\|_{M, B_r(\hat{x})} |x - y| \\ &\leq \sum_{\ell=1}^{nk} \|\partial_\ell g_{ij}\|_{B_r(\hat{x})} |x - y| \\ &\leq \left(\sum_{\ell=1}^{nk} \|\partial_\ell \partial_j F_i\|_{B_r(\hat{x})} \right) |x - y|, \end{aligned} \tag{5.7}$$

by the Mean Value Theorem. Then

$$\begin{aligned} \|DF(x) - DF(y)\|_M \equiv & \sup_{\substack{v \in \mathbb{R}^{nk} \\ |v| = 1}} |[DF(x) - DF(y)]v| \\ &\leq \max_{1 \leq i \leq nk} \sum_{1 \leq j \leq nk} |[DF(x) - DF(y)]_{ij}| \\ &= \max_{1 \leq i \leq nk} \sum_{1 \leq j \leq nk} |\partial_j F_i(x) - \partial_j F_i(y)| \\ &\leq \left(\max_{1 \leq i \leq nk} \sum_{j=1}^{nk} \sum_{\ell=1}^{nk} \|\partial_\ell \partial_j F_i\|_{B_r(\hat{x})} \right) |x - y|, \end{aligned} \tag{5.8}$$

where we have used the estimate of Inequality 5.7.

Note that from 7 of Definition 5.1 and the Cauchy Bounds of Lemma 4.3 we have that for any $1 \leq i \leq n$

$$\begin{aligned} \|\partial_\ell \partial_j h_i\|_{B_r(\hat{x})} &= \|\partial_\ell \partial_j h_i\|_{B_r(\hat{\theta})} \\ &\leq \|\partial_\ell \partial_j h_i\|_{\nu_u e^{-\sigma_u}} \\ &\leq \frac{2\pi^2}{\nu_u^2 \sigma_u^2} \delta_u, \end{aligned}$$

and similarly

$$\|\partial_\ell \partial_j g_i\|_{B_r(\hat{x})} \leq \frac{2\pi^2}{\nu_s^2 \sigma_s^2} \delta_s.$$

Using these estimates and considering the second partial derivatives of F one component at a time we recall 8, 9, and 10 of Definition 5.1 and obtain that

$$\max_{1 \leq i \leq nk} \sum_{j=1}^{nk} \sum_{\ell=1}^{nk} \|\partial_\ell \partial_j F_i\|_{B_r(\hat{x})} \leq N_f C_\beta + C_P = \kappa.$$

Combining this with Inequality (5.8) gives (ii) of Theorem 4.3.

For (I) of Theorem 4.3 we use the notation $[DF_N^{-1}(\hat{x})]_{(a:b)}$ as above and have that

$$\begin{aligned}
\left|DF_N^{-1}(\hat{x})H(\hat{x})\right| &= \left| DF_N^{-1}(\hat{x}) \left[\begin{array}{c} h(\hat{\theta}) \\ 0 \\ \vdots \\ 0 \\ g(\hat{\phi}) \end{array} \right] \right| \\
&= \left| [DF_N^{-1}(\hat{x})]_{(1:n)}\ h(\hat{\theta}) + [DF_N^{-1}(\hat{x})]_{(nk-n+1:nk)}\ g(\hat{\phi}) \right| \\
&\leq \left(\max_{1 \leq i \leq nk} \sum_{j=1}^{n} \left|[DF_N^{-1}(\hat{x})]_{ij}\right| \right) \|h\|_{\nu_u} \\
&+ \left(\max_{1 \leq i \leq nk} \sum_{j=nk-n+1}^{nk} \left|[DF_N^{-1}(\hat{x})]_{ij}\right| \right) \|g\|_{\nu_s} \\
&\leq \hat{\delta}, \qquad (5.9)
\end{aligned}$$

where we have used 6 of Definition 5.1. Then, recalling Equation 5.5 and Inequality 5.6 we have

