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Connecting Orbits for Compact Infinite Dimensional Maps: Computer Assisted Proofs of Existence*

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Abstract. We develop and implement a computer assisted argument for proving the existence of heteroclinic/homoclinic connecting orbits for compact infinite dimensional maps. The argument is based on a posteriori analysis of a certain "discrete time boundary value problem," and a key ingredient is representing the local stable/unstable manifolds of the fixed points. For a compact mapping the stable manifold is infinite dimensional, and an important component of the present work is the development of computer assisted error bounds for numerical approximation of infinite dimensional stable manifolds. As an illustration of the utility of our method we prove the existence of some connecting orbits for a nonlinear dynamical system which appears in mathematical ecology as a model of a spatially distributed ecosystem with population dispersion.

Key words. computer assisted proof, connecting orbits, compact infinite dimensional dynamical systems, stable manifold theory, Galerkin projections

AMS subject classifications. 37L25, 37M99, 65P99, 65G20

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1. Introduction.

The qualitative theory of dynamical systems is concerned with the global orbit structure of nonlinear maps and flows. When we study a specific fixed dynamical system (away from a perturbative regime) it can be difficult to obtain global information by purely analytical methods. Numerical simulations provide insight into the behavior of the system, and in recent years there has been much interest in developing computer assisted methods of proof which distill numerical computations into mathematically rigorous theorems. Such computer assisted arguments are built on some form of a posteriori analysis, and in practice their implementation requires a delicate blend of pen and paper estimates with deliberate control of numerical round-off errors. Several now-classic examples of this program are the works of [66, 67, 72, 73, 32] on the dynamics of the Lorenz equations. The interested reader may also want to consult the review articles [55, 49, 64] for more thorough discussion of the methods and many accomplishments in the area of computer assisted proof for dynamical systems.

Connecting orbits play an important role in dynamical systems theory, as they form a bridge between local and global phenomena. Computer assisted methods for proving the existence of connecting orbits in planar discrete time dynamical systems appear in the literature

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as early as [68]. By now there is a small industry devoted to rigorous computer assisted analysis of connecting orbits for more general finite dimensional maps and ordinary differential equations. A thorough survey of this literature is beyond the scope of the present work, but the reader interested in this research will want to consult [70, 3, 1, 63, 4, 71, 83, 81, 82, 80, 83, 84, 76, 74, 57, 60] and the references therein.

In the present work we develop tools for computer assisted proof of heteroclinic/homoclinic connecting orbits for compact infinite dimensional maps. More precisely, the main contrubutions of the present work are the following:

(I) We develop an a posteriori version of the stable manifold theorem suitable for application to compact infinite dimensional maps. The main result here is Theorem 3.8. The hypotheses of Theorem 3.8 are formulated so that they can be checked for specific infinite dimensional maps using a computer assisted argument. In addition to giving abstract existence of the stable manifold, the conclusions of the theorem provide explicit bounds on the location of the manifold and on its derivative.

(II) We develop a functional analytic, a posteriori framework for studying intersections of stable/unstable manifolds of compact infinite dimensional maps. The main result is Theorem 2.10, which provides explicit sufficient conditions for the existence of a connecting orbit. The theorem accounts for all truncation and discretization errors which go along with projecting the infinite dimensional system for numerical computations. Again, the hypotheses of this theorem are checked via finite computations in specific examples.

(III) We combine (I) and (II) above with the techniques for studying unstable manifolds of compact infinite dimensional maps developed in [45] and implement several computer aided existence proofs for infinite dimensional homoclinic orbits. These results are discussed in section 5.

We illustrate the use of our method for the Kot–Schaffer system. This is a model of dispersive spatiotemporal population dynamics which has received considerable attention in the mathematical ecology literature. See, for example, [50, 51, 79, 58, 88] and the references discussed therein. Previous computer assisted results for the Kot–Schaffer system are discussed further in Remarks 1.1 and 1.2 below. However we note that at present we know of no other successful computer assisted existence proof for intersections of stable/unstable manifolds of infinite dimensional maps. We remark that in principle our arguments apply to other cases of current interest, notably time one maps of parabolic PDE and also dynamics of renormalization operators.

Our argument is based on the method of projected boundaries [21, 27, 26, 5]. (Note, however, that these references deal with finite dimensional systems.) The method of projected boundaries is a functional analytic framework where a connecting orbit for a map is reformulated as the solution of a certain discrete time boundary value problem. This formulation is well suited to numerical continuation schemes based on Newton's method and moreover is amenable to mathematically rigorous computer assisted validation. The methods of projected boundaries is used in the earlier successful computer assisted existence proofs for heteroclinic and homoclinic connecting orbits in finite dimensional maps and flows found in the studies [76, 74, 57, 63]. The present work extends these arguments to certain infinite dimensional settings.

More precisely, let $F\colon \mathcal{X} \to \mathcal{X}$ be a smooth mapping and suppose that we wish to study heteroclinic connecting orbits between some fixed points of the discrete time (semi)dynamical system generated by F. An outline of our computer assisted argument is as follows.

- We begin by formulating a nonlinear operator $\Phi\colon \mathcal{W} \to \mathcal{W}$ having that $\Phi(w) = 0$ is equivalent to a solution of the projected boundary problem. $\mathcal{W}$ will be a product of copies of the phase space $\mathcal{X}$. It is critical that Φ and $\mathcal{W}$ have good finite dimensional projections, and here compactness of the evolution map F plays a role. The map Φ and the space $\mathcal{W}$ are discussed in detail in section 2.4.
- Next we compute numerical approximations of the stable/unstable manifolds and of a connecting orbit for a finite dimensional projection of the dynamical system. We include the numerical connecting orbit into $\mathcal{W}$ and obtain an approximate zero $\bar{w}$ of Φ. The projected problem is discussed in section 2.5.
- We define a "Newton-like" operator $T\colon \mathcal{W} \to \mathcal{W}$ of the form

$$T(w) = w - A\Phi(w),$$

where $A\colon \mathcal{W} \to \mathcal{W}$ is a left approximate inverse of the linear operator $D\Phi(\bar{w})$. A will be obtained by numerical methods and is required to be a one-to-one map. This Newton like operator is discussed further in sections 2.2 and 2.5.
- Finally we develop a posteriori analysis giving sufficient conditions that T is a contraction mapping in a closed neighborhood $B_r(\bar{w}) \subset \mathcal{W}$. Since A is one-to-one the resulting (unique) fixed point $w_* \in B_r(\bar{w})$ is a zero of the map Φ and hence a solution of the projected boundary problem. Then w_* will provide a connecting orbit for the underlying dynamical system generated by the map F. The main result is Theorem 2.2.

This last step of showing that T is a contraction requires checking that a number of inequalities are satisfied in a neighborhood of the approximate solution $\bar{w}$. Since $\bar{w}$ is obtained via numerical methods it is natural that these inequalities will be checked with computer assistance. It is desirable to have $r > 0$ as small as possible, as this provides the validated bound

$$\|w_* - \bar{w}\| \leq r,$$

between the numerical approximation and the true connecting orbit.

To implement this scheme and efficiently manage the (many) resulting inequalities we employ the method of radii-polynomials. This is a toolkit for mathematically rigorous numerical study of Newton-like operators which has been developed by a number of authors over the last several years and is applied successfully to a host of problems in nonlinear analysis. Originally proposed in [85], we refer the interested reader to the works of [56, 44, 13, 7, 6, 75, 39, 38, 35, 36, 37, 34, 18] for applications and extensions of the method and also for more complete discussion of the literature.

An important feature of the radii-polynomial approach is that it provides a family of neighborhoods (about the approximate solution) on which the Newton-like operator is a contraction. The smallest of these neighborhoods provides the tightest error bounds, while the largest provides information about the isolation (i.e., neighborhood of uniqueness) for the solution. While tight error bounds are of most immediate interest, we remark that having more isolation of the solution can be useful as well. For example, in connecting orbit problems isolation provides a measure of transversality [63, 60, 57], while in continuation problems the isolation can be used to maximize continuation step size [18, 38, 39].

Remark 1.1 (fixed and periodic points for Kot–Schaffer). Fixed and periodic orbits for the Kot–Schaffer mapping were studied via rigorous numerical arguments in [33, 31] using a posteriori analysis based on a Newton–Krawczyk operator. Showing the existence of fixed points is of course a necessary prequel to studying connecting orbits. In the present work we use a modification of the arguments of [33, 31] in order to establish the existence of a fixed point and also to analyze its stability. Our modification was discussed in detail (by the second author) in the appendices of [45]. Our arguments differ from those of [33, 31] in that we obtain analytic rather than C^k regularity results and in that we study spatial inhomogeneities given by arbitrary analytic functions rather than trigonometric polynomials.

Remark 1.2 (topological methods/Conley index theory for Kot–Schaffer). Closely related to our study is the work of [16, 17] on the global dynamics of infinite dimensional discrete time dynamical systems. The authors of the works just mentioned use topological methods to compute coverings, or outer enclosures, of the global attractor of the Kot–Schaffer system with various nonlinearities. In addition, the authors use discrete Conley index arguments to establish semi-conjugacies to symbolic dynamics. These semi-conjugacies provide the existence of chaotic motions in the global attractor and yield bounds on the topological entropy of the system. We stress that the authors of [16, 17] take pains to ensure that they enclose the global attractor of the system, and hence their results provide rigorous lower bounds on the complexity of the whole infinite dimensional system.

That being said, a C^0 conjugacy to symbolic dynamics does not establish the existence of invariant stable/unstable manifolds, nor strictly speaking does it establish the existence of connecting orbits between fixed/periodic orbits of the infinite dimensional mapping. The semi-conjugacy forces the existence (in the Kot–Schaffer system) of orbits which “shadow” the connecting orbits of the symbolic system, but these shadows need not be connecting orbits of the Kot–Schaffer system, i.e., they need not accumulate at the fixed/periodic orbit. One concludes only that their forward and backward itineraries are forever confined within a neighborhood of (actually an index pair containing) the fixed/periodic orbit. The arguments of the present work complement the topological methods of [16, 17] and provide means to study C^k phenomena such as transverse intersections of stable/unstable manifolds in infinite dimensional phase space. (See also Remark 1.5 below).

Remark 1.3 (parameterization of unstable manifolds). A critical ingredient in the method of projected boundaries is knowledge of the local stable/unstable manifolds. These are typically represented numerically via some polynomial approximation (perhaps even linear). When developing a posteriori validation for the projected boundary construction it is necessary to have rigorous error bounds on the truncation errors associated with any approximations. (Such bounds are needed even if we use the linear approximation.)

The parameterization method of [8, 9, 10] provides a functional analytic framework for studying the local stable/unstable manifolds of smooth maps on Banach spaces under certain nonresonance conditions on the spectrum of the fixed point. The method yields the dynamics on the manifold as well as the embedding. Efficient numerical schemes based on the parameterization method are discussed, for example, in [43, 41, 12, 6, 14, 59, 11, 62, 24]. See also the recent book on the parameterization method [42] for much more thorough discussion of applications and the surrounding literature.

Because the parameterization method is based on the study of certain functional equations it leads quite naturally to a posteriori analysis. Indeed a number of works develop validated numerical methods based on the parameterization method, and we refer the reader to [76, 25, 77, 61, 60, 63, 24]. The recent work [45] develops a method for validated computation of local unstable manifolds for compact infinite dimensional maps. This work is based on the parameterization method and is implemented for the Kot–Schaffer system studied here. In what follows we employ the techniques of [45] in order to compute polynomial approximations of the unstable manifold for the Kot–Schaffer mapping with rigorously validated error bounds.

Remark 1.4 (stable manifolds of finite co-dimension). The nonresonance conditions required for straightforward application of the parameterization method seem to fail quite generally for stable manifolds associated with infinitely many eigenvalues. Yet such manifolds are the rule rather than the exception for compact infinite dimensional maps. Then, in the present work we study the stable manifold by other arguments.

In section 3 we develop an argument which can be used to obtain validated error bounds associated with the finite dimensional approximation of the stable manifold. These error bounds are especially convenient when applied to the linear approximation of the stable manifolds by the stable eigenspace of a finite dimensional projection map. This argument provides bounds on the derivative of the truncation error as well. Our argument is based on studying a completely different invariance equation than the invariance equation used in the parameterization method. This argument amounts to a reproof of the stable manifold theorem in an explicitly fixed neighborhood of the fixed point, under assumptions which are checked via computer assistance. The invariance equation studied in this case does not yield a conjugacy to the linear dynamics. Our argument exploits in a fundamental way that the stable manifold has finite co-dimension (as is typical for compact maps).

Remark 1.5 (transversality). One feature of a posteriori validation schemes based on the method of projected boundaries mentioned above is that for finite dimensional diffeomorphisms and flows one obtains "for free" the transversality of the connecting orbit as a result of the computer assisted existence proof (for an example, we refer again to [63, 57, 60]). In the present infinite dimensional setting the case for transversality is somewhat more subtle, and we will treat the transversality argument in a separate upcoming work.

Remark 1.6 (slow stable manifolds). The implementation of the projected boundary method in the present work exploits high order approximation of the local unstable manifold but only linear approximation of the stable manifold. This means that we have to study approximate/numerical connecting orbits involving a high number of iterates of the map (i.e., many forward iterates are required before the orbit is close enough to the fixed point that the linear approximation is good enough).

The recent work [65] shows how to parameterize slow stable manifolds and their stable invariant linear bundles. Then one can parameterize only some slow stable directions to high order and approximate the remaining stable directions using the invariant linear bundles. The authors incorporate this construction into a projected boundary setup for computing connecting orbits. A future project is to develop a posteriori validation arguments for this scheme and implement computer assisted proofs for the error bounds. This would substantially decrease the "time of flight" (the value of K in what follows) needed in the arguments here. Of course all of this should be done for maps as well as flows.

Remark 1.7 (renormalization theory). While we focus on an ecological application problem in the present work, we remark that the techniques developed here could be useful in other problems involving fixed points of infinite dimensional operators. We recall, for example, that the seminal work of [53, 23, 78] on a computer assisted proof of the Feigenbaum conjectures was based on the study of the existence and stability of certain nontrivial fixed points of an infinite dimensional renormalization operator. Since then many problems in global analysis have been treated by computer assisted argument applied to renormalization operators. See, for example, the works of [28, 29, 30, 47, 22] and the references discussed therein.

The recent work of [20] provides a theoretical study of connecting orbits between fixed points of renormalization operators and shows that these connecting orbits give quantitative information about combinations of rescaling exponents for renormalization problems. One could obtain mathematically rigorous numerical theorems by combining the ideas of [20] with the validated numerical methods of the present work. In particular the work of [48] develops general computer assisted methods for proving the hyperbolicity of fixed points in renormalization theory. We hope to take up this line of research in a future study.

Remark 1.8 (parabolic partial differential equations). Recent work by a number of authors construct computer assisted proofs based on time τ maps and Poincare sections for parabolic equations [86, 87, 15, 2]. These works develop rigorous numerical integrators for infinite dimensional flows. This facilitates the construction of discrete time maps, such as Poincare sections and time τ maps, which have the compactness properties needed to apply the arguments of the present work. Computer assisted proof for connecting orbits of partial differential equations is a topic of much interest and will be the topic of future studies.

The remainder of the paper is organized as follows. In section 2 we develop the functional analytic setup for studying connecting orbits for compact infinite dimensional maps, as well as the a posteriori theory needed to validate approximate solutions. In section 3 we develop a constructive stable manifold theorem for maps on Banach spaces. This analysis is tailored to computer assisted validation arguments. In section 4 we apply the machinery of sections 2 and 3 and give some computer assisted proofs of connecting orbits for some infinite dimensional maps. All of the programs which implement the computer assisted proofs can be obtained at the Web page for this paper [46].

2. A functional analytic approach to the connecting orbit problem.

2.1. Notation and background.

Suppose that $\mathcal{X}, \mathcal{Y}$ are normed linear spaces. For $p \in \mathcal{X}$, $r > 0$ we write

$$B_r(p) := \{x \in \mathcal{X} : \|x - p\|_{\mathcal{X}} < r\}$$

to denote the ball of radius r centered at p. A linear operator $A \colon \mathcal{X} \to \mathcal{Y}$ is said to be bounded if

$$\sup_{\|x\|_{\mathcal{X}}=1} \|Ax\|_{\mathcal{Y}} < \infty.$$

Let $B(\mathcal{X}, \mathcal{Y})$ denote the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{Y}$ is a Banach space, then $B(\mathcal{X}, \mathcal{Y})$ is a Banach space as well, with norm given by

$$\|A\|_{B(\mathcal{X},\mathcal{Y})} := \sup_{\|x\|_{\mathcal{X}}=1} \|Ax\|_{\mathcal{Y}}.$$

When $\mathcal{Y} = \mathcal{X}$ we write $B(\mathcal{X}) := B(\mathcal{X}, \mathcal{X})$. Throughout the remainder of the present work we take $\mathcal{X}$ and $\mathcal{Y}$ both Banach spaces.

We will be interested in spaces with well behaved finite dimensional projections. More precisely, let $\pi_N \colon \mathcal{X} \to \mathcal{X}$ be a projection operator and suppose that $\mathcal{X}^N := \pi_N(\mathcal{X})$ is a finite dimensional subspace. Let $v_0, \ldots, v_N$ be a basis for $\mathcal{X}^N$ so that for any point $x^N \in \mathcal{X}^N$ there are unique numbers $a_0, \ldots, a_N \in \mathbb{R}$ with

$$x^N = \sum_{n=0}^{N} a_n v_n.$$

Let $L \colon \mathcal{X}^N \to \mathbb{R}^{N+1}$ denote this coordinate identification map. We now endow $\mathbb{R}^{N+1}$ with the norm

$$\|a^N\|_{\mathbb{R}^{N+1}} := \|L^{-1}(a^N)\|_{\mathcal{X}}$$

and have that L is an isometric isomorphism of $\mathcal{X}^N$ into $\mathbb{R}^{N+1}$ with the indicated norm. Let A^N be an $(N+1) \times (N+1)$ matrix. The operator norm on $\mathcal{X}$ induces a matrix norm on the operator A^N by

$$\|A_{ij}\|_{B(\mathbb{R}^{N+1})} := \sup_{\|x^N\|_{\mathcal{X}}=1} \|A_{ij} L(x^N)\|_{\mathcal{X}} = \sup_{\|x^N\|_{\mathbb{R}^{N+1}}=1} \|A_{ij}^N x^N\|_{\mathbb{R}^{N+1}},$$

where $\|\cdot\|_{\mathbb{R}^{N+1}}$ is the norm inherited from $\mathcal{X}$, and $x^N \in \mathcal{X}^N$. We use the shorthand

$$\|x^N\|_N := \|x^N\|_{\mathbb{R}^{N+1}} \qquad \text{and} \qquad \|A^N\|_N := \|A^N\|_{B(\mathbb{R}^{N+1})}.$$

Define the complementary projection $\pi_\infty \colon \mathcal{X} \to \mathcal{X}$ by $\pi_\infty = \mathrm{Id}_{\mathcal{X}} - \pi_N$, and let $\mathcal{X}^\infty := \pi_\infty(\mathcal{X})$. We note that while π_N is always a bounded linear operator, it could be that π_∞ is unbounded. Hence we make the following assumption.

Assumption 2.1. π_∞ is a bounded linear operator.

Then $\mathcal{X}^\infty := \pi_\infty(\mathcal{X})$ is a Banach space and $\mathcal{X} = \mathcal{X}^N \oplus \mathcal{X}^\infty$. We write x^N and x^∞ to denote elements of $\mathcal{X}^N$ and $\mathcal{X}^\infty$, respectively.

In what follows we are often concerned with spaces which are themselves products of Banach spaces. Let $\mathcal{X}_1, \ldots, \mathcal{X}_{K+1}$ be a collection of $K+1$ Banach spaces and define the product space

$$\mathcal{W} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_{K+1},$$

endowed with the norm

$$\|w\|_{\mathcal{W}} := \max_{1 \le j \le K+1} \|x_j\|_{\mathcal{X}_j},$$

where $w = (x_1, \ldots, x_{K+1})$. The product $\mathcal{W}$ is itself a Banach space under this norm. We write $[w]_j = x_j$, $1 \le j \le K+1$ to denote the components of w.

Let $A_{jk}\colon \mathcal{X}_k \to \mathcal{X}_j$ with $1 \le j,k \le K+1$ be bounded linear operators. Define the bounded linear operators $A_j\colon \mathcal{W} \to \mathcal{X}_j$ by

$$A_j(w) = \sum_{k=1}^{K+1} A_{jk} x_k$$

for $1 \le j \le K+1$, and $A\colon \mathcal{W} \to \mathcal{W}$ by

$$[Aw]_j = A_j w, \qquad 1 \le j \le K+1.$$

The product space structure induces the operator norm

$$\|A\|_{B(\mathcal{W})} = \max_{1\le j\le K+1} \sum_{k=1}^{K+1} \|A_{ij}\|_{B(\mathcal{X}_k,\mathcal{X}_j)}.$$

The projections π_N and π_∞ and the component map L induce projections $\pi_N^K, \pi_\infty^K\colon \mathcal{W} \to \mathcal{W}$ given by $\pi_N^K(w)_j := \pi_N(x_j)$, $\pi_\infty^K(w)_j := \pi_\infty(x_j)$, and a component map $\mathcal{L}\colon \mathcal{W}^N \to \mathbb{R}^{(N+1)(K+1)}$ given by

$$\mathcal{L}(w^N) = \begin{pmatrix} L(x_1^N) \\ \vdots \\ L(x_{K+1}^N) \end{pmatrix}.$$

Write $\mathcal{W}^N := \pi_N^K(\mathcal{W})$ and note that $\mathcal{W}^N$ is isometrically isomorphic to $\mathbb{R}^{(N+1)(K+1)}$ with the given norms. If A^{NK} is an $(N+1)(K+1)\times(N+1)(K+1)$ matrix, we write A_{jk}^N to denote the $(N+1)\times(N+1)$ submatrices.

2.2. A posteriori analysis of a "Newton-like" operator on a product space. The following abstract theorem is the main workhorse for our computer assisted existence arguments. Later we will have to verify the hypotheses for specific examples. The proof is identical to the proofs of Theorems 2.1 in [18] or Proposition 1 of [44]. See also [85]. For $U \subset \mathcal{X}$ open we let $C^r(U)$ (or simply C^r when U is understood) denote the space of functions with r continuous and uniformly bounded derivatives on U.

Theorem 2.2. *Suppose that $\Phi\colon \mathcal{W} \to \mathcal{W}$ is a C^2 map. Let $\bar{w} \in \mathcal{W}$ and $A\colon \mathcal{W} \to \mathcal{W}$ be a one-to-one bounded linear operator. Suppose that there are constants r_*, Y_j, Z_j, δ_j, and C_j with $1 \le j \le K+1$ having*

- $$\text{(1)} \qquad \|A_j \Phi(\bar{w})\|_{\mathcal{X}_j} \le Y_j \qquad \textit{(approximate solution)}$$

  for each $1 \le j \le K+1$,

- $$\text{(2)} \qquad \sum_{k=1}^{K+1} \left\| \delta_{jk} Id_{\mathcal{X}_j} - \sum_{l=1}^{K+1} A_{jl} [D\Phi(\bar{w})]_{lk} \right\|_{B(\mathcal{X}_k,\mathcal{X}_j)} \le Z_j \qquad \textit{(approximate inverse)}$$

  for each $1 \le j \le K+1$ (here the Kroneker delta δ_{jk} is one when $j = k$ and zero *otherwise), and that for $w \in B_r(\bar{w})$ we have*

•

$$\|A_j[D\Phi(\bar{w}) - D\Phi(w)]\|_{B(\mathcal{W},\mathcal{X}_j)} \leq \delta_j + C_j r \qquad \textit{(approximately Lipschitz)} \tag{3}$$

whenever $0 < r \leq r_*$ and $1 \leq j \leq K+1$.
Define the $K+1$ polynomials p_j by

$$p_j(r) = C_j r^2 - (1 - Z_j - \delta_j)r + Y_j$$

and suppose that for some $0 < r \leq r_*$ we have

$$p_j(r) \leq 0$$

for each $1 \leq j \leq K+1$. Then there exists a unique $w_* \in B_r(\bar{w})$ so that $\Phi(w_*) = 0$.

The polynomials $p_j(r)$ are sometimes called radii-polynomials and were already mentioned in section 1. The idea of the proof of Theorem 2.2 is to consider the Newton-like operator

$$T(w) = w - A\Phi(w),$$

on a neighborhood of $B_r(\bar{w})$ of the approximate solution $\bar{w}$. Fixed points of T are equivalent to zeros of Φ as A is one-to-one. The radii-polynomials provide sufficient conditions ensuring that T is a contraction on $B_r(\bar{w})$.

Remark 2.3. The δ_j and C_j deserve some additional explanation. Since Φ is C^2 we could choose the C_j as uniform bounds on second the derivatives of Φ over $B_{r_*}(\bar{w})$. In this case the δ_j would be zero and (3) would reduce to a true Lipschitz condition on the first derivative by applying the mean value theorem (as in the usual statement of a Newton–Kantorovich type theorem).

However we allow for the possibility that $D\Phi$ contains terms which we prefer not to differentiate further. This can be managed as long as such terms are themselves small. So suppose, for example, that

$$D\Phi(w) = D\Phi_1(w) + D\Phi_2(w),$$

where $D\Phi_1(w)$ consists of "large" terms and $D^2\Phi_1(w)$ is known and can be bound, while $\|D\Phi_2(w)\| \ll 1$ for all $w \in B_{r_*}(\bar{w})$. Then we apply the mean value theorem to large terms coming from $D\Phi_1$ and the triangle inequality to small terms coming from $D\Phi_2$ and obtain estimates of the form required by (3).

This flexibility is especially useful when dealing with stable manifolds in what follows. We will see that even if the chart maps for the local stable manifold are C^2, in practice we may only have explicit bounds on the first derivative and it will be important that these bounds are not too large (in the precise sense measured by δ_j).

2.3. Hyperbolic fixed points and local stable/unstable manifolds. Let $F\colon \mathcal{X} \to \mathcal{X}$ be a compact C^2 mapping whose second derivative is uniformly bounded in some ball. More precisely we make the following assumptions.

Assumption 2.4.

(i) For some $\bar{x} \in \mathcal{X}$ and $0 < r < \infty$, F is uniformly C^2 on $B_r(\bar{x})$, i.e.,

$$C = C(r, \bar{x}) := \sup_{x \in B_r(\bar{x})} \|D^2F(x)\| < \infty.$$

Then if $x_1, x_2 \in B_r(\bar{x})$, we have

$$\|DF(x_1) - DF(x_2)\| \le C\|x_1 - x_2\|,$$

by the mean value theorem.

(ii) For any $x \in \mathcal{X}$ the linear operator $DF(x)$ is compact. Hence $DF(x)$ has countable spectrum accumulating only at the origin in $\mathbb{C}$. Moreover, any nonzero element of the spectrum is an eigenvalue of finite multiplicity. Since the spectrum of $DF(x)$ is countable and accumulates only at the origin, it follows that $DF(x)$ has at most finitely many unstable eigenvalues, i.e., eigenvalues outside the unit circle in $\mathbb{C}$.

Define the projected maps $F^N \colon \mathcal{X}^N \to \mathcal{X}^N$ by

$$F^N(x^N) := \pi_N(F(x^N))$$

for $x^N \in \mathcal{X}^N$, and $F^\infty \colon \mathcal{X} \to \mathcal{X}$ by

$$F^\infty(x) = F(x) - F^N(\pi_N(x)),$$

and have the decomposition $F(x) = F^N(\pi_N(x)) + F^\infty(x)$. Composing the projected map with the coordinate map gives $\hat{F}^N : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ defined by

$$\hat{F}(a^N) = L\left(F^N\left(L^{-1}(a^N)\right)\right).$$

Now let $p, q \in B_r(\bar{x})$ be hyperbolic fixed points of F, i.e., suppose that $F(p) = p$, $F(q) = q$ and that $DF(p), DF(q)$ have no eigenvalues on the unit circle. By Assumption 2.4(ii) (the compactness assumption) each fixed point has at most a finite number of unstable eigenvalues each of finite multiplicity. Let $\mathbb{E}^u(p), \mathbb{E}^s(p)$ denote the stable and unstable eigenspaces of $DF(p)$ and $\mathbb{E}^u(q), \mathbb{E}^s(q)$ denote the stable and unstable eigenspaces of $DF(q)$ so that

$$\mathbb{E}^u(p) \oplus \mathbb{E}^s(p) = \mathbb{E}^u(q) \oplus \mathbb{E}^s(q) = \mathcal{X}$$

and

$$DF(p)\mathbb{E}^{u,s}(p) \subset \mathbb{E}^{u,s}(p), \qquad DF(q)\mathbb{E}^{u,s}(q) \subset \mathbb{E}^{u,s}(q).$$

Again, Assumption 2.4(ii) gives that $\mathbb{E}^u(p)$ and $\mathbb{E}^u(q)$ are finite dimensional vector spaces.

In the present work we are interested in studying zero dimensional intersections of the stable and unstable manifolds and make the following additional assumptions.

Assumption 2.5. Assume that $DF(p)$ and $DF(q)$ have the same number of unstable eigenvalues, i.e., that

$$\dim\left(\mathbb{E}^u(p)\right) = \dim\left(\mathbb{E}^u(q)\right) = m$$

for some $m \in \mathbb{N}$. Note also that $\mathbb{E}^s(p)$ and $\mathbb{E}^s(q)$ are then co-dimension m vector spaces, as we suppose that p, q are hyperbolic.

For the remainder of the discussion we focus on the unstable manifold of p and the stable manifold of q. To emphasize that the unstable eigenspace is m-dimensional and hence isomorphic to $\mathbb{R}^m$ we write

$$\mathbb{R}^m \approx \mathbb{E}^u = \mathbb{E}^u(p) \qquad \text{and} \qquad \mathbb{E}^s = \mathbb{E}^s(q)$$

and suppress the p and q dependence. The unstable manifold theorem implies that $W^u_{\text{loc}}(p)$ is an immersed m-dimensional disk, while the stable manifold theorem implies that $W^s_{\text{loc}}(q)$ is an immersed disk of co-dimension m. In both cases the immersions are as smooth as the map, i.e., C^2.

Let

$$B_{\tilde{r}} := \{\theta \in \mathbb{R}^m \,:\, \|\theta\| < \tilde{r}\}$$

and

$$\hat{U} \subset \mathbb{E}^s$$

be neighborhoods of the origin in $\mathbb{R}^m$ and $\mathbb{E}^s$, respectively, and let $P\colon B_{\tilde{r}} \subset \mathbb{R}^m \to \mathcal{X}$ and $Q\colon \hat{U} \subset \mathbb{E}^s \to \mathcal{X}$ denote chart maps for the local unstable and stable manifolds, respectively, i.e., suppose that

$$P[B_{\tilde{r}}] = W^u_{\text{loc}}(p) \qquad \text{and} \qquad Q[\hat{U}] = W^s_{\text{loc}}(q).$$

Assumption 2.6. Let $P^N\colon \mathbb{R}^m \to \mathcal{X}^N$, $Q^N\colon \mathbb{R}^{N+1-m} \to \mathcal{X}^N$ be polynomial maps and $P^\infty\colon B_{\tilde{r}} \subset \mathbb{R}^m \to \mathcal{X}$, $Q^\infty\colon \hat{U} \subset \mathbb{R}^{N+1-m} \times \mathcal{X}^\infty \to \mathcal{X}$ be C^1 maps so that

$$P(\theta) = P^N(\theta) + P^\infty(\theta)$$

for all $\theta \in B_{\tilde{r}}$ and

$$Q(\phi^N, \phi^\infty) = Q^N(\phi^N) + Q^\infty(\phi^N, \phi^\infty)$$

for all $\phi = (\phi^N, \phi^\infty) \in \hat{U}$.

Assume in addition that

$$\pi_\infty(Q^\infty(\phi^N, \phi^\infty)) = \pi_\infty(q) + \phi^\infty,$$

where $q \in \mathcal{X}$ is the fixed point with stable manifold parameterized by Q.

From Assumption 2.6 we have that

$$\pi_\infty(Q^\infty(\phi^N, 0)) = \pi_\infty(q) \tag{4}$$

and also that

$$\pi_\infty DQ^\infty(\phi^N, \phi^\infty) = \text{Id}_{\mathcal{X}^\infty}. \tag{5}$$

Remark 2.7 (parameterizations of the local invariant manifolds). Assumption 2.6 deserves some clarification. In the applications to follow we obtain P^N and P^∞ using computer assisted techniques based on the parameterization methods of [8, 9, 10]. In particular we exploit the a posteriori theory and numerical implementation developed in [45]. On the other hand, in section 3 we develop a stable manifold theorem and validated error bounds which allow us to compute Q^N and obtain bounds on Q^∞. See also Remarks 1.3 and 1.4 in the introduction.

2.4. Formalizing the connecting orbit problem: The method of projected boundaries. The notation and assumptions of sections 2.1 and 2.3 are in force throughout this section. In particular p, q are hyperbolic fixed points of the compact map F, and P, Q parameterize the local unstable/stable manifolds of p and q, respectively. We propose the following "multiple shooting" scheme for connecting orbits and suppose that for some fixed $K \in \mathbb{N}$ there exist $\theta^* \in B_{\tilde{r}} \subset \mathbb{R}^m$, $x_1^*, \ldots, x_K^* \in B_r(\bar{x}) \subset \mathcal{X}$ and $\phi^* \in \hat{U} \subset \mathbb{E}^s$ satisfying the system of equations

$$\begin{aligned} P(\theta) &= x_1, \\ F(x_1) &= x_2 \\ &\vdots \\ F(x_{K-1}) &= x_K, \\ F(x_K) &= Q(\phi). \end{aligned}$$

Then

- $F^K(P(\theta^*)) = Q(\phi^*)$,
- $P(\theta^*) \in W^u_{\text{loc}}(p)$, and
- $Q(\phi^*) \in W^s_{\text{loc}}(q)$.

(Here F^K denotes the composition of the map F with itself K times). A solution of this system of equations corresponds to a heteroclinic orbit from p to q. More precisely, let $y^* = x_j^*$ for any $1 \le j \le K$ or $y^* = Q(\phi)$. Since $P(\theta^*) \in W^u_{\text{loc}}(p)$ is a preimage of y^* we have that there exists a sequence $\{y_n\}_{n \le 0} \subset \mathcal{X}$ so that $y_0 = y^*$, $F(y_{n-1}) = y_n$ for all $n \le 0$, and

$$\lim_{n \to -\infty} y_n = p,$$

i.e., there exists a backward orbit of y^* accumulating at p.

Similarly, there is an iterate of y^* in the local stable manifold of q. Then

$$\lim_{n \to \infty} F^k(y^*) = q,$$

i.e., the orbit of y^* is a heteroclinic from p to q. The state of affairs is illustrated in Figure 1.

Based on these remarks we define the map $\Phi \colon B_{\tilde{r}} \times \hat{U} \times \mathcal{X}^K \to \mathcal{X}^{K+1}$ by

$$\Phi(\theta, \phi, x_1, \ldots, x_K) = \begin{bmatrix} P(\theta) - x_1 \\ F(x_1) - x_2 \\ \vdots \\ F(x_{K-1}) - x_K \\ F(x_K) - Q(\phi) \end{bmatrix}. \tag{6}$$

The condition $x_1^*, \ldots, x_K^* \in B_r(\bar{x})$ just recalls Assumption 2.4 and guarantees that the entire discussion takes place in a domain where we have a uniform bound on the second derivative of F. By the discussion above the existence of a zero of Φ implies the existence of a heteroclinic orbit from p to q. We refer to Φ as the *connecting orbit operator*, and in the sequel we develop numerical and a posteriori analysis for Φ.

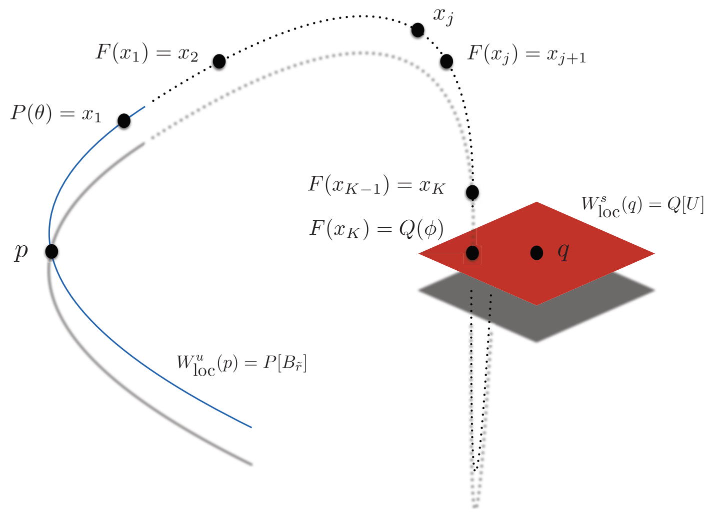

Figure 1. *In this picture P represents the image of the parameterization of the unstable manifold and Q represents the image of the stable parameterization. The picture suggests that the unstable manifold is low dimensional and represented to high order while the stable manifold is high (infinite) dimensional and represented to low (linear) order, as this is the situation which we treat in the applications.*

If $x_2, \dots, x_K$ are not in the image of either P or Q we refer to K as the *time of flight* relative to the parameterizations P and Q, i.e., it takes K iterates for this orbit segment to get from the local unstable to the local stable manifold. Of course, the choice of K is a matter of practical convenience as all the points x_j are in the stable manifold and could be included in the parameterization. In practice one has to choose the parameterizations in a neighborhood of the fixed point so that they are easy to handle, and hence K may be somewhat large.

Remark 2.8 (domain of the connecting orbit operator). By Assumption 2.5 we have an unstable manifold of dimension m at p and a stable manifold of co-dimension m at q. Then $\mathcal{X}$ is isomorphic to $\mathbb{R}^m \times \mathbb{E}^s$ and we have that

$$\mathbb{R}^m \times \mathbb{E}^s \times \mathcal{X}^K \equiv \mathcal{X}^{K+1},$$

in the sense of vector space isomorphisms. In other words the domain and range of Φ are isomorphic, and hence Newton's method is an appropriate tool for studying the zeros of Φ. Another way to say this is that we are looking for zero dimensional intersections between the stable and unstable manifolds.

2.5. A posteriori validation for the connecting orbit operator. We now instantiate the abstract Theorem 2.2 for the specific case of the connecting orbit operator defined in section 2.4. We take $F\colon \mathcal{X} \to \mathcal{X}$ as in section 2.3, and $\mathcal{X}_j = \mathcal{X}$ for all $1 \le j \le K+1$, i.e., for the remainder of the paper let

$$\mathcal{W} = \mathcal{X}^{K+1}.$$

We decompose the connecting orbit operator as

$$\Phi(w) = \Phi^N \left(\pi_N^K(w)\right) + \Phi^\infty(w),$$

where $\Phi^N\colon (\mathcal{X}^N)^{K+1} \to (\mathcal{X}^N)^{K+1}$ is given by

$$\Phi^N(w^N) = \Phi^N(\theta, \phi^N, x_1^N, \dots, x_K^N) = \begin{pmatrix} P^N(\theta) - x_1^N \\ F^N(x_1^N) - x_2^N \\ \vdots \\ F^N(x_{K-1}^N) - x_K^N \\ F^N(x_K^N) - Q^N(\phi^N) \end{pmatrix},$$

and $\Phi^\infty\colon \mathcal{W} \to \mathcal{W}$ is given by

$$\Phi^\infty(w) = \Phi^\infty(\theta, \phi^N, \phi^\infty, x_1, \dots, x_K) = \begin{pmatrix} P^\infty(\theta) - x_1^\infty \\ F^\infty(x_1) - x_2^\infty \\ \vdots \\ F^\infty(x_{K-1}) - x_K^\infty \\ F^\infty(x_K) - Q^\infty(\phi^N, \phi^\infty) \end{pmatrix}.$$

Then the map $\hat{\Phi}^N\colon \mathbb{R}^{(N+1)(K+1)} \to \mathbb{R}^{(N+1)(K+1)}$ defined by

$$\hat{\Phi}^N(\bar{w}^N) = \mathcal{L}\Phi^N\left(\mathcal{L}^{-1}(\bar{w}^N)\right)$$

gives an approximation of the connecting orbit operator as a map between Euclidean spaces. Note that L and $\mathcal{L}$ are bounded linear operator and hence Fréchet differentiable. Then the maps $\hat{F}^N$ and $\hat{\Phi}^N$ are differentiable (in fact C^2).