$$\begin{aligned}
\left|DF(\hat{x})^{-1}F(\hat{x})\right| &\leq \left|[I + DF_N(\hat{x})^{-1}DH(\hat{x})]^{-1}\ DF_N(\hat{x})^{-1}F(\hat{x})\right| \\
&= \left|[I + DF_N(\hat{x})^{-1}DH(\hat{x})]^{-1}\ DF_N(\hat{x})^{-1}\left(F_N(\hat{x}) + H(\hat{x})\right)\right| \\
&\leq \frac{1}{1 - M_N}\left(\left|DF_N^{-1}(\hat{x})F_N(\hat{x})\right| + \left|DF_N^{-1}(\hat{x})H(\hat{x})\right|\right) \\
&\leq \frac{1}{1 - M_N}\left(\hat{\epsilon} + \hat{\delta}\right) \\
&\leq \epsilon_{NK}, \qquad (5.10)
\end{aligned}$$

where we have used 2 of Definition 5.1, the Estimate given by Inequality 5.9, and the the defintion of ϵ_{NK} given by Equation 5.2. This establishes condition (I) of Theorem 4.3. Finally note that (III) of Theorem 4.3 follows directly from assumption C of the present theorem and Inequality 5.6, while (II) of Theorem 4.3 is assumption B of the present Theorem.

Then the conditions of Theorem 4.3 are satisfied and we obtain the existence of a unique $x_* \in B_r(\hat{x})$ so that $F(x_*) = 0$. Note that since $x_* \neq x_0$ by 1 of Definition 5.1, we obtain a non-trivial homoclinic orbit.

Now we turn to the question of transversality of the intersection at x_*. An argument similar to the one used to derive Equation 5.5, except with $\hat{x}$ replaced by a variable $x \in B_r(\hat{x})$ shows that $DF(x)$ is invertible for all $x \in B_r(\hat{x})$ as long as $DF_N(x)$ is invertible for all $x \in B_r(\hat{x})$ and the condition given by Equation 5.3 is met. Since we have assumed that both of these conditions are met, it follows that $DF(x_*)$ is non-singular.

What remains is to show is that the non-singularity of $DF(x_*)$ implies that the homoclinic orbit is transverse. Assume for the moment that $k = 1$, so that the local manifolds $W^u_{\text{loc}}(p) = P[B_{\nu_u}(0)]$ and $W^s_{\text{loc}}(p) = Q[B_{\nu_s}(0)]$ intersect at x_*. In this case the operator F reduces to

$$F(\theta, \phi) = P(\theta) - Q(\phi).$$

and we have a solution $x_* = (\theta_*, \phi_*) \in B_r(\hat{x})$. Since $DF(x_*)$ is non-singular, the columns of

$$DF(x_*) = [D_\theta P(\theta_*)| - D_\phi Q(\phi_*)]$$

span $\mathbb{R}^n$. But the columns of $D_\theta P(\theta_*)$ and $D_\phi Q_{\phi_*}$ span $T_{P(\theta_*)}W^u(p)$ and $T_{Q(\phi_*)}W^s(p)$ respectively. It follows that $T_{P(\theta_*)}W^u(p)$ and $T_{Q(\phi_*)}W^s(p)$ span $\mathbb{R}^n$, which is to say that x_* is a point of transverse intersection.

Now suppose $K > 1$, and $x_* \in \mathbb{R}^{nk}$ is the solution of $F = 0$. Since any f-iterate of a local unstable manifold is again a local unstable manifold, and any f-iterate of a homoclinic point is another homoclinic point, we have that the local unstable manifold $f^k[W^u_{\text{loc}}(p)] = f^k[P(B_{n_u}(0))]$ intersects $W^s_{\text{loc}}(p) = Q[B_{\nu_s}(0)]$ at the phase space point $Q(\phi_*) = f^k[P(\theta_*)]$. Then we are in exactly the same situation as above, and the intersection is transverse if and only if the matrix

$$[-D_\theta f^k[P(\theta_*)]\,|D_\phi Q(\phi_*)] = [-D_x f^k[P(\theta_*)]D_\theta P(\theta_*)|D_\phi Q(\phi_*)]$$

is non-singular. Note that $D_x f^k(x)$ is non-singular for any $x \in \mathbb{R}^n$ as f is a diffeomorphism.