Definition 2.9 (validation values for the connecting orbit operator). *The fixed data* $\bar{\theta} \in \mathbb{R}^m$, $\bar{x}_1^N, \dots, \bar{x}_K^N \in \mathbb{R}^{N+1}$, $\bar{\phi}^N \in \mathbb{R}^{N+1-m}$, *the* $(N+1)(K+1) \times (N+1)(K+1)$ *matrix* A^{NK}, *and the collection of positive constants* r_*, r_q, r_P, r_Q, δ_P, δ_Q, $\{C_F^j\}_{j=1}^{K+1}$ *are referred to as validation values for the connecting orbit operator if*

1. $\|q - \pi_N(q)\|_{\mathcal{X}} \le r_q$,
2. A^{NK} *is invertible,*
3. $$B_{r_*}(\bar{\theta}) \subset B_{\tilde{r}} \subset \mathbb{R}^m,$$
4. $$B_{r_*}(\bar{\phi}^N) \subset \pi_N(\hat{U}) \subset \mathbb{R}^{N+1-m},$$
5. $$\sup_{\theta \in B_{\tilde{r}}} \|P^\infty(\theta)\|_{\mathcal{X}} \le r_P \qquad \text{and} \qquad \sup_{\phi \in \hat{U}} \|Q^\infty(\phi^N, \phi^\infty)\|_{\mathcal{X}} \le r_Q,$$
6. $$\sup_{\theta \in B_{\tilde{r}}} \|DP^\infty(\theta)\|_{B(\mathbb{R}^m, \mathcal{X})} \le \delta_P,$$

   and

   $$\sup_{\phi \in \hat{U}} \left\|\pi_N\left(DQ^\infty(\phi^N, \phi^\infty)\right)\right\|_{B(\mathbb{R}^{N+1-m} \oplus \mathcal{X}^\infty, \mathbb{R}^{N+1})} \le \delta_Q;$$

7. *for each* $1 \leq j \leq K+1$ *we have*

$$\|DF(\bar{x}_j) - DF(x)\|_{B(\mathcal{X})} \leq C_F^j \|\bar{x}_j - x\|_{\mathcal{X}},$$

whenever $x \in B_r(\bar{x}_j) \subset \mathcal{X}$ *and* $0 < r \leq r_*$*;*

8. *if* $p = q$*, then we ask in addition that* $r_* > r_p = r_q$ *and*

$$\pi_N(p) \notin B_{r_*}(\bar{x}_j^N) \subset \mathbb{R}^{N+1}$$

for some $1 \leq j \leq K$*,*

We note that Condition 1 of Definition 2.9 asks that r_q is an error estimate on q. Condition 2 requires that A^{NK} is an invertible matrix, a fact which is used below to construct a one-to-one approximate inverse for $D\Phi$ at the approximate solution. Conditions 3 and 4 simply ask that Φ is well defined on $B_{r_*}(\bar{w})$, i.e., that this ball is in the domain of the stable/unstable manifold parameterizations. Condition 5 says that r_P and r_Q bound the unstable and stable truncation errors while condition 6 says that δ_P and δ_Q bound the derivatives of these errors. Condition 7 asks that C_F^j are Lipschitz constants for DF as in (i) of section 2, and finally condition 8 rules out the possibility of a "trivial homoclinic connection," i.e., $x_j \neq p$, $\theta, \phi = 0$.

Theorem 2.10 (radii-polynomials for the connecting orbit operator). *Suppose that* $\bar{\theta} \in \mathbb{R}^m$*,* $\bar{x}_1^N, \ldots, \bar{x}_K^N \in \mathbb{R}^{N+1}$*,* $\bar{\phi}^N \in \mathbb{R}^{N+1-m}$*,* A^{NK}*,* r_**,* r_q*,* r_P*,* r_Q*,* δ_P*,* δ_Q*, and* C_F^1*, …,* C_F^{K+1} *are a collection of validation values for the connecting orbit operator, in the sense of Definition 2.9. Define* $\bar{w}^N \in \mathbb{R}^{(N+1)(K+1)}$ *by*

$$\bar{w}^N = (\bar{\theta}, \bar{\phi}^N, \bar{x}_1^N, \ldots, \bar{x}_{K+1}^N),$$

$\bar{x}_1, \ldots, \bar{x}_K \in \mathcal{X}^N$ *by*

$$\bar{x}_j = L(\bar{x}_j^N),$$

and $\bar{w} \in \mathcal{W}^N$ *by*

$$\bar{w} = \mathcal{L}(\bar{w}^N).$$

Let $A\colon \mathcal{W} \to \mathcal{W}$ *be the one-to-one bounded linear operator given by*

$$Aw = \mathcal{L}^{-1} A^{NK} \mathcal{L} \pi_N^K(w) + \pi_\infty^K(w).$$

Below we abuse notation and write $A^{NK} x^N$ *for* $\mathcal{L}^{-1} A^{NK} \mathcal{L} x^N$*, i.e., we think of the matrix* A^{NK} *as acting on* $\mathcal{X}^N$*.*

Now assume that for each $1 \leq j \leq K+1$ *there are positive constants* Y_j^0, Y_j^1, Y_j^2, Z_j^0, Z_j^1, Z_j^2, C_j^1, C_j^2, *and* δ_j *satisfying*

$$\left\| A_j^{MN} \hat{\Phi}^N \left(\bar{w}^N \right) \right\|_N \leq Y_j^0,$$

$$r_P \|A_{j1}^N\|_N + \sum_{i=2}^{K+1} \|A_{ji} \pi_N (F^\infty(\bar{x}_{i-1}))\|_{\mathcal{X}} + r_Q \|A_{jK+1}^N\|_N \leq Y_j^1,$$

$$\max \left(\begin{cases} r_P & j = 1 \\ \|\pi_\infty(F^\infty(\bar{x}_{j-1}))\|_\infty & 2 \leq j \leq K \\ \|\pi_\infty(F^\infty(\bar{x}_K))\|_\infty + r_q & j = K+1 \end{cases} \right) \leq Y_j^2,$$

$$\sum_{k=1}^{K+1} \left\| \delta_{jk} Id_{\mathbb{R}^{N+1}} - \sum_{l=1}^{K+1} A_{jl}^N D\hat{\Phi}_{lk}^N(\bar{w}^N) \right\|_N \leq Z_j^0$$

(where the Kronecker delta δ_{jk} *is one of* $j = k$ *and zero otherwise),*

$$\delta_P \|A_{j1}\|_N + \sum_{k=2}^{K+1} \|A_{jk}^N \pi_N (DF^\infty(\bar{x}_{k-1}))\|_{\mathcal{X}} + \delta_Q \, \|A_{jK+1}^N\| \leq Z_j^1,$$

$$\max \left(\begin{cases} \delta_P & j = 1 \\ \|\pi_\infty(DF^\infty(\bar{x}_{j-1}))\|_{\mathcal{X}} & 2 \leq j \leq K \\ \|\pi_\infty(DF^\infty(\bar{x}_K))\|_{\mathcal{X}} & j = K+1 \end{cases} \right) \leq Z_j^2,$$

$$\|A_{j1}^N\|_N \sup_{\theta \in B_{r_*}(\bar{\theta})} \|D^2 P^N(\theta)\| + \|A_{jK+1}^N\|_N \sup_{\phi \in B_{r_*}(\bar{\phi})} \|D^2 Q^N(\phi)\| \leq C_j^1,$$

$$\sum_{k=2}^{K+1} \|A_{jk}\|_N \, C_F^{k-1} \leq C_j^2,$$

and

$$2\delta_P \|A_{j1}^N\|_N + 2\delta_Q \|A_{jK+1}^N\|_N \leq \delta_j.$$

Then the numbers

$$Y_j := Y_j^0 + Y_j^1 + Y_j^2,$$
$$Z_j = Z_j^0 + Z_j^1 + Z_j^2,$$
$$C_j = C_j^1 + C_j^2,$$

and δ_j *satisfy the hypotheses of Theorem* 2.2*: i.e., the polynomials*

$$p_j(r) := C_j r^2 - (1 - Z_j - \delta_j) r + Y_j$$

are radii-polynomials for the connecting orbit operator.

Proof. We begin by considering the defect, or a posteriori error. Note that $\pi_\infty^K(\bar{w}) = 0$. Now consider

$$
\begin{aligned}
A_j\Phi(\bar{w}) &= A_j[\Phi^N(\bar{w}) + \Phi^\infty(\bar{w})] \\
&= A_j^N\Phi^N(\bar{w}) + A_j^N\pi_N^K(\Phi^\infty(\bar{w})) + \pi_\infty^K(\Phi^\infty(\bar{w})_j).
\end{aligned} \tag{7}
$$

We have that

$$
\begin{aligned}
\Phi^\infty(\bar{w})_j &= \begin{cases} P^\infty(\bar{\theta}) - \bar{x}_1^\infty, & j = 1, \\ F^\infty(\bar{x}_{j-1}) - \bar{x}_j^\infty, & j = 2, \dots K, \\ F^\infty(\bar{x}_K) - Q^\infty(\bar{\phi}^N, 0), & j = K+1, \end{cases} \\
&= \begin{cases} P^\infty(\bar{\theta}), & j = 1, \\ F^\infty(\bar{x}_{j-1}), & j = 2, \dots K, \\ F^\infty(\bar{x}_K) - Q^\infty(\bar{\phi}^N, 0), & j = K+1, \end{cases}
\end{aligned}
$$

and it follows that

$$
\|\pi_\infty^K(\Phi^\infty(\bar{w})_j)\| \le Y_j^2, \tag{8}
$$

where we employ the property given by (4) to bound the norm of the π_∞ projection of Q^∞. Next we note that

$$
\begin{aligned}
A_j^N\pi_N^K(\Phi^\infty(\bar{w})) &= \sum_{k=1}^{K+1} A_{jk}^N\pi_N(\Phi^\infty(\bar{w})_k) \\
&= A_{j1}\pi_N(P^\infty(\bar{\theta})) + \sum_{k=2}^{K+1} A_{jk}^N\pi_N(F^\infty(\bar{x}_{k-1})) + A_{jK+1}^N\pi_N(Q^\infty(\bar{\phi}^N, \bar{\phi}^\infty)).
\end{aligned}
$$

Note also that

$$
\|\Phi^N(\bar{w})\|_{\mathcal{X}} = \|\hat{\Phi}^N(\bar{w}^N)\|_N,
$$

as $\mathcal{L}$ is an isometric isomorphism. Then, recalling (7) and the bound of (8) we have

$$
\begin{aligned}
\|A_j\Phi(\bar{w})\| &\le \|A_j^N\hat{\Phi}^N(\bar{w}^N)\|_N + \|A_{j1}^N\|r_P + \sum_{k=2}^{K+1} \|A_{jk}^N\pi_N F^\infty(\bar{x}_{k-1})\| + \|A_{jK+1}^N\|r_Q + Y_j^2 \\
&\le Y_j^0 + Y_j^1 + Y_j^2 \\
&= Y_j
\end{aligned}
$$

and see that Y_j satisfies the condition required by (1) of Theorem 2.2.

Moving on to the approximate inverse bound we begin with the finite projection and observe that

$$
\pi_N\left(\delta_{jk}\mathrm{Id}_{\mathcal{X}} - \sum_{l=1}^{K+1} A_{jl}[D\Phi(\bar{w})]_{lk}\right) = \delta_{jk}\mathrm{Id}_N - \sum_{l=1}^{K+1} A_{jl}^N[D\Phi^N(\bar{w}))]_{lk} - \sum_{l=1}^{K+1} A_{jl}^N\pi_N[D\Phi^\infty(\bar{w})]_{lk}. \tag{9}
$$

The estimates are aided substantially by the fact that $D\Phi(\bar{w})$ has an almost diagonal structure. For example, summing over all $1 \le k \le K+1$ for the last term in (9) gives

$$\sum_{k=1}^{K+1}\sum_{l=1}^{K+1} A_{jl}^N \pi_N [D\Phi^\infty(\bar{w})]_{lk}$$
$$= A_{j1}^N \pi_N(DP^\infty(\bar{\theta})) + \sum_{k=2}^{K+1} A_{jk}^N \pi_N(DF^\infty(\bar{x}_{k-1})) + A_{jK+1}^N \pi_N(DQ^\infty(\bar{\phi}^N, \bar{\phi}^\infty)).$$

For the π_∞ projection we recall that

$$\pi_\infty(DQ^\infty(\bar{\phi}^N, \bar{\phi}^\infty)) = \mathrm{Id}_{\mathcal{X}^\infty}$$

(from (5)) and

$$\sum_{k=1}^{K+1} \pi_\infty\left([D\Phi(\bar{w})]_{jk}\right) = \begin{cases} \pi_\infty DP^\infty(\bar{\theta}) - \mathrm{Id}_{\mathcal{X}^\infty}, & j=1, \\ \pi_\infty DF^\infty(\bar{x}_{j-1}) - \mathrm{Id}_{\mathcal{X}^\infty}, & j=2,\dots,K, \\ \pi_\infty DF^\infty(\bar{x}_K) - \mathrm{Id}_{\mathcal{X}^\infty}, & j=K+1. \end{cases} \tag{10}$$

Since

$$\pi_\infty\left(\delta_{jk}\mathrm{Id}_{\mathcal{X}_j} - \sum_{l=1}^{K+1} A_{jl}[D\Phi(\bar{w})]_{lk}\right) = \delta_{jk}\mathrm{Id}_{\mathcal{X}_j^\infty} - \pi_\infty\left([D\Phi(\bar{w})]_{jk}\right) \tag{11}$$

and recalling (10) and (11) we see that

$$\sum_{k=1}^{K+1} \pi_\infty\left(\delta_{jk}\mathrm{Id}_{\mathcal{X}_j} - \sum_{l=1}^{K+1} A_{jl}[D\Phi(\bar{w})]_{lk}\right) = \mathrm{Id}_{\mathcal{X}^\infty} - \sum_{k=1}^{K+1} \pi_\infty\left([D\Phi(\bar{w})]_{jk}\right)$$
$$= \begin{cases} \pi_\infty DP^\infty(\bar{\theta}), & j=1, \\ \pi_\infty DF^\infty(\bar{x}_{j-1}), & j=2,\dots,K, \\ \pi_\infty DF^\infty(\bar{x}_K), & j=K+1. \end{cases}$$

Of course we have that

$$\|D\Phi^N(\bar{w})\|_{B(\mathcal{W})} = \|D\hat{\Phi}^N(\bar{w}^N)\|_N,$$

again by the fact that $\mathcal{L}$ is an isometric isomorphism. Combining the expressions for the π_N and π_∞ projections gives

$$\left\|\delta_{jk}\mathrm{Id}_{\mathcal{X}} - \sum_{l=1}^{K+1} A_{jl}[D\Phi(\bar{w})]_{lk}\right\|_{\mathcal{X}} \le Z_j^0 + Z_j^1 + Z_j^2 = Z_j,$$

as required by (2) of Theorem 2.2.

Finally let $w \in B_r(\bar{w})$ and $h \in \mathcal{W}$. We write

$$h = (h_u, h_s, h_1, \dots, h_K)$$

with $h_u \in \mathbb{R}^m$, $h_s \in \mathbb{R}^{N+1-m} \oplus \mathcal{X}^\infty$, and $h_j \in \mathcal{X}$ for $1 \le j \le K$. Then

$$
\begin{aligned}
&[(D\Phi(\bar{w}) - D\Phi(w))\, h]_j \\
&= \begin{cases} \left(DP(\bar{\theta}) - DP(\theta)\right) h_u, & j = 1, \\ \left(DF(\bar{x}_{j-1}) - DF(x_{j-1})\right) h_{j-1}, & 2 \le j \le K, \\ \left(DF(\bar{x}_K) - DF(x_K)\right) h_K + \left(DQ(\phi^N, \phi^\infty) - DQ(\bar{\phi}^N, \bar{\phi}^\infty)\right) h_s, & j = K+1. \end{cases}
\end{aligned}
$$

We consider these component by component. For $j = 1$ we have

$$
\begin{aligned}
\|(DP(\bar{\theta}) - DP(\theta)\| &\le \|DP^N(\bar{\theta}) - DP^N(\theta)\| + \|DP^\infty(\bar{\theta}) - DP^\infty(\theta)\| \\
&\le \sup_{\theta \in B_r(\bar{\theta})} \|DP^N(\theta)\| r + \|DP^\infty(\bar{\theta})\| + \|DP^\infty(\theta)\| \\
&\le \sup_{\theta \in B_{r_*}(\bar{\theta})} \|DP^N(\theta)\| r + 2\delta_P.
\end{aligned}
$$

For the DQ bounds in the $j = K+1$ component we consider projections. The π_N projection is bounded as

$$
\begin{aligned}
\|\pi_N(DQ(\bar{\phi}) - DQ(\phi))\| &\le \|DQ^N(\bar{\phi}^N) - DQ^N(\phi^N)\| + \|\pi_N\left(DQ^\infty(\bar{\phi}) - DQ^\infty(\phi)\right)\| \\
&\le \sup_{\phi^N \in B_r(\bar{\phi}^N)} \|DQ^N(\theta)\| r + \|\pi_N DQ^\infty(\bar{\theta})\| + \|\pi_N DQ^\infty(\theta)\| \\
&\le \sup_{\phi^N \in B_{r_*}(\bar{\phi}^N)} \|DQ^N(\theta)\| r + 2\delta_Q,
\end{aligned}
$$

while for the π_∞ projection we have

$$
\|\pi_\infty(DQ(\bar{\phi}) - DQ(\phi))\| = 0,
$$

as $\pi_\infty DQ(\phi) = \pi_\infty DQ^\infty(\phi)$ and recalling again (5). Now for terms involving the map F we simply employ the C_F^j bounds. Then

$$
\begin{aligned}
&\| \left[(D\Phi(\bar{w}) - D\Phi(w))\, h\right]_j \| \\
&\le \begin{cases} \left(\sup_{\theta \in B_{r_*}(\bar{\theta})} \|DP^N(\theta)\| r + 2\delta_P\right) \|h_u\|, & j = 1, \\ C_F^{j-1} r \|h_{j-1}\|, & 2 \le j \le K+1, \\ \left(C_F^K + \sup_{\phi^N \in B_{r_*}(\bar{\phi}^N)} \|DQ^N(\theta)\|\right) r\|h_K\| + 2\delta_Q \|h_s\|, & j = K+1. \end{cases}
\end{aligned}
\tag{12}
$$

Supposing that $\|h\| = 1$ we have

$$
\begin{aligned}
\|A_j[D\Phi(\bar{w}) - D\Phi(w)]h\| &\le \sum_{k=1}^{K+1} \|A_{jk}[D\Phi(\bar{w}) - D\Phi(w))h]_k\| \\
&\le (C_j^1 + C_j^2) r + \delta_j \\
&= C_j r + \delta_j.
\end{aligned}
$$

Taking the supremum over all such h shows that the C_j and δ_j satisfy the conditions required by (3) of Theorem 2.2. ■

3. Validated C^0 and C^1 error bounds for a local representation of the stable manifold of a compact infinite dimensional map. We begin by remarking that the material in this section is largely independent from the remainder of the paper. In particular some notation is used in this section which has a different meaning in other sections of the paper. (One should be especially careful about the meanings of $\mathcal{X}$, K, and Ψ in different parts of the paper.) However we feel that there is little possibility of confusion once this remark is taken into account.

Let $\mathcal{X} = \mathbb{R}^{N_u} \times \mathbb{R}^{N_s} \times \mathcal{X}^\infty$ with $\mathcal{X}^\infty$ a (possibly infinite dimensional) Banach space. Then we write $x \in \mathcal{X}$ as

$$x = (v^u, v^s, v^\infty) \qquad \text{with} \qquad v^u \in \mathbb{R}^{N_u}, v^s \in \mathbb{R}^{N_s}, v^\infty \in \mathcal{X}^\infty$$

and employ the norm on $\mathcal{X}$ given by

$$\|x\|_{\mathcal{X}} := \|v^u\|_{\mathbb{R}^{N_u}} + \|v^s\|_{\mathbb{R}^{N_u}} + \|v^\infty\|_{\mathcal{X}^\infty}.$$

We do not require the norms on $\mathbb{R}^{N_u,N_s}$ to be the Euclidean norms. (In fact in the applications we have in mind they will be weighted sum norms inherited from the ℓ^1_ν structure of the overall problem.)