Now, by hypothesis the matrix

$$DF(x_*) = \begin{pmatrix} -D_\theta P(\theta_*) & & 0 \\ \vdots & \mathbf{A} & \vdots \\ 0 & & -D_\phi Q(\phi_*) \end{pmatrix},$$

is non-singular, so that if we construct the non-singular matrix

$$\mathbf{B} = \begin{pmatrix} D_x f^k[P(\theta_*)] & \mathbf{0} \\ \mathbf{0} & Id_{n(k-1)\times n(k-1)} \end{pmatrix}$$

and multiply, we have that the product

$$\mathbf{B}DF(x_*) = \begin{pmatrix} -D_x f^k[P(\theta_*)]D_\theta P(\theta_*) & & 0 \\ \vdots & \mathbf{C} & \vdots \\ 0 & & -D_\phi Q(\phi_*) \end{pmatrix}$$

is the product of non-singular matrices hence is itself non-singular (here the actual form of $\mathbf{C}$ is unimportant to us). Since $\mathbf{B}DF(x_*)$ is non-singular, it has linearly independent columns. Exploiting this linear independence gives that the columns of

$$[-D_\theta f^k[P(\theta_*)]\,|D_\phi Q(\phi_*)] = [-D_x f^k[P(\theta_*)]D_\theta P(\theta_*)|D_\phi Q(\phi_*)],$$

span $\mathbb{R}^n$, which is to say that the local manifolds $W^s_{\text{loc}}(p) = Q[B_{\nu_s}(0)]$ and $W^u_{\text{loc}}(p) = f^k[P(B_{\nu_u}(0)]$ intersect transversally, as desired.

$\square$

6. Numerical Computations. We begin by considering a Lomelí Map with parameters $a = 0.5$, $b = -0.5$, $c = 1$, $\alpha = -5.34$, and $\tau = 0.8$. These correspond to Dullin-Meiss parameters of $\bar{a} = 1$, $\bar{b} = 0.5$, $\bar{c} = 0.5$, $\mu = -2.4$ and $\epsilon = 5.5$. For these parameters values there is a hyperbolic fixed point at $p = (x_-, x_-, x_-)$ with $x_- = -2.745207879911715$. Then $Df(p)$ has unstable complex conjugate eigenvalues $-0.402451645443971 \pm i2.035392592347574$ and stable eigenvalue 0.232299350932085. Table 6 illustrates the results of the parameterization computations, which are carried out using the rigorous interval arithmetic library IntLab (which runs under Matlab).

The table records the dimension of the manifolds, the approximation order N used in each case, the time taken to compute the coefficients of the polynomial approximations P_N and Q_N, the time taken to a-posteriori validated the approximations, the magnitudes of the resulting bounds on the truncation errors $\|h\|_{\nu_u} = \delta_u$ and $\|g\|_{\nu_s} = \delta_s$, the size of the parameter domain radii ν_u and ν_s, the size of the eigenvector scaling, and finally a rigorous bound on the size of the local manifolds in the sigma-norm.

Dim	Order	Approx Time	Valid Time	Validated Error	Radius	$\lvert\xi\rvert$	$\lVert\cdot\rVert_{\nu,\Sigma}$
1	50	5.16 sec	0.40 sec	8.71×10^{-13}	0.9	2	1.96
2	25	1.68 min	2.84 sec	5.67×10^{-12}	0.4	1.5	1.21

TABLE 6.1

Manifold Validation Performance: Example 1 ($\epsilon = 5.5, \mu = -2.4$)

K	$\hat{x}_1$	r
6	$(-1.648314148155201, -3.605864990373435, -2.750773367689280)$	1.1×10^{-11}
6	$(-1.692334813290302, -3.652591337627915, -2.718741184627647)$	1.06×10^{-11}

TABLE 6.2

Primary Intersection Validation ($\epsilon = 5.5, \mu = -2.4$): 3.21 sec for proof of both orbits. Chaos confirmed in both cases.