Suppose that $F\colon \mathcal{X} \to \mathcal{X}$ is a C^2 map with $F(0) = 0$. Moreover assume that F is decomposed as $F = (F_1, F_2, F_3)$, where these component maps take values in $\mathbb{R}^{N_u}$, $\mathbb{R}^{N_s}$, and $\mathcal{X}^\infty$, respectively. The main requirement on the decomposition is that each component map is expressed as a linear self map, plus a small linear coupling term, plus quadratically small nonlinearities. More precisely we assume that F is given by

$$\begin{aligned} F_1(v^u, v^s, v^\infty) &= A_u^N v^u + E_1(v^u, v^s, v^\infty) + H_1(v^u, v^s, v^\infty), \\ F_2(v^u, v^s, v^\infty) &= A_s^N v^s + E_2(v^u, v^s, v^\infty) + H_2(v^u, v^s, v^\infty), \\ F_3(v^u, v^s, v^\infty) &= A_s^\infty v^\infty + E_3(v^u, v^s, v^\infty) + H_3(v^u, v^s, v^\infty) \end{aligned} \tag{13}$$

with $F_1\colon \mathcal{X} \to \mathbb{R}^{N_u}$, $F_2\colon \mathcal{X} \to \mathbb{R}^{N_s}$, and $F_3\colon \mathcal{X} \to \mathcal{X}^\infty$. Here

$$A_u^N \in B\left(\mathbb{R}^{N_u}, \mathbb{R}^{N_u}\right), \qquad A_s^N \in B\left(\mathbb{R}^{N_s}, \mathbb{R}^{N_s}\right), \qquad \text{and} \qquad A_s^\infty \in B\left(\mathcal{X}^\infty, \mathcal{X}^\infty\right)$$

are the maps describing the linear "self coupling" of $\mathbb{R}^{N_u}$, $\mathbb{R}^{N_s}$, and $\mathcal{X}^\infty$ into themselves, i.e., these maps are endomorphisms. We assume that A_u^N is invertible and that

$$\left\|\left(A_u^N\right)^{-1}\right\| < 1, \qquad \|A_s^N\| < 1, \qquad \text{and} \qquad \|A_s^\infty\| < 1.$$

Similarly the terms

$$E_1 \in B(\mathcal{X}, \mathbb{R}^{N_u}), \qquad E_2 \in B(\mathcal{X}, \mathbb{R}^{N_s}), \qquad \text{and} \qquad E_3 \in B(\mathcal{X}, \mathcal{X}^\infty)$$

are assumed to be bounded linear operators, and we assume further that there are real numbers $\epsilon_1, \epsilon_2, \epsilon_3 \geq 0$ with

$$\|E_1\| \leq \epsilon_1, \qquad \|E_2\| \leq \epsilon_2, \qquad \text{and} \qquad \|E_3\| \leq \epsilon_3.$$

We think of $E_{1,2,3}$ as being small in norm (in a sense to be made precise below). The maps $E_{1,2,3}$ describe the linear coupling of the full space $\mathcal{X}$ into the component spaces $\mathbb{R}^{n_s,n_u}$ and $\mathcal{X}^\infty$.

Suppose that $r, R > 0$ and let

$$B_r = \left\{ v^s \in \mathbb{R}^{N_s} \, : \, \|v^s\| < r \right\}$$

and

$$U_R = \left\{ v^\infty \in \mathcal{X}^\infty \, : \, \|v^\infty\| < R \right\}.$$

Assume that

$$H_1 \colon \mathcal{X} \to \mathbb{R}^{N_u}, \qquad H_2 \colon \mathcal{X} \to \mathbb{R}^{N_s}, \qquad \text{and} \qquad H_3 \colon \mathcal{X} \to \mathcal{X}^\infty$$

are two times Fréchet differentiable functions and that there is a $\rho > 0$ so that for any

$$x \in \mathcal{B}_\rho(0) = \left\{ x \in \mathcal{X} : \|x\| \leq \rho \right\},$$

the nonlinearities satisfy the bounds

$$\begin{aligned} \|H_{1,2,3}(x)\| &\leq C_{1,2,3}\|x\|^2, \\ \|DH_{1,2,3}\| &\leq \tilde{C}_{1,2,3}\|x\|, \\ \|D^2 H_{1,2,3}\| &\leq \hat{C}_{1,2,3} \end{aligned} \tag{14}$$

for some constants $C_{1,2,3}, \tilde{C}_{1,2,3}, \hat{C}_{1,2,3} \geq 0$. Of course, since $H_{1,2,3}(0,0,0) = 0$, $DH_{1,2,3}(0,0,0) = 0$ we have the obvious bounds, $C_i \leq \frac{1}{2}\hat{C}_i(\max r, R)^2$, $\tilde{C}_i \leq \hat{C}_i \max r, R$. But these bounds may not be saturated and, in concrete cases, we can obtain better bounds.

We now look for a smooth function $\omega \colon B_r \times U_R \to \mathbb{R}^{N_u}$ so that for all $(v^s, v^\infty) \in B_r \times U_R$ we have

$$\omega \left[F_2(\omega(v^s, v^\infty), v^s, v^\infty), F_3(\omega(v^s, v^\infty), v^s, v^\infty) \right] = F_1(\omega(v^s, v^\infty), v^s, v^\infty). \tag{15}$$

The heuristic meaning of (15) is that the graph of ω is invariant and, moreover, that points on the graph of ω are contracted upon application of the evolution F and hence accumulate at the origin. This heuristic observation must be made precise, and in addition the existence (or lack thereof) of such an ω depends strongly on the choice of r and R. These technicalities are treated in the lemmas to follow.

3.1. Lemmas. In this section we formulate a fixed point operator Φ for ω and develop sufficient lemmas which enable us to validate the existence of fixed points of the operator Φ (introduced in (16)). We also show that a fixed point of Φ is a function whose graph is an invariant manifold for F. Basically, we follow the scheme in [54] and prove that the operator maps to itself some ball in spaces of functions with 1 (or 2) derivatives. We also show that it is a contraction in C^0.

The Banach fixed point theorem now yields the existence of a unique fixed point, and the regularity results of [54] (see also Theorem 3.6 below) yield the desired bounds on derivatives. Note also that by Hadamard's interpolation theorem [40, 52, 19] we obtain that the fixed point operator is a contraction in higher regularity spaces. Therefore (in all the norms for which we have contraction), the distance of the fixed point to the initial guess is bound by the distance between the initial guess and the application of the operator.

The main difference with the treatment of [54] is that we have to segregate the infinitely many variables in the stable manifold into two groups: a finite dimensional set that will be dealt with numerically and an infinite dimensional set of variables for which we only have rough estimates. The hope is that, in some problems of interest, when the number of variables studied numerically is large enough, the "truncation effect" is small. The goal of this section is to provide rather explicit sufficient conditions that can verified in concrete problems, and indeed in section 4, we will prove a theorem in a concrete model. Since we verify the conditions in concrete problems using computers we have to be rather explicit in the conditions. We just note that all the bounds we will have to verify are algebraic expressions (which we give explicitly) on some basic bounds on the norms of the linearizations and the sups of the nonlinear parts on a ball. The explicit choice of this ball is part of our argument.

3.1.1. Formulation of the fixed point equation. Define the function space

$$\mathcal{H}_{r,R} = \left\{ \omega \in C^0\left(B_r \times U_R, \mathbb{R}^{N_u}\right) \;:\; \omega(0,0) = 0, \quad \text{and} \sup_{(v^s,v^\infty)\in B_r\times U_R} \|\omega(v^s,v^\infty)\|_{\mathbb{R}^{N_u}} < \infty \right\}.$$

With $\omega \in \mathcal{H}_{r,R}$ fixed, let $\Gamma\colon \mathbb{R}^{N_s}\times\mathcal{X}^\infty \to \mathbb{R}^{N_s}\times\mathcal{X}^\infty$ and $\Psi\colon \mathbb{R}^{N_s}\times\mathcal{X}^\infty \to \mathcal{X}$ be the operators

$$\Psi(v^s,v^\infty) = \begin{bmatrix} \omega(v^s,v^\infty) \\ v^s \\ v^\infty \end{bmatrix}$$

and

$$\Gamma(v^s,v^\infty) = \begin{bmatrix} (F_2\circ\Psi)(v^s,v^\infty) \\ (F_3\circ\Psi)(v^s,v^\infty) \end{bmatrix}.$$

Then, (15) becomes

$$\omega\circ\Gamma(v^s,v^\infty) = F_1\circ\Psi(v^s,v^\infty)$$

or

$$\omega\circ\Gamma(v^s,v^\infty) = A_u^N\omega(v^s,v^\infty) + E_1\circ\Psi(v^s,v^\infty) + H_1\circ\Psi(v^s,v^\infty).$$

The invertibility of A_u^N lets us write

$$\omega(v^s,v^\infty) = \left(A_u^N\right)^{-1}\left[(\omega\circ\Gamma)(v^s,v^\infty) - (E_1\circ\Psi)(v^s,v^\infty) - (H_1\circ\Psi)(v^s,v^\infty)\right],$$

which suggests that we study fixed points of the operator

$$\Phi[\omega](v^s,v^\infty) = \left(A_u^N\right)^{-1}\left[(\omega\circ\Gamma)(v^s,v^\infty) - (E_1\circ\Psi)(v^s,v^\infty) - (H_1\circ\Psi)(v^s,v^\infty)\right]. \tag{16}$$

3.1.2. Basic properties of Φ. The next several lemmas establish some useful properties of Φ. In particular Lemma 3.1 establishes conditions under which Φ is well defined, Lemma 3.2 establishes conditions under which a fixed point of Φ is a C^0 stable manifold for F, and

Lemma 3.4 establishes conditions ensuring that Φ maps a C^1 ball to itself. Lemma 3.5 establishes conditions ensuring that Φ maps a C^2 ball to itself. Finally Lemma 3.7 establishes conditions which ensure that Φ is a contraction mapping in C^0 norm. Throughout this section the results are formulated with an eye toward numerical validation, i.e., the lemmas are formulated in such a way that for a particular mapping F we can check the hypotheses using the computer.

Lemma 3.1 (composition lemma). *Let $\hat{R} > 0$ and*

$$\mathcal{U}_{r,R,\hat{R}} := \left\{ \omega \in \mathcal{H}_{r,R} \ : \ \sup_{(v^s,v^\infty)\in B_r\times U_R} \|\omega(v^s,v^\infty)\|_{\mathbb{R}^{N_u}} \leq \hat{R} \right\}.$$

Assume that $r + R + \hat{R} < \rho$ and that

1.
$$\|A_s^N\| + \frac{\epsilon_2(r+R+\hat{R}) + C_2(r+R+\hat{R})^2}{r} \leq 1,$$

2.
$$\|A_s^\infty\| + \frac{\epsilon_3(r+R+\hat{R}) + C_3(r+R+\hat{R})^2}{R} \leq 1.$$

Let $\omega \in \mathcal{U}_{r,R,\hat{R}}$ and Ψ, Γ be defined as above. Then for all $(v^s, v^\infty) \in B_r \times U_R$ we have that

$$F_2 \circ \Psi(v^s, v^\infty) \in B_r$$

and

$$F_3 \circ \Psi(v^s, v^\infty) \in U_R,$$

so that $\Gamma(v^s, v^\infty) \in B_r \times U_R$. It follows that $\omega \circ \Gamma$, and hence $\Psi[\omega]$ are well defined.

Proof. Let $x = (v^u, v^s, v^\infty) \in \mathcal{X}$ with $\|v^s\| \leq r$, $\|v^\infty\| \leq R$ and $\|v^u\| \leq \hat{R}$. Then

$$\|x\| = \|v^u\| + \|v^s\| + \|v^\infty\| \leq r + R + \hat{R}.$$

Since $r + R + \hat{R} < \rho$ we have that

$$\|E_{1,2,3}x\| \leq \|E_{1,2,3}\|\|x\| \leq \epsilon_{1,2,3}(r+R+\hat{R})$$

and

$$\|H_{1,2,3}(x)\| \leq C_{1,2,3}(r+R+\hat{R})^2.$$

Then

$$\begin{aligned} \|F_2(x)\| &\leq \|A_s^N v^s\| + \|E_2 x\| + \|H_2(x)\| \\ &\leq \|A_s^N\| r + \epsilon_2(r+R+\hat{R}) + C_2(r+R+\hat{R})^2 \\ &\leq r, \end{aligned}$$

and

$$\begin{aligned}\|F_3(x)\| &\le \|A_s^\infty v^\infty\| + \|E_3 x\| + \|H_3(x)\| \\ &\le \|A_s^\infty\| R + \epsilon_3(r + R + \hat{R}) + C_3(r + R + \hat{R})^2 \\ &\le R,\end{aligned}$$

by assumptions 1 and 2. From these we have that F_2, F_3 map into B_r, U_R, respectively.

Now if $\omega \in \mathcal{U}_{r,R,\hat{R}}$, then for any $(v^s, v^\infty) \in B_r \times U_R$ we have $\|\omega(v^s, v^\infty)\| \le \hat{R}$. It follows that $F_2 \circ \Psi(v^s, v^\infty) \in B_r$ and $F_3 \circ \Psi(v^s, v^\infty) \in U_R$, i.e., these maps go into the domain of ω. Combining these observations we see that indeed $\omega \circ \Gamma$ and $\Psi[\omega]$ are well defined. ■

In the next lemma we assume strict inequalities and obtain better properties for Φ. In this lemma we also assume that $\hat{R}$ is a function of r, R. More precisely let $K > 0$ and assume that $\hat{R} = K(r + R)$. In other words we suppose that if $\omega \in \mathcal{U}_{r,R,K(r+R)}$, we have that

$$\|\omega(v^s, v^\infty)\| \le K(r + R)$$

for all $\|v^s\| \le r, \|v^\infty\| \le R$. So with $(v^s, v^\infty) \in B_r \times U_R$ we let $x = (v^s, v^\infty, \omega(v^s, v^\infty))$ and have

$$\|E_{1,2,3}(x)\| \le \epsilon_{1,2,3}(r + R + K(r + R)) = \epsilon_{1,2,3}(K + 1)(r + R) \tag{17}$$

and

$$\|H_{1,2,3}(x)\| \le C_{1,2,3}\|r + R + K(r + R)\|^2 = C_{1,2,3}(K + 1)^2(r + R)^2. \tag{18}$$

These bounds play an important role in the following lemma.

Lemma 3.2 (C^0 stable manifold). *Suppose that $(r + R)(K + 1) < \rho$ and consider*

$$\tilde{\mathcal{U}}_{r,R,K} := \mathcal{U}_{r,R,K(r+R)} = \{\omega \in \mathcal{H}_{r,R} \,:\, \|\omega\| \le K(r + R)\}\,.$$

Assume that $\lambda_u, \lambda_s^N, \lambda_s^\infty$ are positive constants with

1.
$$\left\|(A_u^N)^{-1}\right\| + \left\|(A_u^N)^{-1}\right\| \frac{\left(\epsilon_1(K + 1) + C_1(K + 1)^2(r + R)\right)}{K} \le \lambda_u \le 1,$$

2.
$$\|A_s^N\| + \frac{\epsilon_2(K + 1)(r + R) + C_2(K + 1)^2(r + R)^2}{r} \le \lambda_s^N < 1,$$

3. *and*
$$\|A_s^\infty\| + \frac{\epsilon_3(K + 1)(r + R) + C_3(K + 1)^2(r + R)^2}{R} \le \lambda_s^\infty < 1.$$

Then $\Phi\colon \tilde{\mathcal{U}}_{r,R,K} \to \tilde{\mathcal{U}}_{r,R,K}$ defined by (16) is well defined.

Moreover if $\omega^* \in \tilde{\mathcal{U}}_{r,R,K}$ is a fixed point of Φ (so in particular ω^* is continuous), then the graph of ω^* is a local stable manifold of the origin in $\mathcal{X}$.

Proof. First observe that for any $\omega \in \mathcal{H}_{r,R}$ we have that $\Phi[\omega](0,0) = 0$. This follows by inspection of (16). Next consider $x = (v^u, v^s, v^\infty) \in \mathcal{X}$ with $\|v^s\| \le r, \|v^\infty\| \le R$ and $\|v^u\| \le K(r+R)$. From (17) and (18) and the fact that $r + R + K(r+R) < \rho$ we have that

$$\|E_{1,2,3}x\| \le \epsilon_{1,2,3}(K+1)(r+R)$$

and

$$\|H_{1,2,3}(x)\| \le C_{1,2,3}(K+1)^2(r+R)^2.$$

Proceeding as in Lemma 3.1 but employing the strict bounds given by assumptions 1 and 2 we obtain

$$\begin{aligned} \|F_2(x)\| &\le \|A_s^N\|\|v^s\| + \|\epsilon_2 x\| + \|H_2(x)\| \\ &\le \|A_s^N\| r + \epsilon_2(K+1)(r+R) + C_2(K+1)^2(r+R)^2 \\ &\le \lambda_s^N r \\ &< r \end{aligned} \tag{19}$$

and

$$\begin{aligned} \|F_3(x)\| &\le \|A_s^\infty\|\|v^\infty\| + \|E_3 x\| + \|H_3(x)\| \\ &\le \|A_s^\infty\| R + \epsilon_3(K+1)(r+R) + C_3(K+1)^2(r+R)^2 \\ &\le \lambda_s^\infty R \\ &< R. \end{aligned} \tag{20}$$

From these we recover the fact that F_2, F_3 map strictly into B_r and U_R, respectively. (Momentarily we will actually obtain strict contraction rates.)

First let $\omega \in \tilde{\mathcal{U}}_{r,R,K}$ and choose $v = (v^s, v^\infty) \in B_r \times U_R$. Take $v^u = \omega(v^s, v^\infty)$ so that $\|v^u\| = \|\omega(v^s, v^\infty)\| \le K(r+R)$. With $x = (v^u, v^s, v^\infty)$ and from (19) and (20) it follows again that $F_2(x) \in B_r$ and $F_3(x) \in U_R$. Then $\|\omega(F_2(x), F_3(x))\| \le K(r+R)$ and we have

$$\begin{aligned} \|\Phi[\omega](v^s, v^\infty)\| &\le \left\| \left(A_u^N\right)^{-1} \right\| [\|\omega(F_2(x), F_3(x))\| + \|E_1 x\| + \|H_1(x)\|] \\ &\le \left\| \left(A_u^N\right)^{-1} \right\| (K(r+R) + \epsilon_1(K+1)(r+R) + C_1(K+1)^2(r+R)^2) \\ &\le \frac{\left\| \left(A_u^N\right)^{-1} \right\| (K + \epsilon_1(K+1) + C_1(K+1)^2(r+R))}{K} K(r+R) \\ &\le \lambda_u K(r+R) \\ &\le K(r+R), \end{aligned}$$

by assumption 1. Then $\Phi(\omega)$ is indeed in $\tilde{\mathcal{U}}_{r,R,K}$ and the operator Φ is well defined.

Now assume that $\omega^* \in \tilde{\mathcal{U}}_{r,R,K}$ is a fixed point of Φ and choose $v_0^s \in B_r$ and $v_0^\infty \in U_R$. Define

$$\omega_0^* = \omega^*(v_0^s, v_0^\infty).$$

For $n \ge 1$ define

$$\omega_n^* := F_1(\omega_{n-1}^*, v_{n-1}^s, v_{n-1}^\infty),$$

$$v_n^N := F_2(\omega_{n-1}^*, v_{n-1}^s, v_{n-1}^\infty),$$

$$v_n^\infty = F_3(\omega_{n-1}^*, v_{n-1}^s, v_{n-1}^\infty),$$

and

$$x_n := (\omega_n^*, v_n^s, v_n^\infty).$$

Note that for all $n \geq 1$ we have that $\|v_n^s\| \leq r, \|v_n^\infty\| \leq R$, so that (v_n^s, v_n^∞) is always in the domain of ω and the sequence is well defined. Moreover, inductive application of the bounds (19) and (20) lead to

$$\|F_2(x_n)\| = \|F_2^n(x_0)\| \leq (\lambda_s^N)^{n-1} r$$

and

$$\|F_3(x_n)\| = \|F_3^n(x_0)\| \leq (\lambda_s^\infty)^{n-1} R.$$

From this it follows that $v_n^s, v_n^\infty \to 0$ as $n \to \infty$. From the continuity of ω^* it now follows that

$$\lim_{n\to\infty} \omega_n^* = \lim_{n\to\infty} \omega^* \left(v_n^N, v_n^\infty\right) = \omega^*(0,0) = 0.$$

Then

$$\lim_{n\to\infty} F^n(x_0) = \lim_{n\to\infty} (\omega_n^*, v_n^N, v_n^\infty) = 0,$$

so that $(\omega_0^*, v_0^N, v_0^\infty) \in W^s_{\mathrm{loc}}(0)$ by definition. Since $(v_0^N, v_0^\infty) \in B_r \times U_R$ was arbitrary we have that the graph of ω^* is a local stable manifold. ∎

Remark 3.3 (linear conjugacy versus linear decay rates). Note that while we do have the "decay rates" $\|F_2^n(x)\| \leq (\lambda_s^N)^{n-1}\|x\|$ and $\|F_3^n(x)\| \leq (\lambda_s^\infty)^{n-1}\|x\|$ in the domain of ω, we do not have a conjugacy to linear dynamics. In order to track the asymptotics of the stable orbit it is necessary to iterate the full nonlinear mappings $F_{2,3}$. This is in stark contrast to results obtained using the parameterization method (see also Remark 1.3).

The earlier Lemma 3.2 shows that a fixed point of Φ provides a representation of the local stable manifold. To establish the existence of a fixed point having certain regularity properties we use a classical argument as in [54], proving C^0 contraction and C^1 (or C^2) propagated bounds. Propagated bounds refer to the fact that the operator maps a ball in spaces of differentiable functions to themselves.

The space of differentiable functions satisfying a uniform bound is directly related to $\hat{U}_{r,R,\hat{R}}$. To see this assume that ω is Fréchet differentiable and that K is a uniform bound having

$$\sup_{v^s\in B_r} \sup_{v^\infty \in U_R} \|D\omega(v^s, v^\infty)\| \leq K.$$

Combining the above bound with the fact that $\omega(0,0) = 0$ gives

$$\|\omega(v^s, v^\infty)\| \leq K(r+R) \qquad \text{for} \quad (v^s, v^\infty) \in B_r \times U_R,$$

by the mean value theorem. This observation is used in the following.

Lemma 3.4 (C^1 propagated bounds). *Let Φ be as defined in* (16) *and suppose that* r, R, B_r, $K, U_R, \tilde{\mathcal{U}}_{r,R,K}, \lambda_s^N, \lambda_s^\infty$, *and* λ_u *are as in Lemma* 3.2 *so that in particular* $\Phi\colon \tilde{\mathcal{U}}_{r,R,K} \to \tilde{\mathcal{U}}_{r,R,K}$. *Now define*

$$\mathcal{V}_K = \left\{ \omega \in C^1\left(B_r \times U_R, \mathbb{R}^{N_u}\right) \ : \ \sup_{B_r \times U_R} \|D\omega\| \leq K \right\},$$

and assume that

$$\left\| \left(A_u^N\right)^{-1} \right\| \|A_s^N\| + \left\| \left(A_u^N\right)^{-1} \right\| \left(\|A_s^\infty\| + (\epsilon_2 + \epsilon_3)(K+1) + (\tilde{C}_2 + \tilde{C}_3)(K+1)^2(r+R) \right)$$

$$+ \left\| \left(A_u^N\right)^{-1} \right\| \frac{\epsilon_1(K+1) + \tilde{C}_1(K+1)^2(r+R)}{K} \leq 1. \tag{21}$$

Define

$$\mathcal{H}_{r,R,K} = \tilde{\mathcal{U}}_{r,R,K} \cap \mathcal{V}_K.$$

Then

$$\Phi\colon \mathcal{H}_{r,R,K} \to \mathcal{H}_{r,R,K}.$$

Proof. Let $\omega \in \mathcal{H}_{r,R,K}$ and $\hat{\omega} = \Phi[\omega]$. By Lemma 3.2 we have that $\hat{\omega} \in \tilde{\mathcal{U}}_{r,R,K}$ and want to establish that $\hat{\omega} \in \mathcal{V}_K$. Let $h \in \mathbb{R}^{N_s}$, $H \in \mathcal{X}^\infty$ and note that

$$D\Psi(v^s, v^\infty)(h, H) = \begin{bmatrix} D\omega(v^s, v^\infty) \\ \mathrm{Id}_{\mathbb{R}^{N_s}} \\ \mathrm{Id}_{\mathcal{X}^\infty} \end{bmatrix} (h, H) = \begin{bmatrix} D\omega(v^s, v^\infty)(h, H) \\ h \\ H \end{bmatrix}$$

and

$$\begin{aligned} &D\Gamma(v^s, v^\infty)(h, H) \\ &\quad = \begin{bmatrix} D(F_2 \circ \Psi)(v^s, v^\infty) \\ D(F_3 \circ \Psi)(v^s, v^\infty) \end{bmatrix} (h, H) \\ &\quad = \begin{bmatrix} (D_{\mathcal{X}} F_2)(\Psi(v^s, v^\infty)) \cdot D\Psi(v^s, v^\infty)(h, H) \\ (D_{\mathcal{X}} F_3)(\Psi(v^s, v^\infty)) \cdot D\Psi(v^s, v^\infty)(h, H) \end{bmatrix} \\ &\quad = \begin{bmatrix} A_s^N h + E_2 \cdot D\Psi(v^s, v^\infty)(h, H) + DH_2(\Psi(v^s, v^\infty)) \cdot D\Psi(v^s, v^\infty)(h, H) \\ A_s^\infty H + E_3 \cdot D\Psi(v^s, v^\infty)(h, H) + DH_3(\Psi(v^s, v^\infty)) \cdot D\Psi(v^s, v^\infty)(h, H) \end{bmatrix}. \end{aligned}$$

Suppressing the (h, H) dependence we have that

$$\begin{aligned} &D\hat{\omega}(v^s, v^\infty) \\ &\quad = \left(A_u^N\right)^{-1} \left(D\omega(\Gamma(v^s, v^\infty)) \cdot D\Gamma(v^s, v^\infty) - E_1 D\Psi(v^s, v^\infty) \right. \\ &\qquad\qquad \left. - DH_1(\Psi(v^s, v^\infty)) \cdot D\Psi(v^s, v^\infty) \right) \\ &\quad = \left(A_u^N\right)^{-1} \left(D\omega(\Gamma(v^s, v^\infty)) \cdot \begin{bmatrix} (D_{\mathcal{X}} F_2)(\Psi(v^s, v^\infty)) \cdot D\Psi(v^s, v^\infty) \\ (D_{\mathcal{X}} F_3)(\Psi(v^s, v^\infty)) \cdot D\Psi(v^s, v^\infty) \end{bmatrix} \right. \\ &\qquad\qquad \left. - E_1 D\Psi(v^s, v^\infty) - DH_1(\Psi(v^s, v^\infty)) \cdot D\Psi(v^s, v^\infty) \right). \end{aligned} \tag{22}$$

Now for any $(v^s, v^\infty) \in B_r \times U_R$

$$\begin{aligned} \|D\Psi(v^s,v^\infty)\|_{B(\mathbb{R}^{N_s}\times\mathcal{X}^\infty,\mathcal{X})} &:= \sup_{\|h\|+\|H\|=1} \|D\Psi(v^s,v^\infty)(h,H)\|_{\mathcal{X}} \\ &= \sup_{\|h\|+\|H\|=1} \left\| \begin{bmatrix} D\omega(v^s,v^\infty)(h,H) \\ h \\ H \end{bmatrix} \right\|_{\mathcal{X}} \\ &\leq \|D\omega(v^s,v^\infty)\| + \|h\| + \|H\| \\ &\leq K+1. \end{aligned} \tag{23}$$

Then

$$\begin{aligned} &\|D\Gamma(v^s,v^\infty)\|_{B(\mathbb{R}^{N_s}\times\mathcal{X}^\infty)} \\ &\quad\leq \left\| \begin{bmatrix} (D_{\mathcal{X}}F_2)(\Psi(v^s,v^\infty))\cdot D\Psi(v^s,v^\infty) \\ (D_{\mathcal{X}}F_3)(\Psi(v^s,v^\infty))\cdot D\Psi(v^s,v^\infty) \end{bmatrix} \right\|_{B(\mathbb{R}^{N_s}\times\mathcal{X}^\infty)} \\ &\quad\leq \|D_{\mathcal{X}}F_2(\Psi(v^s,v^\infty))\cdot D\Psi(v^s,v^\infty)\| + \|D_{\mathcal{X}}F_3(\Psi(v^s,v^\infty))\cdot D\Psi(v^s,v^\infty)\| \\ &\quad\leq \|A_s^N\| + \|A_s^\infty\| + (\epsilon_2+\epsilon_3)(K+1) + (\tilde{C}_2+\tilde{C}_3)(K+1)^2(r+R). \end{aligned} \tag{24}$$

Combining these estimates with (22) and the inequality of assumption (21) gives

$$\begin{aligned} &\|D\hat{\omega}(v^s,v^\infty)\| \\ &\quad\leq \left\|\left(A_u^N\right)^{-1}\right\| \Big[K\left(\|A_s^N\|+\|A_s^\infty\|\right) + (\epsilon_2+\epsilon_3)K(K+1) + (\tilde{C}_2+\tilde{C}_3)K(K+1)^2(r+R) \\ &\qquad\qquad + \epsilon_1(K+1) + \tilde{C}_1(K+1)^2(r+R)\Big] \leq K, \end{aligned}$$

so that $\hat{\omega} \in \mathcal{V}_K$ as claimed. But $\omega \in \mathcal{V}_K$ was arbitrary so that Φ indeed maps $\mathcal{V}_K$ into itself. ■

A slight variation of the argument for C^1 propagated bounds applies to the second derivative as well. For the sake of completeness we include the following lemma.

Lemma 3.5 (C^2 **propagated bounds**). *Let Φ be as defined in (16) and suppose that r, R, B_r, $U_R, K, \tilde{\mathcal{U}}_{r,R,K}, \lambda_s^N, \lambda_s^\infty$, and λ_u satisfy Lemmas 3.2 and 3.4. In particular we assume that*

$$\tau := \|A_s^N\| + \|A_s^\infty\| + (\epsilon_2+\epsilon_3)(K+1) + (\tilde{C}_2+\tilde{C}_3)(K+1)^2(r+R) < 1,$$

so that $\Phi\colon \mathcal{H}_{r,R,K} \to \mathcal{H}_{r,R,K}$ as in Lemma 3.4.

Define

$$\beta := \|(A_u^N)^{-1}\| \left[\tau^2 + K(\epsilon_2+\epsilon_3) + K(\tilde{C}_2+\tilde{C}_3)(r+R) + \epsilon_1 + \tilde{C}_1(r+R)\right]$$

and

$$\alpha := \|(A_u^N)^{-1}\| \left[(\hat{C}_2+\hat{C}_3)K(K+1)^2 + \hat{C}_1(K+1)^2\right].$$

Assume that

$$\beta < 1, \tag{25}$$

and choose

$$L > \frac{\alpha}{1-\beta}.$$

Define

$$\tilde{\mathcal{V}}_L = \left\{ \omega \in C^2\left(B_r \times U_R, \mathbb{R}^{N_u}\right) \ : \ \sup_{B_r \times U_R} \|D^2\omega\| \leq L, \right\}, \tag{26}$$

and let

$$\mathcal{H}_{r,R,K,L} = \mathcal{H}_{r,R,K} \cap \tilde{\mathcal{V}}_L.$$

Then,

$$\Phi\colon \mathcal{H}_{r,R,K,L} \to \mathcal{H}_{r,R,K,L}.$$

The key is that we obtain bounds of the form

$$\|\Phi[\omega]\| \leq \beta L + \alpha$$

with β, α independent of L. Then when $\beta < 1$ we see that by taking L sufficiently large we get the propagation of the bound. In fact we need precisely that

$$L \geq \frac{\alpha}{1-\beta}$$

in order to ensure

$$\beta L + \alpha \leq L.$$

In our case we have $\beta = \|(A_u^N)^{-1}\|(\|A_s\| + \|A_\infty\|)^2+$ "small" to first order, so that in practice Lemma 3.5 can always be satisfied. Indeed, in theoretical applications [54] shows that we can have propagated bounds of derivatives at any order. Of course in a computer assisted proof we must still verify that $\beta < 1$.

Proof. Let $\omega \in \mathcal{H}_{r,R,K,L}$ and consider $\Phi[\omega]$. Using the chain rule and that E_1 and $(A_u^N)^{-1}$ are linear operators, we have

$$\begin{aligned} D^2\Phi[\omega] = (A_u^N)^{-1}\big[&(D^2\omega \circ \Gamma)\,(D\Gamma)^{\otimes 2} + (D\omega \circ \Gamma)D^2\Gamma - E_1 D^2\Psi \\ &- (D^2 H_1 \circ \Psi)(D\Psi)^{\otimes 2} - (DH_1 \circ \Psi)D^2\Psi\big], \end{aligned} \tag{27}$$

so that

$$\begin{aligned} \|D^2\Phi[\omega]\| \leq \|(A_u^N)^{-1}\| \big(&\|D^2\omega\|\|D\Gamma\|^2 + \|D\omega\|\|D^2\Gamma\| \\ &+ \epsilon_1\|D^2\Psi\| + \|D^2H_1 \circ \Psi\|\|D\Psi\|^2 + \|DH_1 \circ \Psi\|\|D^2\Psi\|\big). \end{aligned}$$

By now, we have estimates of almost all the elements appearing in (27). See (24), (23), and the definition of the space $\mathcal{V}_{K,L}$ in (26). In order to complete the argument we observe that

$$D^2\Psi = \begin{bmatrix} D^2\omega \\ 0 \\ 0 \end{bmatrix}, \tag{28}$$

and hence,

$$\|D^2\Psi\| \leq L. \tag{29}$$

Now recall that

$$\|D\Gamma\| \leq \tau,$$

by the estimate of (24). We compute

$$D^2\Gamma = \begin{pmatrix} \epsilon_2 D^2\Psi + (D^2H_2 \circ \Psi)(D\Psi)^{\otimes 2} + (DH_2 \circ \Psi)D^2\Psi \\ \epsilon_3 D^2\Psi + (D^2H_3 \circ \Psi)(D\Psi)^{\otimes 2} + (DH_3 \circ \Psi)D^2\Psi \end{pmatrix}$$

and see that

$$\begin{aligned} \|D^2\Gamma\| &\leq (\epsilon_2 + \epsilon_3)\|D^2\omega\| + (\hat{C}_2 + \hat{C}_3)\|D\Psi\|^2 + (\tilde{C}_2 + \tilde{C}_3)\|\Psi\|\|D^2\Psi\| \\ &\leq \left[\epsilon_2 + \epsilon_3 + (\tilde{C}_2 + \tilde{C}_3)(r + R)\right] L + (\hat{C}_2 + \hat{C}_3)(K+1)^2. \end{aligned} \tag{30}$$

Hence, putting together the obvious bounds for all the elements in the same order as in (27), we obtain

$$\begin{aligned} \|D^2\Phi[\omega]\| \leq\ & \|(A_u^N)^{-1}\| \left[\tau^2 + K(\epsilon_2 + \epsilon_3) + K(\tilde{C}_2 + \tilde{C}_3)(r + R) + \epsilon_1 + \tilde{C}_1(r + R)\right] L \\ & + \|(A_u^N)^{-1}\| \left[(\hat{C}_2 + \hat{C}_3)K(K+1)^2 + \hat{C}_1(K+1)^2\right] \\ \leq\ & \beta L + \alpha \\ \leq\ & L, \end{aligned}$$

as $\beta < 1$ and

$$L \geq \frac{\alpha}{1 - \beta},$$

by hypothesis. ■

The following Theorem, adapted from [54, Proposition A2], will be key in the discussion of the regularity of the stable manifold. In fact the result in [54] is even more general than stated here, and we take only what we need for what follows.

Theorem 3.6 (a theorem of Lanford). *Let $\omega_n : U \subset \mathcal{X} \to \mathcal{Y}$ be a sequence of functions with $\mathcal{X}$, $\mathcal{Y}$ Banach spaces and U a convex subset of $\mathcal{X}$. Suppose that for all $n \geq 0$ and $x \in U$ we have that*

$$\sup_{x \in U} \|D\omega_n(x)\| \leq K$$

and that each $D\omega_n$ is Lipschitz continuous with Lipschitz constant L. Assume that for each x the sequence $\omega_n(x)$ converges weakly to $\omega^*(x)$. Then ω^* has Lipschitz continuous derivative with Lipschitz constant L, and $D\omega_n(x)$ converges weakly to $D\omega^*(x)$ for ally $x \in B$.

Next we show that Φ is a Lipschitz operator and obtain explicitly the bound.

Lemma 3.7 (Lipschitz constant). *Suppose that Φ is as defined in* (16) *and that r, R, B_r, U_R, $\mathcal{H}_r$, λ_u, λ_s^N, λ_s^∞ are as in Lemma* 3.2. *Let*

$$\mathcal{L}_{r,R,K} = \left\{\omega \in C^0(B_r \times U_r, \mathbb{R}^{n_u}) : \ \omega(0,0) = 0, \text{ and } \omega \text{ is Lipschitz with constant } K\right\}.$$

Then Φ is a Lipschitz mapping with constant

$$\left\|\left(A_u^N\right)^{-1}\right\| \left[1 + K\left(\epsilon_2 + \epsilon_3 + (\tilde{C}_2 + \tilde{C}_3)(K+1)(r+R)\right) + \epsilon_1 + \tilde{C}_1(K+1)(r+R)\right] =: \kappa.$$

Proof. Let $\omega_1, \omega_2 \in \mathcal{H}_{r,R,K}$ and define the functions $\Psi_{1,2} \colon \mathbb{R}^{N_s} \times \mathcal{X}^\infty \to \mathcal{X}$ and $\Gamma_{1,2} \colon \mathbb{R}^{N_s} \times \mathcal{X}^\infty \to \mathbb{R}^{N_s} \times \mathcal{X}^\infty$ by

$$\Psi_1(v^s, v^\infty) = \begin{bmatrix} \omega_1(v^s, v^\infty) \\ v^s \\ v^\infty \end{bmatrix}, \qquad \Psi_2(v^s, v^\infty) = \begin{bmatrix} \omega_2(v^s, v^\infty) \\ v^s \\ v^\infty \end{bmatrix}$$

and

$$\Gamma_1(v^s, v^\infty) = \begin{bmatrix} F_2 \circ \Psi_1(v^s, v^\infty) \\ F_3 \circ \Psi_1(v^s, v^\infty) \end{bmatrix}, \qquad \Gamma_2(v^s, v^\infty) = \begin{bmatrix} F_2 \circ \Psi_2(v^s, v^\infty) \\ F_3 \circ \Psi_2(v^s, v^\infty) \end{bmatrix}.$$

Then for any $(v^s, v^\infty) \in B_r \times U_R$ we have

$$\begin{aligned} &\Phi[\omega_1] - \Phi[\omega_2] \\ &\quad = \left(A_u^N\right)^{-1}[\omega_1 \circ \Gamma_1 - E_1 \circ \Psi_1 - H_1 \circ \Psi_1] - \left(A_u^N\right)^{-1}[\omega_2 \circ \Gamma_2 - E_1 \circ \Psi_2 - H_1 \circ \Psi_2] \\ &\quad = \left(A_u^N\right)^{-1}[\omega_1 \circ \Gamma_1 - \omega_2 \circ \Gamma_2 + E_1(\Psi_2 - \Psi_1) + H_1 \circ \Psi_2 - H_1 \circ \Psi_1]. \end{aligned}$$

There are three terms in this expression, and we bound these one at at time. First note that

$$\begin{aligned} \|\Psi_1 - \Psi_2\| &= \left\| \begin{bmatrix} \omega_1 \\ v^s \\ v^\infty \end{bmatrix} - \begin{bmatrix} \omega_2 \\ v^s \\ v^\infty \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} \omega_1 - \omega_2 \\ 0 \\ 0 \end{bmatrix} \right\| \\ &= \|\omega_1 - \omega_2\|. \end{aligned}$$

From this we see that the second and third terms satisfy

$$\|E_1(\Psi_2 - \Psi_1)\| \leq \|E_1\|\|\Psi_1 - \Psi_2\| \leq \epsilon_1\|\omega_1 - \omega_2\| \tag{31}$$

and

$$\begin{aligned} \|H_1 \circ \Psi_2 - H_1 \circ \Psi_1\| &\leq \|DH_1\|\|\Psi_1 - \Psi_2\| \\ &\leq \tilde{C}_1(K+1)(r+R)\|\Psi_1 - \Psi_2\| \\ &\leq \tilde{C}_1(K+1)(r+R)\|\omega_1 - \omega_2\|. \end{aligned} \tag{32}$$

Next we observe that

$$
\begin{aligned}
\|\Gamma_1-\Gamma_2\| &\leq \left\| \begin{bmatrix} F_2\circ\Psi_1(v^s,v^\infty) \\ F_3\circ\Psi_1(v^s,v^\infty) \end{bmatrix} - \begin{bmatrix} F_2\circ\Psi_2(v^s,v^\infty) \\ F_3\circ\Psi_2(v^s,v^\infty) \end{bmatrix} \right\| \\
&\leq \|F_2\circ\Psi_1 - F_2\circ\Psi_2\| + \|F_3\circ\Psi_1 - F_3\circ\Psi_2\| \\
&\leq \left\|A_s^N v^s + E_2\Psi_1 + H_2\Psi_1 - \left(A_s^N v^s + E_2\Psi_2 + H_2\Psi_2\right)\right\| \\
&\quad + \|A_s^\infty v^\infty + E_3\Psi_1 + H_3\Psi_1 - (A_s^\infty v^\infty + E_3\Psi_2 + H_3\Psi_2)\| \\
&\leq \|E_2\|\|\Psi_1-\Psi_2\| + \|DH_2\|\|\|\Psi_1-\Psi_2\| + \|E_3\|\|\Psi_1-\Psi_2\| + \|DH_3\|\|\|\Psi_1-\Psi_2\| \\
&\leq \left(\epsilon_2+\epsilon_3+(\tilde{C}_2+\tilde{C}_3)(K+1)(r+R)\right)\|\omega_1-\omega_2\|
\end{aligned}
$$

and obtain the following bound on the first term:

$$
\begin{aligned}
\|\omega_1\circ\Gamma_1-\omega_2\circ\Gamma_2\| &\leq \|\omega_1\circ\Gamma_1-\omega_2\circ\Gamma_1\| + \|\omega_2\circ\Gamma_1-\omega_2\circ\Gamma_2\| \\
&\leq \|\omega_1-\omega_2\| + K\|\Gamma_1-\Gamma_2\| \\
&\leq \|\omega_1-\omega_2\| + K\left(\epsilon_2+\epsilon_3+(\tilde{C}_2+\tilde{C}_3)(K+1)(r+R)\right)\|\omega_1-\omega_2\| \\
&\leq \left(1+K\left[\epsilon_2+\epsilon_3+(\tilde{C}_2+\tilde{C}_3)(K+1)(r+R)\right]\right)\|\omega_1-\omega_2\|. \qquad (33)
\end{aligned}
$$

Combining (31), (32), and (33) gives

$$
\begin{aligned}
&\|\Phi[\omega_1]-\Phi[\omega_2]\| \\
&\quad\leq \left\|\left(A_u^N\right)^{-1}\right\| \left[1+K\left(\epsilon_2+\epsilon_3+(\tilde{C}_2+\tilde{C}_3)(K+1)(r+R)\right)\right. \\
&\qquad\qquad \left. +\,\epsilon_1+\tilde{C}_1(K+1)(r+R)\right]\|\omega_1-\omega_2\| \\
&\quad\leq \kappa\|\omega_1-\omega_2\|,
\end{aligned}
$$

as desired. ∎

3.2. Validated error bounds for the linear approximation of the infinite dimensional local stable manifold. The following theorem and its corollary collect the Lemmas above into our main result.