We then use a classical, numerical Newton scheme to find an approximate numerical solution to the discretized homoclinic functional equaiton $F_N(x) = 0$ with $k = 6$ and of course $n = 3$. This leads to an approximate zero

$$\hat{x} = \begin{bmatrix} \hat{\theta} \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{x}_5 \\ \phi \end{bmatrix} = \begin{bmatrix} (-0.337379322019076, 0.088431234641040) \\ (-1.648314148155201, -3.605864990373435, -2.750773367689280) \\ (1.979508268106647, -1.648314148155201, -3.605864990373435) \\ (-1.054666610773029, 1.979508268106647, -1.648314148155201) \\ (-2.313572985270695, -1.054666610773029, 1.979508268106647) \\ (-2.642742570718999, -2.313572985270695, -1.054666610773029) \\ 0.228218016117584 \end{bmatrix}$$

Using Theorem 5.1 we can validate that there is a true solution of the homoclinic functional equation in a polydisk $B_r(\hat{x})$ with $r = 1.1 \times 10^{-11}$. Table 6 gives computation data for the proof just described, and also for the proof of a second distinct solution of the homoclinic operator equation for $k = 6$. In each case only the $\hat{x}_1$ data is recorded. Figure 6.1 shows the time series data for the x component of the first of these two orbits. Black dots represent points in $\hat{x}$. Red points represent iterates on the local manifolds.

We note that in these first two proofs is that the time taken to compute the rigorous interval enclosures of the coefficients for the two variable polynomial P_N is 1 minute 68 seconds, while the validation of the two homoclinic orbits takes only 3.21 seconds. Since we can use the same polynomial approximations P_N and Q_N in any homoclinic functional equation, regardless of the size of k, we compute 32 more distinct homoclinic orbits with k varying. The results are tabulated in Table 6, and again only $\hat{x}_1$ components are recorded. Note that the time required to validate all 34 of orbits is a little less than the time needed to compute the rigorous approximation of the stable manifold. This suggests that high order approximation of the manifolds is most useful when computing many distinct homoclinic orbits at a given parameter set. Figures 6.2 and 6.3 show time series data for the x-component of the shortest and longest homoclinic orbits validated.

We note that in the previous example the dynamics is “fast” in the sense that as few as 6 iterates are needed in order to transition from the local unstable to the local stable manifold. In order to compute orbits with longer ‘time of flight’ (higher k) we consider a Lomelí map with parameters a, b, c, and τ as before, but with $\alpha = -0.04$. This corresponds to a Dullin-Meiss value of $\epsilon = 0.2$ with all other parameters as above. At these parameter values we study the fixed point $p = (x_-, x_-, x_-)$ with $x_- = -0.847213595499957$. The differential $Df(p)$ has unstable complex conjugate eigenvalues of $-0.150742620101308 \pm i1.205183554810613$ and a stable eigenvalue of 0.677878442452638. The data for the parameterization computations is given in Table 6, with format identical to before. Table 6 gives data for the results of the