Theorem 3.8 (a stable manifold theorem). *Suppose that the C^2 map $F\colon \mathcal{X}\to\mathcal{X}$ and positive constants r,R,K,ρ, $C_{1,2,3}$, $\tilde{C}_{1,2,3}$, $\hat{C}_{1,2,3}$, $\epsilon_{1,2,3}$, $\|(A_u^N)^{-1}\|$, $\|A_s^N\|$, and $\|A_s^\infty\|$ are as discussed in the intro to section 3, and assume that these constants satisfy the hypotheses of Lemmas 3.1, 3.2, 3.4, and 3.7. Assume in addition that κ from Lemma 3.7 has*

$$\kappa < 1.$$

Then there exists a unique $\omega^*\in C^0\left(B_r\times U_R,\mathbb{R}^{n_u}\right)$ having that

1. $\omega^*(0,0)=0$,
2. $$\sup_{v^s\in B_r}\sup_{v^\infty\in U_R}\|\omega^*(v^s,v^\infty)\|\leq K(r+R), \qquad (34)$$

3. *ω is Lipschitz with constant K,*
4. Graph$(\omega^*) \subset \mathcal{X}$ *is a local stable manifold of the origin in $\mathcal{X}$.*

Proof. Under the present hypotheses Φ maps $\mathcal{L}_{r,R,K}$ into itself with contraction constant $\kappa < 1$. Since $\mathcal{L}_{r,R,K}$ is closed under the uniform norm the contraction mapping theorem gives that Φ has a unique fixed point $\omega^* \in \mathcal{L}_{r,R,K}$. (Of course this gives the zero behavior, continuity, and the Lipschitz constant for ω^* as claimed.) Lemma 3.2 gives that the graph of ω^* is a local stable manifold. ∎

Corollary 3.9. *If in addition to the hypotheses of Theorem 3.8 we have the the hypotheses of Lemmas 3.4 and 3.5 are satisfied, then ω^* from Theorem 3.8 is differentiable with*

$$\sup_{v^s \in B_r} \sup_{v^\infty \in U_R} \|D\omega^*(v^s, v^\infty)\| \leq K.$$

Moroever $D\omega(v^s, v^\infty)$ is Lipschitz continuous on $B_r \times U_R$ with Lipschitz constant L. Here K and L are as given in the Lemmas 3.4 and 3.5.

Proof. Choose any

$$\omega_0 \in \mathcal{H}_{r,R,K,L} \subset \mathcal{L}_{r,R,K,L}.$$

(For example, ω_0 identically zero is one such). Consider the sequence

$$\omega_n = \Phi^n[\omega_0].$$

By Theorem 3.8 we have that

$$\omega_n \to \omega^*,$$

the unique fixed point of $\Phi \in \mathcal{L}_{r,R,K,L}$. Moreover the convergence is in the C^0 (uniform) norm. By Lemmas 3.5 and 3.4 we have that

$$\omega_n = \Phi^n[\omega_0] \in \mathcal{H}_{r,R,K,L}$$

for all $n \geq 0$, as Φ propagates the C^1 and C^2 bounds. Applying the mean value theorem in conjunction with the uniform bounds on the second derivative gives that each ω_n is C^1 and, moreover, that each $D\omega_n$ is Lipschitz continuous with Lipschitz constant L. Now Theorem 3.6 gives that ω^* is differentiable, that

$$D\omega_n \to D\omega^*$$

weakly (a fact we note but do not use), and that $D\omega^*$ is Lipschitz with constant L as claimed. But Theorem 3.8 already provides ω^* Lipschitz with bound K, and the bound on $D\omega^*$ cannot exceed the bound on the Lipschitz constant of ω_*. Hence we obtain

$$\|D\omega^*\| \leq K,$$

i.e.,

$$\omega^* \in \mathcal{H}_{r,R,K},$$

as desired. ∎

Remark 3.10 (changing to good coordinates). Consider a mapping $F\colon \mathcal{X} \to \mathcal{X}$ and $\bar{a} \in \mathcal{X}$ with $F(\bar{a}) = \bar{a}$. Then the map

$$\bar{F}(a) = F(\bar{a} + a) - \bar{a}$$

has a fixed point at $a = 0$. Suppose now that $V, W\colon \mathcal{X} \to \mathcal{X}$ are bounded linear isomorphisms which put $\bar{F}$ into the desired form, i.e., for $v \in \mathcal{X}$ let $a = Vv$ and suppose that

$$W\bar{F}(Vv) = \begin{pmatrix} A_u v^u + \epsilon_1(v^u, v^s, v^\infty) + H_1(v^u, v^s, v^\infty) \\ A_s v^s + \epsilon_2(v^u, v^s, v^\infty) + H_2(v^u, v^s, v^\infty) \\ A^\infty v^\infty + \epsilon_3(v^u, v^s, v^\infty) + H_3(v^u, v^s, v^\infty) \end{pmatrix}$$

with the $A_{u,s}$, $\epsilon_{1,2,3}$, and $H_{1,2,3}$ mappings as in section 3. Then we say that the coordinates v given by V and W are *good coordinates* for F. The term is appropriate because these coordinates "almost diagonalize" the linear part of F near the fixed point $\bar{a}$.

Now suppose that we apply Theorem 3.8 to the map $\bar{F}$ so that the local stable manifold in the good coordinates is given by the graph of a function ω_* as in Theorem 3.8. Then the local stable manifold for the original mapping F is parameterized by the function

$$Q(v^s, v^\infty) = \bar{a} + V \begin{pmatrix} \omega_*(v^s, v^\infty) \\ v^s \\ v^\infty \end{pmatrix}.$$

Remark 3.11 (heuristics for choosing r, R, and K). When applying Theorem 3.8 at a known fixed point of a given map the free parameters are r, R, and K. Ideally we would like to have r as big as possible, R as small as possible, and r/K as close to one as possible. To see why $K \approx r$ is desirable note that in that case (34) gives a bound on ω_* which is quadratic in r. We would also like that $K < 1$, so that at worst we have

$$1 + K < 2, \qquad \text{and} \qquad (1+K)^2 < 4,$$

and we will assume that $\|\epsilon_1\|$ and $\|\epsilon_2\|$ are negligible.

Then assumption (2) in Lemma 3.2 combined with $\|\epsilon_2\|, R \approx 0$ suggests

$$r < \frac{1 - \|A_s^N\|}{4C_2},$$

while bound number (1) of Lemma 3.2 combined with $\|\epsilon_1\|, R \approx 0$ suggests

$$\frac{4C_1 r}{\frac{1 - \|(A_u^N)^{-1}\|}{\|(A_u^N)^{-1}\|}} < K,$$

and last bound number (3) of Lemma 3.2 with $R \ll r$ suggests

$$\frac{8C_3 r^2}{1 - \|A_s^\infty\|} < R \qquad \text{and} \qquad \frac{4\|\epsilon_3\| r}{1 - \|A_s^\infty\|} < R.$$

Considering the definition of κ in Lemma 3.7 under the same constraints on R and r leads to

$$r < \frac{1 - \|(A_u^N)^{-1}\|}{4\tilde{C}_1 \|(A_u^N)^{-1}\|}$$

and

$$\frac{2(\tilde{C}_2+\tilde{C}_3)\|(A_u^N)^{-1}\|}{1-\|(A_u^N)^{-1}\|}r < K.$$

Finally, the constraint given by (21) suggests we choose also

$$r < \frac{\frac{1}{\|(A_u^N)^{-1}\|} - \|A_s^N\| - \|A_s^\infty\|}{8(\tilde{C}_2+\tilde{C}_3)}$$

and

$$\frac{8\tilde{C}_1}{\frac{1}{\|(A_u^N)^{-1}\|} - \|A_s^N\| - \|A_s^\infty\|}r < K.$$

Combining these observations leads to the following procedure.

- *Step* 1: Let r_{tol} be a fixed numerical tolerance which we think r should not exceed and choose

$$\hat{r} = \min\left(r_{\mathrm{tol}}, \frac{1-\|A_s^N\|}{4C_2}, \frac{1-\|(A_u^N)^{-1}\|}{4\tilde{C}_1\|(A_u^N)^{-1}\|}, \frac{\frac{1}{\|(A_u^N)^{-1}\|} - \|A_s^N\| - \|A_s^\infty\|}{8(\tilde{C}_2+\tilde{C}_3)}\right).$$

- *Step* 2: Choose

$$\hat{R} = \frac{4}{1-\|A_s^\infty\|}\max\left(2C_3\hat{r}^2, \|\epsilon_3\|\hat{r}\right).$$

- *Step* 3: Choose

$$\hat{K} = \hat{r}\max\left(\frac{4C_1}{\frac{1-\|(A_u^N)^{-1}\|}{\|(A_u^N)^{-1}\|}}, \frac{2(\tilde{C}_2+\tilde{C}_3)\|(A_u^N)^{-1}\|}{1-\|(A_u^N)^{-1}\|}, \frac{8\tilde{C}_1}{\frac{1}{\|(A_u^N)^{-1}\|} - \|A_s^N\| - \|A_s^\infty\|}\right).$$

- *Step* 4: Choose

$$r = 0.9\hat{r},$$

$$R = 1.1\hat{R},$$

  and

$$K = 1.1\hat{K},$$

  i.e., choose r, R, and K to be ninety percent of their optimal values.
- *Step* 5: Check (using interval arithmetic) r, R, and K so chosen actually satisfy the inequalities given by the hypotheses of Theorem 3.8 and Corollary 3.9, i.e., check that the ninety percent rule of thumb above was indeed sufficient to cover the terms ignored in the heuristic discussion above. If these conditions are satisfied, then we have the conclusions of Theorem 3.8 and Corollary 3.9.

Of course the procedure could fail at step 5. If this happens one must look more closely at the actual values of the constants and see where the problem is. We remark that one could also write programs to choose r, R and K by some more sophisticated nonlinear optimization scheme, but the heuristics given above are sufficient for the applications to follow.

4. Analysis of the Kot–Schaffer model. The Kot–Schaffer mapping is a model for the population dynamics of a spatially extended species. The version of the model considered here assumes that the local population dynamics are governed by a logistic growth law but that there is spatial dispersion from one season to the next. The spatial dispersion is modeled by convolution of the population profile against a fixed dispersion kernel.

Heuristically speaking we refer to $[0, \pi]$ as "an environment" populated by some species and of $u\colon [0, \pi] \to \mathbb{R}$ as the "present" or "current" population distribution or population profile of that species throughout the enviornment. Then for $x \in [0, \pi]$ a "site" in the environment, $u(x)$ is the population of the species at site x.

Consider the map $\mathfrak{F}\colon L^2(0, \pi) \to L^2(0, \pi)$ given by

$$\mathfrak{F}[u](x) := \frac{1}{\pi} \int_0^\pi K(x-y) N[u](y)\, dy, \tag{35}$$

where $K \in L^2(0, \pi)$ is referred to as the dispersion kernel and models how populations disperse into neighboring sites, and $N\colon L^2(0, \pi) \to L^2(0, \pi)$ is a nonlinear function describing the "uncoupled" population dynamics at x, i.e., the dynamics in the absence of dispersion. In what follows we suppose local logistic dynamics given by

$$N[u](x) = \mu u(x)(1 - c(x)u(x)),$$

where μ is the reproductive rate of the species and $c(x)$ denotes the "carrying capacity" at the site x. Then $\mathfrak{F}(u)$ is the population profile "next season." (Reference to the ecological literature was made in section 1.)

We make additional regularity assumptions concerning K and c. Namely, we assume that K and c are even, 2π periodic, and they can be extended analytically to some complex strip containing the real axis. Such functions have convergent Fourier-cosine series, and we let $\{b_n\}_{n=0}^\infty$ and $\{c_n\}_{n=0}^\infty$ denote the cosine coefficients of $K(x)$ and $c(x)$, respectively, so that

$$K(x) = b_0 + 2\sum_{n=1}^\infty b_n \cos(nx) \qquad \text{and} \qquad c(x) = c_0 + 2\sum_{n=1}^\infty c_n \cos(nx)$$

for all x on a complex strip containing the real axis. Note that we define the cosine series with the factor of 2 in front of the sum so that we obtain a Banach algebra under discrete convolution when we formalize the sequence space of Fourier-cosine sequences in section 4.1.

Remark 4.1. Since we choose an analytic convolution kernel we have that $\mathfrak{F}(u)$ is analytic as long as $u \in L^1(0, \pi)$. In other words, any function with a preimage under $\mathfrak{F}$ is in fact analytic. In particular if u is a point in an invariant set (such as a stable/unstable manifold or a connecting orbit), then u is analytic. Then when we study invariant dynamics we restrict the domain of $\mathfrak{F}$ to a space of analytic functions.

Let $\{a_n\}_{n=0}^\infty$ denote the Fourier-cosine coefficients of u, i.e., we suppose that

$$u(x) := a_0 + 2\sum_{n=1}^\infty a_n \cos(nx).$$

The Kot–Schaffer map projected onto the sequence space of cosine coefficients is given by

$$F(a)_n = \mu b_n a_n - \mu b_n (c * a * a)_n \tag{36}$$

with discrete triple convolution product

$$(c * a * a)_n = \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3 \in \mathbb{Z}}} c_{|k_1|} a_{|k_2|} a_{|k_3|}$$

for $n \geq 0$.

In order to have a specific instantiation of the map for numerical consideration we fix the reproductive rate $\mu = 3.5$ and choose convolution kernel and spatial inhomogeneity given by

$$K(x) = \frac{1-r^2}{1-2r\cos(x)+r^2} = 1 + 2\sum_{n=1}^{\infty} r^n \cos(nx)$$

with $r = 1/2$, and

$$c(x) = \frac{1-s^2}{2\,(1-2s\cos(x)+s^2)} = \frac{1}{2} + \sum_{n=1}^{\infty} s^n \cos(nx)$$

with $s = 1/5$, i.e., we take both the dispersion kernel and the carrying capacity to be given by Poision kernels and have

$$b_n = r^n \qquad \text{and} \qquad c_n = \frac{s^n}{2}.$$

A typical orbit of the system with these parameter choices is illustrated in Figure 2, and we observe the seemingly chaotic behavior.

Remark 4.2 (nontrivial fixed points, stability analysis, and parameterization of the unstable manifold). The techniques of [45] are used in order to show that there exists a nontrivial fixed point with exactly one unstable eigenvalue and to compute a parameterization of the unstable manifold with validated error bounds $r_P > 0$. A bound δ_P on the derivative of the truncation error is obtained from r_P by using Cauchy estimates (see [63, 45, 57, 60]). Then r_P and δ_P provide the data needed for the validation values as in Definition 2.9.

4.1. Banach algebras associated with even periodic analytic functions. Consider a sequence

$$a = \{a_n\}_{n \in \mathbb{N}}, \qquad a_n \in \mathbb{C},$$

of complex numbers (with a "lower index"). For any $\nu \geq 1$ define the weighted little ℓ^1 norm

$$\|a\|_\nu^1 := |a_0| + 2\sum_{n=1}^{\infty} |a_n| \nu^n$$

and the Banach Space

$$\ell_\nu^1 := \{a = \{a_n\} : \|a\|_\nu^1 < \infty\}.$$

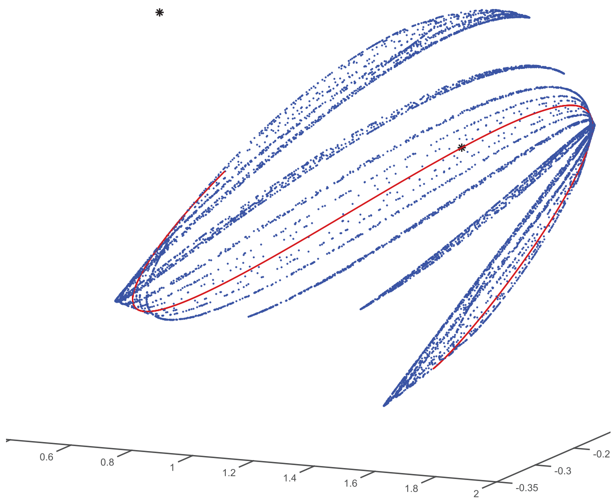

Figure 2. ***Attracting Set For Kot–Schaffer:*** *(All references to color refer to the online version). Dynamics of the Kot–Schaffer map Fourier space with $\mu = 3.5$, $b_n = 2^{-n}$, and $c_n = 5^{-n}/2$. The coordinates of the figure are the first three Fourier modes. The trivial fixed point appears as a black star in the upper left of the figure. A nontrivial fixed point appears as a black star in the middle of the attractor. The one dimensional unstable unstable manifold of the fixed point is illustrated as a red curve through the nontivial fixed point. The attractor is obtained by iteration of an arbitrary initial condition. The figure shows* 10,000 *iterates of the initial point. The computations are carried out projecting to $N = 25$ modes, i.e., the truncated phase space is* 26 *dimensional. The red curve illustrates the local unstable manifold.*

An "upper indexed" sequence of complex numbers $\{c^n\}_{n\in\mathbb{N}}$, satisfying the bounds below, defines a linear functional on ℓ^1_ν given by the formula

$$l(a) = \sum_{n=0}^{\infty} c^n a_n. \tag{37}$$

More precisely, define the norm

$$\|c\|^\infty_\nu := \max\left(|c^0|, \sup_{n\geq 1} \frac{|c^n|}{2\nu^n}\right)$$

and note that

$$\ell^\infty_\nu := \{c = \{c^n\} : \|c\|^\infty_\nu < \infty\}$$

is a Banach space. In fact $\ell^\infty_\nu = (\ell^1_\nu)^*$ with the isometric isomorphism given explicitly by (37). The proof follows as in the usual case of $\nu = 1$, i.e., we recall the standard fact that $\ell^\infty = (\ell^1)^*$. We sometimes employ the estimates

$$\|a\|^1_\nu \leq 2\sum_{n=0}^{\infty} |a_n|\nu^n \qquad \text{and} \qquad \|c\|^\infty_\nu \leq \sup_{n\geq 0} \frac{|c^n|}{\nu^n},$$

which cost a factor of two but are typographically pleasing.