K	$\hat{x}_1$	r	time
8	$(-1.878269557294666 - 3.704360821688669 - 2.644177124855255)$	1.0×10^{-11}	3.13 sec
·	$(-1.598486534326447 - 3.712394711133192 - 2.715338895232408)$	1.1×10^{-11}	·
9	$(-1.693365888596068 - 3.516449414154529 - 2.776271298390562)$	1.05×10^{-11}	4.95 sec
·	$(-2.033965491062911 - 3.691036738831221 - 2.607784382423848)$	1.05×10^{-11}	·
·	$(-3.649752275192224 - 2.876479215542708 - 2.487231377447373)$	1.00×10^{-11}	·
11	$(-1.724921906236488 - 3.503098391735548 - 2.773685700840596)$	1.06×10^{-11}	10.1 sec
·	$(-2.089900084565888 - 3.686144425839955 - 2.594568562106802)$	1.0×10^{-11}	·
·	$(-3.634873256589227 - 2.906134859387798 - 2.482573305537549)$	1.0×10^{-11}	·
·	$(-3.620917995724487 - 2.915222901577827 - 2.483676082433866)$	1×10^{-11}	·
·	$(-2.114585182128023 - 3.679401143907701 - 2.591096188024756)$	1.03×10^{-11}	·
·	$(-1.768297176557754 - 3.496683655844906 - 2.765421447288818)$	1.05×10^{-11}	·
12	$(-1.613946132963925 - 3.601054205346514 - 2.761528716808955)$	1.1×10^{-11}	6.8 sec
·	$(-1.672093712060165 - 3.527103879468739 - 2.777334962197874)$	1.06×10^{-11}	·
·	$(-2.122097145983667 - 3.674130503709248 - 2.591708802528494)$	1.04×10^{-11}	·
·	$(-1.822510500455057 - 3.571768208555173 - 2.720873332899369)$	1.1×10^{-11}	·
13	$(-3.644121861531430 - 2.872709464400592 - 2.489856336907984)$	1×10^{-11}	10.35 sec
·	$(-1.720320939862523 - 3.656391590596805 - 2.709687450800172)$	1.0×10^{-11}	·
·	$(-1.972320520664557 - 3.693712699582179 - 2.623618282117915)$	1.0×10^{-11}	·
·	$(-3.647170226591191 - 2.867372482172479 - 2.490553201321108)$	1×10^{-11}	·
·	$(-1.582489566947040 - 3.527839146851471 - 2.799122415227350)$	1.07×10^{-11}	·
·	$(-1.574224069064366 - 3.529792848481951 - 2.800384605320380)$	1.1×10^{-11}	·
20	$(-1.931148725862011 - 3.707646666557216 - 2.627909579872393)$	1.0×10^{-11}	4.01 sec
·	$(-3.638326627639060 - 2.901176380906034 - 2.483107270172258)$	1.0×10^{-11}	
21	$(-3.690719490424216 - 2.823690393936636 - 2.490880594199791)$	1×10^{-11}	16.44 sec
·	$(-1.957194765763665 - 3.705297800511473 - 2.621878341459779)$	1.05×10^{-11}	·
·	$(-1.729640666364290 - 3.510087223951199 - 2.769773329564600)$	1.06×10^{-11}	·
·	$(-1.690639165363386 - 3.669844178437995 - 2.711227287037635)$	1.1×10^{-11}	·
·	$(-1.950380561442004 - 3.705777860019821 - 2.623527619587280)$	1.1×10^{-11}	·
·	$(-3.702924845715120 - 2.791265552326865 - 2.496202529149488)$	1.0×10^{-11}	·
·	$(-3.708117158393551 - 2.774277602263187 - 2.499173703371658)$	0.98×10^{-11}	·
·	$(-1.932029291989042 - 3.707691973193318 - 2.627641932935536)$	1.04×10^{-11}	·
·	$(-3.616786394029812 - 2.918973341522530 - 2.483676503055390)$	1×10^{-11}	·

TABLE 6.3

Secondary Homoclinic Orbits ($\epsilon = 5.5, \mu = -2.4$): Total Time for Proofs: 55.0 sec. Transversality confirmed in all cases.

Dim	Order	Approx Time	Proof Time	Validated Error	Radius	$\|\xi\|$	$\|\cdot\|_\nu$
1	50	4.95 sec	0.45 sec	2.71×10^{-11}	0.9	1.5	5.63
2	25	1.66 min	2.94 sec	4.30×10^{-13}	0.4	0.5	0.32

TABLE 6.4

Manifold Validation Performance: Example 2 ($\epsilon = 0.2, \mu = -2.4$)

homoclinic validation computations for five different orbits with values of k varying between 75 and 121. Figures 6.3 and 6.4 show time series data for the shortest and longest of these homoclinic orbits (x-component in both cases). Note that for the orbit with $k = 121$ the discretized homoclinic functional equation $F_N : \mathbb{R}^{nk} \to \mathbb{R}^{nk}$ has $nk = 121 \times 3 = 363$.

Finally we cary out a similar computation for the map $G : \mathbb{R}^6 \to \mathbb{R}^6$ obtained by a

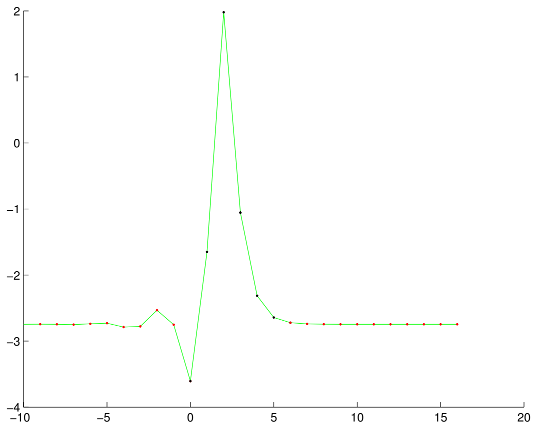

FIG. 6.1. *Time Series Data:*

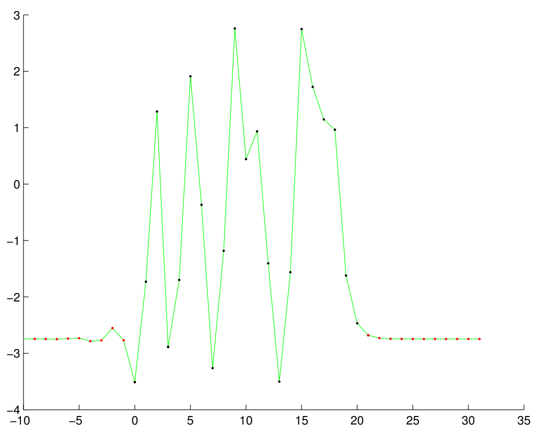

FIG. 6.2. *Time Series Data:*

coupling a pair of Lomelí maps as discussed in Section 2.3. We take parameters $a_1 = a_2 = 0.5$, $b_1 = b_2 = -0.5$, $c_1 = c_2 = 1$, $\tau_1 = \tau_2 = 0.8$, $\alpha_1 = -5.339999999999998$ and $\alpha_2 = -5.939999999999998$ (corresponding to Dullin-Meiss parameters of $\epsilon_1 = 5.5$ and $\epsilon_2 = 6.1$). The maps are coupled with a strength of $\varepsilon = 5 \times 10^{-7}$. (The reason for the small coupling strength is that we obtain a numerical guess by continuing away from the product system having $\varepsilon = 0$. The coupled system is quite sensitive to this parameter, and a tangency develops for coupling strengths much larger than this).

We study the fixed point $p = (x_-^1, x_-^1, x_-^1, x_-^2, x_-^2, x_-^2)$ with $x_-^1 = -2.74507879911714$ and $x_-^2 = -2.869817807045693$. The differential $DG(p)$ has two pair of unstable complex conjugate eigenvalues $-0.428678184042694 \pm i2.076458156435394$ and $-0.402451645448668 \pm i2.035392592342751$, and a pair of real distinct stable eigenvalues 0.232299350933555 and 0.222447464570467. Then fixed point has a four dimensional unstable manifold and a two dimensional stable manifold. We show that these manifolds intersect transversally using the arguments developed above. The results of the computer assisted proofs are recorded in