A bounded linear operator $A\colon \ell_\nu^1 \to \ell_\nu^1$ can be expressed as

$$(Ah)_n = l_n(h)$$

with $l_n \in (\ell_\nu^1)^*$ for each $n \in \mathbb{N}$. Then there are $\{c_n^k\}_{k=0}^\infty \in \ell_\nu^\infty$ so that

$$(Ah)_n = \sum_{k=0}^\infty c_n^k h_k,$$

i.e., we can think of A as an infinite matrix with entries $\{c_n^k\}_{k,n\in\mathbb{N}}$. (Here we use superscript k to denote the columns and subscript n to denote the rows of this infinite matrix.) We bound the operator norm by considering

$$\begin{aligned} \|A\|_{B(\ell_\nu^1)} &:= \sup_{\|h\|_\nu^1=1} \|Ah\|_\nu^1 \\ &\le 2\sum_{n=0}^\infty \sup_{k\ge 0} \frac{|c_n^k|}{\nu^k}\nu^n, \end{aligned} \tag{38}$$

which we can think of as "sup over columns, sum over rows" for the infinite matrix A. (Again this bound costs several factors of two but is typographically pleasant.)

For any $a, b \in \ell_\nu^1$ we define *the discrete cosine convolution product*

$$(a * b)_n := \sum_{\substack{k_1+k_2=n \\ k_1,k_2\in\mathbb{Z}}} a_{|k_1|} b_{|k_2|} = \sum_{k=0}^n a_{n-k} b_k + \sum_{k=1}^\infty (a_{n+k} b_k + b_{n+k} a_k). \tag{39}$$

Note that $a * b \in \ell_\nu^1$ whenever $a, b \in \ell_\nu^1$ and moreover that

$$\|a * b\|_\nu^1 \le \|a\|_\nu^1 \|b\|_\nu^1,$$

so that $(\ell_\nu^1, *)$ is a Banach algebra. The space ℓ_ν^1 so endowed is related to certain spaces of analytic functions as follows.

Lemma 4.3. *Let $a = \{a_n\}, b = \{b_n\} \in \ell_\nu^1$ and define the functions*

$$f(z) := a_0 + 2\sum_{n=1}^\infty a_n \cos(nz) \qquad \textit{and} \qquad g(z) := b_0 + 2\sum_{n=1}^\infty b_n \cos(nz).$$

We write

$$A_r = \{z \in \mathbb{C} \ : \ |\mathrm{imag}(z)| < r\}$$

to denote the complex strip of width r.

- The ℓ_ν^1 norm bounds the C^0 norm. *If $\nu \ge 1$, then f, g are well defined, even, 2π periodic, Lebesgue integrable functions having*

$$\sup_{|\mathrm{imag(z)}|\le\log(\nu)} |f(z)| \le \|a\|_\nu^1,$$

and similarly for g. If $\nu = 1$, then the supremum is over the real axis only.

- Regularity. *If $\nu > 1$, then f, g define analytic functions on the complex strip A_r with $r = \log(\nu)$. Moreover the functions f, g are continuous on the closure of the strip. If a, b are real sequences, then f and g are real analytic.*
- Products. *If $\nu > 1$, then the function fg is analytic on the complex strip A_r and continuous on the closure of the strip with $r = \log(\nu)$. The product function has Fourier coefficients $\{c_n\}$ given by discrete convolution, i.e.,*

$$c_n = (a * b)_n.$$

A partial converse is given by the observation that if f is even, periodic $[0, \pi]$, and real analytic, then there exists a $\nu > 1$ so that the Fourier cosine coefficients $\{a_n\}_{n=0}^{\infty}$of f are in ℓ_ν^1.

Finally we observe that the projected Kot–Schaffer mapping given by (36) is a well defined map on ℓ_ν^1 as

$$\|F(a)\|_\nu^1 \leq |\mu| \|b\|_\nu^\infty \|a\|_\nu^1 + |\mu| \|b\|_\nu^\infty \|c\|_n u^1 (\|a\|_\nu^1)^2 < \infty$$

for all $a, b, c \in \ell_\nu^1$. Moreover the map is Fréchet differentiable with

$$[DF(a)h]_n = \mu b_n h_n - 2\mu b_n (c * a * h)_n$$

for every $a, h \in \ell_\nu^1$, and we have the bound

$$\|DF(a)\|_{B(\ell_\nu^1)} \leq |\mu| \|b\|_\nu^\infty + 2|\mu| \|b\|_\nu^\infty \|c\|_\nu^1 \|a\|_\nu^1.$$

In fact the derivative $DF(a)$ is a compact linear operator due to the smoothing effect of b_n.

4.1.1. Projections and truncations. Fix $N \in \mathbb{N}$, $\nu > 1$, and define the sequences $v^n = \{v_j^n\}_{j=0}^{\infty} \in \ell_\nu^1$ by

$$v_j^n = \begin{cases} v_j^n = 1 & \text{if } n = j, \\ 0 & \text{otherwise.} \end{cases}$$

We define the subspace $\mathcal{X}^N \subset \ell_\nu^1$ by

$$\mathcal{X}^N := \operatorname{span}(v^0, \ldots, v^N).$$

For $a = \{a_n\}_{n=0}^{\infty} \in \ell_\nu^1$ we define the projection operator $\pi_N \colon \ell_\nu^1 \to \ell_\nu^1$ by

$$\pi_N(a)_n = \begin{cases} a_n, & 0 \leq n \leq N, \\ 0, & n \geq N+1, \end{cases}$$

so that $\pi_N\left(\ell_\nu^1\right) = \mathcal{X}^N$. Similarly $\pi_\infty \colon \ell_\nu^1 \to \ell_\nu^1$ is given by

$$\pi_\infty(a) = \begin{cases} 0, & 0 \leq n \leq N, \\ a_n, & n \geq N+1, \end{cases}$$

and define $\mathcal{X}^\infty = \pi_\infty(\ell_\nu^1)$.

Suppose that $a^N \in \mathcal{X}^N = \text{span}(v^0, \dots, v^N)$, i.e.,

$$a^N = \sum_{n=0}^{N} a_n v^n,$$

or in componentwise form

$$a_n^N = \begin{cases} a_n, & 0 \leq n \leq N, \\ 0, & n \geq N+1. \end{cases}$$

Then we have the map $L\colon \mathcal{X}^N \to \mathbb{R}^{N+1}$ given by

$$L(a^N) = \begin{pmatrix} a_0 \\ \vdots \\ a_N \end{pmatrix}.$$

L induces the norm on $\mathbb{R}^{N+1}$ given by

$$\|a^N\|_{\mathbb{R}^{N+1}} := \sum_{n=0}^{N} |a_n| \nu^n.$$

It will be important to bound some linear operators defined in term of numerical matrices. In addition, some linear functionals induced by convolution play an important role in the truncation error analysis to follow. The following lemmas are helpful. The proofs are similar to Corollaries 1 and 3 in [44].

Lemma 4.4 (convolution sums for tails). *Let $0 \leq n \leq N$ and $a^N, c^N \in \ell_\nu^1$ be of the form $a^N = (a_0, \dots, a_N, 0, 0, 0, \dots)$, $c^N = (c_0, \dots, c_N, 0, 0, 0, \dots)$. For any $h \in \ell_\nu^1$ let $h^\infty = (0, \dots, 0, h_{N+1}, h_{N+2}, \dots)$ and define the linear functional $l_n\colon \ell_\nu^1 \to \mathbb{R}$ by*

$$l_n(h) := (a^N * c^N * h^\infty)_n.$$

Let

$$\kappa_n^1 := \max_{N+1 \leq k \leq 2N-n} \frac{|(c^N * a^N)_{n+k}|}{2\nu^k} \qquad \text{and} \qquad \kappa_n^2 := \max_{N+1 \leq k \leq 2N+n} \frac{|(c^N * a^N)_{k-n}|}{2\nu^k}.$$

Then

$$\|l_n\|_{(\ell_\nu^1)^*} \leq \kappa_n^1 + \kappa_n^2.$$

4.2. Good coordinates for a nontrivial fixed point of Kot–Schaffer. Let $\bar{a} \in \ell_\nu^1$ be a nontrival fixed point for the Kot–Shcaffer map. Following the discussion of Remark 3.10 we define the local Kot–Schaffer map $\bar{F}\colon \ell_\nu^1 \to \ell_\nu^1$ relative to the fixed point $\bar{a}$ by the sequence of component maps

$$\overline{F}(a)_n = F(\bar{a} + a)_n - \bar{a}_n = \mu b_n a_n - 2\mu b_n \left(c * \bar{a} * a\right)_n - \mu b_n \left(c * a * a\right)_n \tag{40}$$

for $n \geq 0$. Now we seek linear isomorphisms $V, W\colon \ell_\nu^1 \to \ell_\nu^1$ as in Remark 3.10.

Define the bounded linear operators $L^N\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ and $\epsilon^N\colon \ell^1_\nu \to \mathbb{R}^{N+1}$ by

$$L^N(a^N) := \mu b_n a_n^N - 2\mu b_n (c^N * \bar{a}^N * a^N)_n$$

and

$$\epsilon^N(a^N, a^\infty)_n = -2\mu b_n \left[(c^N * \bar{a}^N * a^\infty)_n + (c^N * \bar{a}^\infty * a)_n + (c^\infty * \bar{a} * a)_n, \right]$$

for $0 \le n \le N$. Note that since L^N is a bounded linear operator from $\mathbb{R}^{N+1}$ to $\mathbb{R}^{N+1}$ there is an $N+1 \times N+1$ matrix of real numbers M^N so that

$$M^N a^N = L^N(a^N)$$

for all $a^N \in \mathbb{R}^{N+1}$. In fact the columns of M^N are determined by evaluating L^N on the standard basis vectors e_j. We write

$$\bar{F}(a^N, a^\infty) = \left[\begin{array}{c} \bar{F}_1(a^N, a^\infty) \\ \bar{F}_2(a^N, a^\infty) \end{array} \right],$$

where the maps $\bar{F}_1\colon \ell^1_\nu \to \mathbb{R}^{N+1}$ and $\bar{F}_2\colon \ell^1_\nu \to \mathcal{X}^\infty$ are given by

$$\bar{F}_1(a^N, a^\infty)_n = M^N a^N + \epsilon^N(a^N, a^\infty)_n - \mu b_n (c * a * a)_n$$

for $0 \le n \le N$ and

$$\bar{F}_2(a^N, a^\infty)_n = \mu b_n a_n^\infty - 2\mu b_n (c * \bar{a} * a)_n - \mu b_n (c * a * a)_n.$$

Suppose for the sake of simplicity that M^N is diagonalizable, so that there are $N+1 \times N+1$ matrices V^N, W^N, and Σ^N so that

$$M^N = V^N \Sigma^N W^N$$

with

$$\Sigma^N = \begin{pmatrix} \Sigma_u^N & 0 \\ 0 & \Sigma_s^u \end{pmatrix},$$

$$\Sigma_u^N = \begin{pmatrix} \lambda_u^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_u^{n_u} \end{pmatrix},$$

$$\Sigma_s^N = \begin{pmatrix} \lambda_s^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_s^{n_s} \end{pmatrix},$$

and

$$W^N = \left(V^N\right)^{-1}.$$

Here

$$1 < |\lambda_u^1| \le \cdots \le |\lambda_u^{n_u}|,$$
$$|\lambda_s^{n_s}| \le \cdots \le |\lambda_s^1| < 1,$$

and $n_u + n_s = N$. Of course V^N is just the matrix of right eigenvectors for M^N.

Define the coordinates

$$v^N = W^N a^N$$

and

$$v^\infty = a^\infty,$$

so that

$$A^N = V^N v^N = [V_u^N | V_s^N] \begin{pmatrix} v_u \\ v_s \end{pmatrix}$$

with $v_s \in \mathbb{R}^{n_s}$, $v_u \in \mathbb{R}^{n_u}$. The desired isomorphisms are

$$W = \begin{bmatrix} W^N & 0 \\ 0 & \mathrm{Id}^\infty \end{bmatrix} \qquad \text{and} \qquad V = \begin{bmatrix} V^N & 0 \\ 0 & \mathrm{Id}^\infty \end{bmatrix}.$$

Let W_u^N denote the matrix composed of the first n_u rows of W^N and let W_s^N denote the matrix of the remaining n_s rows, i.e.,

$$W^N = \begin{pmatrix} W_u^N \\ W_s^N \end{pmatrix},$$

where W_u^N is an $n_u \times N+1$ and W_s^N is an $n_s \times N+1$ matrix. Then we can write

$$\begin{aligned} &W\bar{F}V(v^N, v^\infty) \\ &= \begin{pmatrix} \Sigma_u v^u + W_u^N \epsilon^N (V^N v^N, v^\infty)^N - \mu W_u^N B^N (c * V^N v^N + v^\infty * V^N v^N + v^\infty)^N \\ \Sigma_s v^s + W_s^N \epsilon^N (V^N v^N, v^\infty)^N - \mu W_s^N B^N (c * V^N v^N + v^\infty * V^N v^N + v^\infty)^N \\ \mu B^\infty v^\infty - 2\mu B^\infty (c * \bar{a} * V^N v^N + \phi^\infty)^\infty - \mu B^\infty (c * V^N v^N + v^\infty * V^N v^N + v^\infty)^\infty \end{pmatrix}. \end{aligned}$$

Defining

$$\begin{aligned} A_u^N &:= \Sigma_u, \\ A_s^N &:= \Sigma_s, \\ A_s^\infty &:= \mu B^\infty, \\ E_1(v^u, v^s, v^\infty) &= W_u^N \epsilon^N (V^N v^N, v^\infty)^N, \\ E_2(v^u, v^s, v^\infty) &= W_s^N \epsilon^N (V^N v^N, v^\infty)^N, \\ E_3(v^u, v^s, v^\infty) &= -2\mu B^\infty (c * \bar{a} * V^N v^N + v^\infty)^\infty, \\ H_1(v^u, v^s, v^\infty) &= -\mu W_u^N B^N (c * V^N v^N + v^\infty * V^N v^N + v^\infty)^N, \\ H_2(v^u, v^s, v^\infty) &= -\mu W_s^N B^N (c * V^N v^N + v^\infty * V^N v^N + v^\infty)^N, \end{aligned}$$

and

$$H_3(v^u, v^s, v^\infty) = -\mu B^\infty (c * V^N v^N + v^\infty * V^N v^N + v^\infty)^\infty,$$

we see that we have a system of “good coordinates” for the Kot–Schaffer system. Indeed, for N large enough we have the bounds

$$
\begin{aligned}
\|A_u^{-1}\| &\le \frac{1}{|\lambda_u^1|} < 1,\\
\|A_s\| &\le |\lambda_s^1| < 1,\\
\|A_\infty\| &\le |\mu||b_{N+1}| < 1,\\
\|E_1\| &\le 2|\mu| \left(\|W_u^N B^N e^\infty\|_\nu^1 + \|W_u^N B^N\| \|c^N\|_\nu^1 \|\bar{a}^\infty\|_\nu^1 \|V^N\| \right.\\
&\qquad \left. + \|W_u^N B^N\| \|\bar{a}\|_\nu^1 \|c^\infty\|_\nu^1 \|V^N\|\right) =: \epsilon_1,\\
\|E_2\| &\le 2|\mu| \left(\|W_s^N B^N e^\infty\|_\nu^1 + \|W_s^N B^N\| \|c^N\|_\nu^1 \|\bar{a}^\infty\|_\nu^1 \|V^N\| \right.\\
&\qquad \left. + \|W_s^N B^N\| \|\bar{a}\|_\nu^1 \|c^\infty\|_\nu^1 \|V^N\|\right) =: \epsilon_2,\\
\|E_3\| &\le 2|\mu||b_{N+1}| \|c\| \|\bar{a}\| \|V^N\| =: \epsilon_3,\\
\|H_1(v)\| &\le |\mu| \|W_u^N B^N\| \|c\| \|V^N\|^2 \|v\|^2 =: C_1 \|v\|^2,\\
\|H_2(v)\| &\le |\mu| \|W_s^N B^N\| \|c\| \|V^N\|^2 \|v\|^2 =: C_2 \|v\|^2,\\
\|H_3(v)\| &\le |\mu||b_{N+1}| \|c\| \|V^N\|^2 \|v\|^2 =: C_3 \|v\|^2,\\
\|DH_1(v)\| &\le 2|\mu| \|W_u^N B^N\| \|c\| \|V^N\|^2 \|v\| =: \tilde{C}_1 \|v\|,\\
\|DH_2(v)\| &\le 2|\mu| \|W_s^N B^N\| \|c\| \|V^N\|^2 \| \|v\| =: \tilde{C}_2 \|v\|,\\
\|DH_3(v)\| &\le 2|\mu||b_{N+1}| \|c\| \|V^N\|^2 \|v\| =: \tilde{C}_3 \|v\|,\\
\|D^2 H_1(v)\| &\le 2|\mu| \|W_u^N B^N\| \|c\| \|V^N\|^2 =: \hat{C}_1,\\
\|D^2 H_2(v)\| &\le 2|\mu| \|W_s^N B^N\| \|c\| \|V^N\|^2 =: \hat{C}_2,\\
\|D^2 H_3(v)\| &\le 2|\mu||b_{N+1}| \|c\| \|V^N\|^2 =: \hat{C}_3,
\end{aligned}
$$

where e^∞ is the vector with components given by the positive numbers

$$e_n^\infty := \kappa_n^1 + \kappa_n^2,$$

$0 \le n \le N$, with the kappas defined as in Lemma 4.4.

Let $v^N = (\theta, \phi^N)$, V_s^N be the $(N+1) \times n_s$ submatrix of V^N containing the n_s stable eigenvectors of M^N, and V_u^N be the $(N+1) \times n_u$ submatrix containing the n_u unstable eigenvectors of M^N. Then we employ the parameterization of the stable manifold of $\bar{a}$ given by

$$Q(\phi^N, \phi^\infty) = Q^N(\phi^N) + Q^\infty(\phi^N, \phi^\infty),$$

where

$$Q^N(\phi^N) := \pi_N(\bar{a}) + V_s^N \phi^N,$$

and

$$Q^\infty(\phi^N, \phi^\infty) = \pi_\infty(\bar{a}) + \begin{pmatrix} V_u^N & 0 \\ 0 & \mathrm{Id}_\infty \end{pmatrix} \begin{bmatrix} \omega(\phi^N, \phi^\infty) \\ \phi^\infty \end{bmatrix}.$$

Error bounds for Q^∞ and DQ^∞ are required in order to define the validation values r_Q and δ_Q of definition 2.9. These bounds are obtained by choosing $\hat{r}$ and $\hat{R}$ as in Remark 3.11 of section 3.2 and using the estimates above to show that the hypotheses of Theorem 3.8 are satisfied. Note that the absolute values of λ_s^1 and $1/\lambda_u^1$, the norms of $W_u^N B^N e^\infty$, $W_u^N B^N$, $W_s^N B^N$, and indeed even the matrices W^N, V^N and the bounds $\kappa_n^{1,2}$ all depend on the fixed point $\bar{a}$ and are all computed with computer assistance.

4.3. Connecting orbit operator validation values for Kot–Schaffer. For $a \in \ell_\nu^1$ and $0 \leq n \leq N$ we decompose the component maps of Kot–Schaffer as

$$\begin{aligned} F(a)_n &= \mu b_n a_n - \mu b_n (c * a * a)_n \\ &= \mu b_n a_n - \mu b_n (c^N * a^N * a^N)_n - \mu b_n \left[2(c^N * a * a^\infty)_n + (c^\infty * a * a)_n \right]. \end{aligned}$$

Following the discussion of section 2.2 we define the map $F^N \colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by

$$F(a^N)_n^N := \mu b_n a_n^N - \mu b_n (c^N * a^N * a^N)_n \qquad \text{with} \qquad 0 \leq n \leq N$$

and the map $F^\infty \colon \ell_\nu^1 \to \ell_\nu^1$ by

$$F^\infty(a)_n := \begin{cases} \mu b_n \left[2(c^N * a * a^\infty)_n + (c^\infty * a * a)_n \right], & 0 \leq n \leq N, \\ \mu b_n a_n - \mu b_n (c * a * a)_n, & n \geq N. \end{cases}$$

Note that if $\bar{a}^N \in \pi_N(\ell_\nu^1)$, i.e., $\pi_\infty(\bar{a}^N) = 0$, then things reduce to

$$F^\infty(\bar{a}^N)_n = \begin{cases} \mu b_n (c^\infty * \bar{a}^N * \bar{a}^N)_n, & 0 \leq n \leq N, \\ -\mu b_n (c * \bar{a}^N * \bar{a}^N)_n, & n \geq N, \end{cases}$$

and we have the bounds

$$\|\pi_N F^\infty(\bar{a}^N)\| \leq |\mu| \sup_{0 \leq n \leq N} |b_n| \|\bar{a}^N\|^2 \|c^\infty\| \tag{41}$$

and

$$\|\pi_\infty F^\infty(\bar{a}^N)\| \leq |\mu| |b_{N+1}| \|c\| \|\bar{a}^N\| \|\bar{a}^N\|. \tag{42}$$

Note that these are exactly the quantities on which we must obtain bounds in order to define the Y_j^1 and Y_j^2 constants in Theorem 2.10.