K	$\hat{x}_1$	r	time
75	$(-0.717248519714197 - 1.043252947479510 - 0.860812112677259)$	1.04×10^{-7}	6.32 sec
76	$(-1.107394504655081 - 0.745731963636135 - 0.642025567084575)$	1.4×10^{-7}	6.15 sec
111	$(-1.104148108665029 - 0.729631044649217 - 0.648872760710501)$	1.05×10^{-7}	15.04 sec
118	$(-1.087535686140795 - 0.715568561563514 - 0.669111490970251)$	1.3×10^{-7}	16.11 sec
121	$(-0.995810895350469 - 0.972045779061998 - 0.671276957464922)$	1.04×10^{-7}	18.6 sec

TABLE 6.5
Homoclinic Orbits ($\epsilon = 0.2, \mu = -2.4$): Transversality confirmed in all cases.

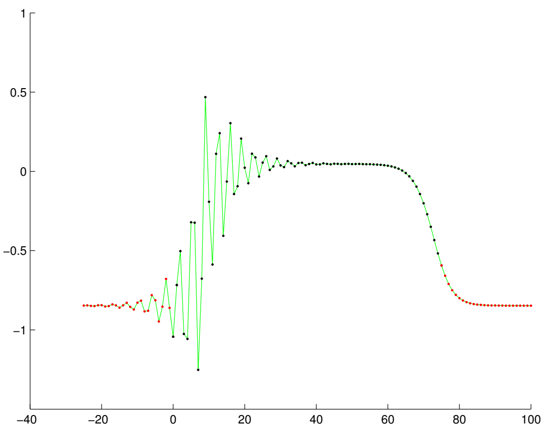

FIG. 6.3. *Time Series Data:*

Tables 6.6 and 6.7. Note that since we are only doing one proof, we use lower order approximations and smaller parameter domains. This helps to mitigate the substantially slower run time of the coefficient computations for the four variable polynomial approximation of the unstable manifold.

7. Conclusions.

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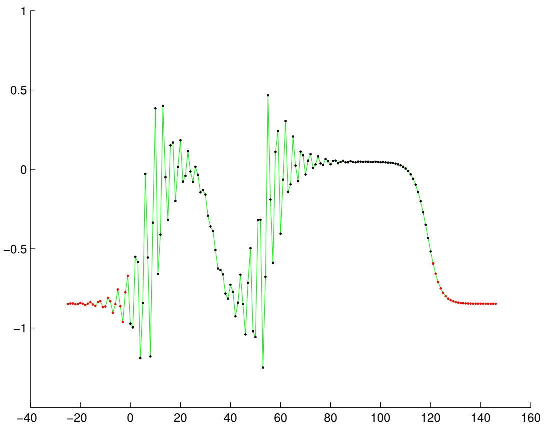

FIG. 6.4. *Time Series Data:*

Dim	Order	Approx Time	Proof Time	Validated Error	Radius	$\lvert\xi\rvert$	$\\| \cdot \\|_\nu$
2	9	25.8 sec	3.36 sec	4.95×10^{-13}	0.001	1.4	1.46×10^{-3}
4	5	27.9 min	2.73 sec	2.96×10^{-12}	0.001	0.4	7.1×10^{-4}

TABLE 6.6
Manifold Validation Performance: Six Dimensional Example.

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K	$\hat{x}_1$	r	time
20	$\begin{pmatrix} -2.743916182731272 \\ -2.745285611304841 \\ -2.745493386071762 \\ -2.868750115142552 \\ -2.869921042268439 \\ -2.870035612699910 \end{pmatrix}$	1.49×10^{-11}	6.32 sec

Table 6.7

Primary Intersection Validation: Six Dimensional Example. Chaos confirmed. The similarity in the coordinates of x_1 is due to the fact that the orbit begins close to the fixed point.

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