Next we note that for any $a, h \in \ell_\nu^1$ we have that

$$[DF(a)h]_n = \mu b_n h_n - 2\mu b_n (c * a * h)_n.$$

Then

$$[DF^N(a^N)h^N]_n = \mu b_n h_n - 2\mu b_n (c^N * a^N * h^N)_n \qquad \text{for} \qquad 0 \leq n \leq N,$$

and

$$[DF^\infty(a)h]_n = \begin{cases} 2\mu b_n \left[(c^N * a * h^\infty)_n + (c^N * h * a^\infty)_n + (c^\infty * a * h)_n \right], & 0 \leq n \leq N, \\ \mu b_n h_n - 2\mu b_n (c * a * h)_n, & n \geq N+1. \end{cases}$$

Again if $\bar{a}^N \in \pi_N(\ell^1_\nu)$, this reduces to

$$[DF^\infty(\bar{a}^N)h]_n = \begin{cases} 2\mu b_n \left[(c^N * \bar{a}^N * h^\infty)_n + (c^\infty * \bar{a}^N * h)_n \right], & 0 \le n \le N, \\ \mu b_n h_n - 2\mu b_n (c * \bar{a}^N * h)_n, & n \ge N+1. \end{cases}$$

Let $\kappa^1_n(\bar{a}^N)$ and $\kappa^2_n(\bar{a}^N)$ be defined as in Lemma 4.4, and write

$$\kappa_n(\bar{a}^N) = \kappa^1_n(\bar{a}^N) + \kappa^2_n(\bar{a}^N).$$

For $0 \le n \le N$ we define the components of a vector $e(\bar{a}^N) \in \mathbb{R}^{N+1}$ by

$$e_n(\bar{a}^N) = 2|\mu||b_n|\, \kappa_n(\bar{a}^N)$$

and have that

$$2|\mu||b_n||(c^N * \bar{a}^N * h^\infty)_n| \le e_n(\bar{a}_N)\|h\|^1_\nu.$$

Then taking the supremum over h of norm one we have that

$$\left\| \pi_N \left(DF^\infty(\bar{a}^N) \right) \right\|_{B(\ell^1_\nu)} \le \|e\|^1_\nu + 2|\mu| \|b^N\|^\infty_\nu \|\bar{a}^N\|^1_\nu \|c^\infty\|^1_\nu.$$

If A^N is an $N+1 \times N+1$ matrix, let $|A|^N$ denote the matrix whose entries are the absolute values of the entries of A^N. Then

$$\left\| A^N \pi_N \left(DF^\infty(\bar{a}^N) \right) \right\|_{B(\ell^1_\nu)} \le \||A|^N e\|^1_\nu + 2|\mu| \|A^N B^N\|^\infty_\nu \|\bar{a}^N\|^1_\nu \|c^\infty\|^1_\nu, \tag{43}$$

where B^N is the diagonal matrix with the nonzero entries $b_0, \dots, b_N$. We see that the expression given in (43) is the term which must be bound by the Z^1_j in the hypotheses of Theorem 2.10. We also note that in the first term, multiplying the matrix of absolute values by the vector of bounds given by e gives better results than would be obtained by $\|A^N\|\|e\|$. Similarily, in the second term a poorer bound would be obtained by taking $\|A^N\|\|b^N\|^\infty_\nu$.

In the tail we have the estimate

$$\left\| \pi_\infty \left(DF^\infty(\bar{a}^N) \right) \right\| \le |\mu||b_{N+1}|(1 + 2\|c\|\|\bar{a}^N\|), \tag{44}$$

as needed in order to define the Z^2_j constants in Theorem 2.10.

Finally for $\bar{a}^N \in \mathcal{X}^N$ and $h, a_1, a_2 \in \ell^1_\nu$ consider

$$[(DF(a_1) - DF(a_2))h]_n = 2\mu b_n (c * a_1 - a_2 * h)_n.$$

Then

$$\|[DF(a_1) - DF(a_2)]h\|_{\ell^1_\nu} \le 2\mu \|b\|^\infty_\nu \|c\| \|a_1 - a_2\| \|h\|.$$

Let $r > 0$ and $a \in B_r(\bar{a}^N)$. Taking the sup over all h with norm one gives

$$\|[DF(\bar{a}^N) - DF(a)\|_{B(\ell^1_\nu)} \le 2|\mu| \|b\|^\infty_\nu \|c\|^1_\nu r. \tag{45}$$

Then we have

$$C_F(a) := 2\mu \|b\|^\infty_\nu \|c\|^1_\nu,$$

a bound which is in fact uniform in a. Then taking $C_F(a) = C^j_F$ gives the constants required by definition 2.9.

Remark 4.5 (numerical implementation). We note that from a numerical point of view the term $(c^N * a^N * a_n^N)_n$ is cubic, even though the nonlinearity is quadratic when c is considered fixed and a is the variable. In other words we have to evaluate a term whose numerical complexity is the same as a general cubic discrete convolution. In order to evaluate this term we use the interval arithmetic fast Fourier transform provided in IntLab. The reader interested in the implementation can examine the program

`kotSchaffer_logistic_intval.m`.

Similarly, the matrix associated with $DF^N(a^N)$ is computed using IntLab's fast Fourier transform and some basic shift operations. The program

`kotSchafferDifferential_logistic_intval.m`

contains the implementation. Implementation of the so-called kappa bounds is in the program

`kappaBounds.m`.

The codes found at [46] also provide implementations of the ℓ_ν^1 and ℓ_ν^∞ norms, as well as the necessary operator norms.

5. Existence of some connecting orbits by computer assisted proof.

All of the computer programs discussed in this section are implemented in the MATLAB programing environment. The IntLab package is used for all interval arithmetic computations [69]. The computer programs discussed in this section are made freely available at [46].

The existence of nontrivial fixed points for Kot–Schaffer, as well as the relevant linear analysis, is provided using the methods discussed in appendices A-F of [45]. However since we are trying to find a connecting orbit and not just compute the unstable manifold we used slightly different computational parameters than those discussed in [45]. The programs used for this analysis are

- `kotSchaffer_validateFP.m` : returns validated error bounds in ℓ_ν^1 for the fixed point.
- `kotSchaffer_validateEig.m` : returns validated error bounds in $\mathbb{C} \times \ell_\nu^1$ for the eigenvalue/eigenvector pair.
- `kot_oneUnstableEig_check.m` : computer assisted proof that the fixed point has exactly one unstable eigenvalue.

We begin by considering the data in the file `tangleProofDataSet1.mat`. Here we have model parameters as in section 4 and project the problem onto $N = 30$ cosine series modes. We note that the numerical approximation $\bar{p}^N \in \mathbb{R}^{31}$ of the fixed point p_1 has $\bar{p}_0^N = 1.530903719863764$ and that the validated error is $r_0 \leq 3.2 \times 10^{-14}$. In particular the fixed point is nontrivial.

$DF(p_1)$ has unstable eigenvalue $\lambda_* \in \mathbb{R}$ with

$$|\lambda_* - (-1.713560724396476)| \leq 7.5 \times 10^{-13},$$

i.e., $\lambda_* \approx -1.714$. The coordinates of the approximate eigenvector can be seen by running the program `a_tangleProofDataSet1.m`, and the eigenvector is validated with an error less than 10^{-12}. Moreover `a_tangleProofDataSet1.m` validates that λ_* is the only unstable eigenvalue at p_1, i.e., p_1 has a one dimensional unstable manifold and co-dimension one stable manifold. Again, the ideas used in these computer assisted validations are discussed in the appendices of [45].

Next we compute a polynomial parameterization of the unstable manifold to order $M = 25$, i.e., we have that

$$P^{MN}(\theta) = \sum_{m=0}^{M} p_m^N \theta^m,$$

where the coefficients $p_m \in \mathbb{R}^{31}$ for $0 \leq m \leq 25$. Using the program:
`validateUnstableManifold_Kot1D_fast.m`
we obtain a validated error bound of $r_P \leq 1.3 \times 10^{-11}$, i.e., $P(\theta) = P^N(\theta) + H(\theta)$ with $H \colon [-1, 1] \to \ell^1_\nu$

$$\sup_{\theta \in [-1,1]} \|H(\theta)\|^1_\nu \leq r_P.$$

We take $\hat{r} = 0.98$ and $B_{\hat{r}} = [-0.98, 0.98]$. Then using Cauchy bounds we obtain that

$$\sup_{\theta \in [-0.98, 0.98]} \|H'(\theta)\|^1_\nu \leq 6.1 \times 10^{-10} =: \delta_P.$$

The numerical computation of P^{MN} and method for obtaining the validated error bounds are the subject of [45]. The reader interested in the details can also consult the program `a_tangleProofDataSet1.m`.

5.1. Validated error bounds for the stable manifold. In order to approximate the stable manifold we implement the "good coordinate" change discussed in section 4.2, and then we employ the validation algorithm discussed in section 3.2. The linear approximation of the manifold is given by the span of the stable eigenvectors of the matrix

$$M = DF^N(\bar{p}_1^N).$$

We compute rigorous enclosures of these eigenvectors using the built in IntLab function `verifyEig`. We obtain validated bounds on the linear approximation of the stable manifold on the neighborhood

$$U_{\tilde{r}, \tilde{R}} = \left\{ (\phi^N, \phi^\infty) : \|\phi^N\| \leq \tilde{r}, \qquad \text{and} \qquad \|\phi^\infty\| \leq \tilde{R} \right\}$$

with

$$\tilde{r} = 3.8 \times 10^{-9} \qquad \text{and} \qquad \tilde{R} = \tilde{r}$$

of magnitude

$$\sup_{(\phi^N, \phi^\infty) \in U_{\tilde{r}, \tilde{R}}} \|Q^\infty(\phi)\|^1_\nu \leq 4.8 \times 10^{-13}$$

and the bound

$$\sup_{(\phi^N, \phi^\infty) \in U_{\tilde{r}, \tilde{R}}} \|DQ^\infty(\phi)\| \leq 6.4 \times 10^{-5}$$

on the derivative. The function which performs these validations is
`validateStableManifold_KotCoDim1.m`.
Again the reader can run the program
`a_tangleProofDataSet1.m`
to obtain the results just mentioned.

5.2. Existence of connecting orbits. Using the data discussed in the previous sections we define the projected connecting orbit operator with $K = 34$, i.e., we look for a numerical orbit which begins on the unstable manifold approximation and terminates on the linear approximation of the stable manifold after 34 iterations. Such a numerical orbit is found using a Newton method and we denote the results by

$$\bar{w}^N = (\bar{\theta}, \bar{\phi}^N, \bar{x}_1^N, \dots, \bar{x}_K^N).$$

We note that

$$\bar{\theta} = -0.768423478031049 \in [-0.98, 0.98]$$

and

$$\|\bar{\phi}^N\| \leq 2.1 \times 10^{-9}, \qquad \text{so that} \qquad (\bar{\phi}^N, 0) \in \hat{U}.$$

Letting $r_* = \hat{r} - \|\bar{\phi}^N\|$ leads to

$$r_* = 1.6569524798823 \times 10^{-9},$$

and we note that

$$B_{r_*}(\bar{\theta}) \subset [-0.98, 0.98] \qquad \text{and} \qquad B_{r_*}(\bar{\phi}^N) \in \pi_N \hat{U}.$$

Since $\tilde{R} = \tilde{r}$ we have that $B_{r_*}(\bar{\phi}^N) \subset \hat{U}$. We do not here reproduce the data for $\bar{x}_1^N, \dots, \bar{x}_K^N \in \mathbb{R}^{31}$, but again the data can be obtained by running the program

`a_tangleProofDataSet1.m`,

and we note for example that $\bar{x}_1^N$ is not in $B_{r_*}(p_1)$, i.e., $\bar{w}^N$ is not the trivial solution. Using the results from the beginning of section 5 and from subsection 5.1 we build the radii-polynomials of Theorem 2.10 and check that for

$$r \in [9.220490251212868 \times 10^{-11}, r_*],$$

the radii-polynomials are well defined and strictly negative. We conclude there exists a

$$w_* \in \mathcal{W},$$

so that $\Phi(w_*) = 0$ and, for example, $\|\bar{w} - w_*\| \leq 10^{-10}$. Then there is a homoclinic connection for the Kot–Schaffer mapping with system parameters as described in section 4, and this connecting orbit is very close to our numerical approximation. See Figure 3 for a depiction of the ℓ_ν^1 norms of the points along this orbit. The validation just described is performed by the computer program `a_tangleProofDataSet1.m`.

Figures 4, 5, 6, and 7 illustrate other orbits whose existence is established using the programs

- `a_tangleProof_3_excursion.m` for the three excursion orbit,
- `a_tangleProof_4_excursion.m` for the four excursion orbit,
- `a_tangleProofLongExcursion.m` for the long excursion orbit.

Theses programs validate the approximate connecting orbit data found in the files

- `tangleProofDataSet_3_excursions.mat`,
- `tangleProofDataSet_4_excursions.mat`, and
- `tangleProofDataSetLongExcursion.mat`,

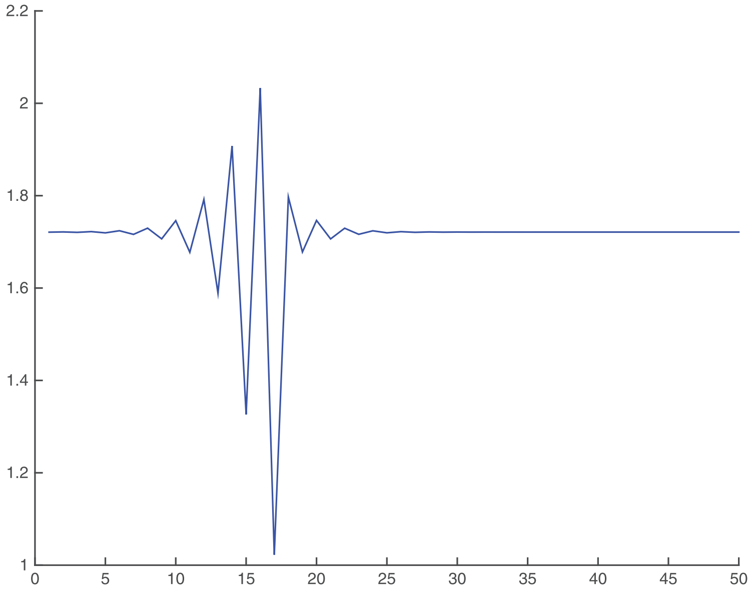

Figure 3. *The figure illustrates the first (and shortest) homoclinic connection for the Kot–Schaffer map whose existence is established as discussed in section 5.2. The proof proceeds by showing that the connecting orbit operator has a zero near an approximate numerical zero. For this orbit we have "time of flight"* $K = 34$*, and we truncate the Kot–Schaffer map to* $N = 30$ *spatial Fourier modes (i.e., the dimension of the projection for the operator is* $(34 + 1) \times (30 + 1) = 1085$*). The unstable manifold is approximated to polynomial order* $M = 25$*, and the stable manifold is approximated using the linear approximation. The proof is successful, and the true solution is no farther than* $r = 10^{-10}$ *away from the numerical approximation (in the* C^0 *norm). The plot shows the* ℓ^1_ν *one norm of the points along the orbit.*

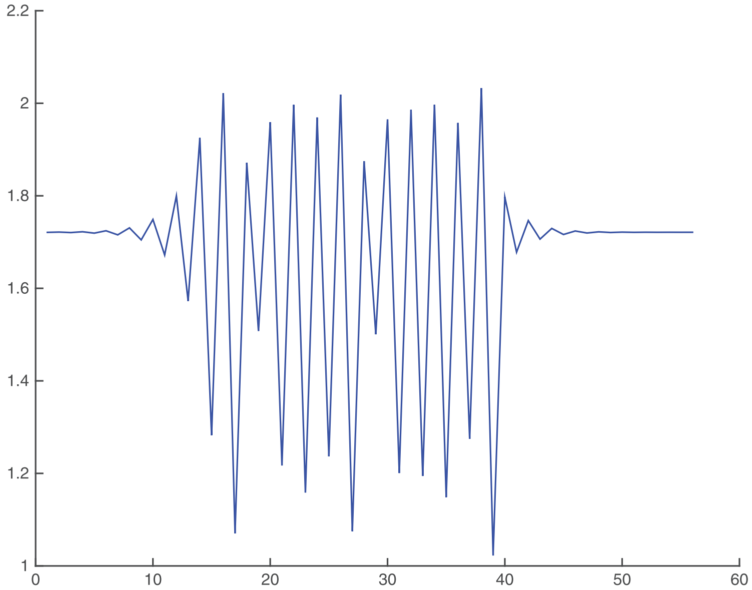

Figure 4. *An orbit with* $K = 51$ *and* $N = 30$ *validated with an accuracy of* $r = 6.4 \times 10^{-11}$*. This connection makes a "long excursion," i.e., it spends more then 20 iterates away from the fixed point.*

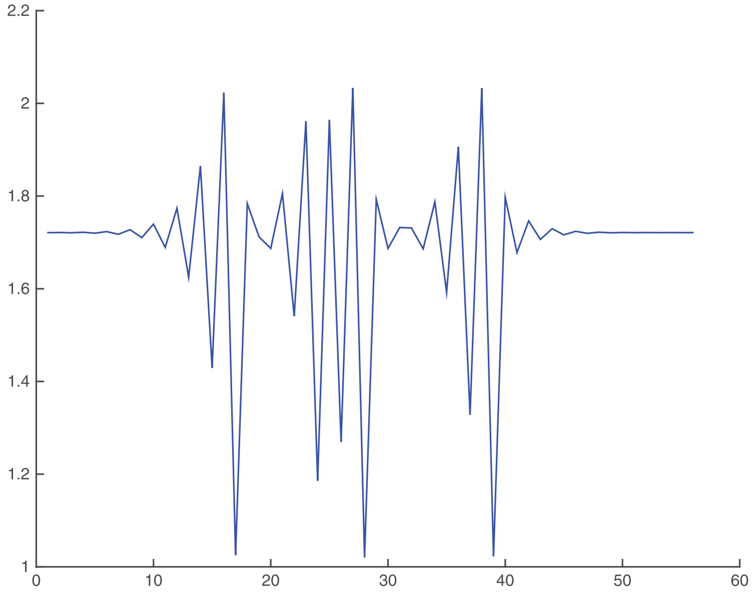

Figure 5. *An orbit with $K = 51$ and $N = 30$ validated with an accuracy of $r = 9.1 \times 10^{-11}$. This connection makes three excursions of about 8 iterates away and back to the fixed point before finally converging.*

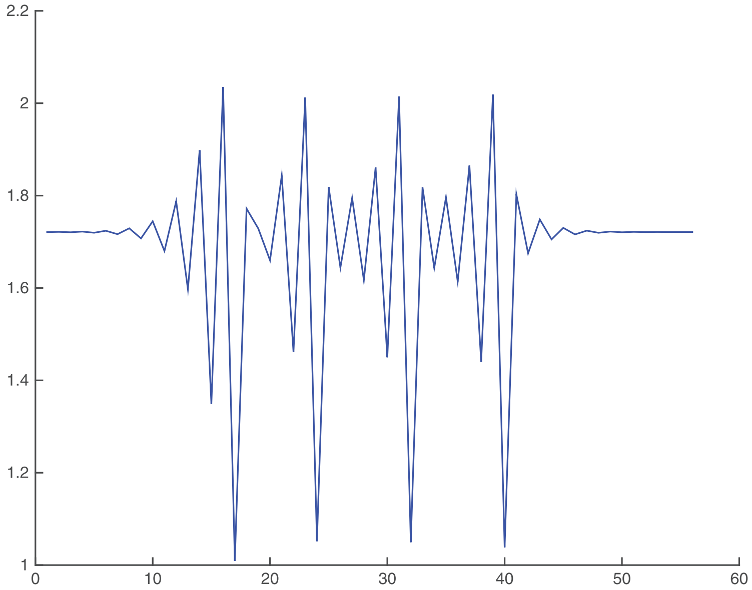

Figure 6. *An orbit with $K = 53$ and $N = 30$ validated with an accuracy of $r = 1.4 \times 10^{-10}$. This connection makes four short excursions away and back to the fixed point before finally converging.*

respectively. The computations are similar to those discussed above and in particular result in mathematically rigorous error bound between the numerical approximate solution and the true solution. The reader interested in the details may run the associated computer programs. We remark that all computations were performed on a Mac Pro desktop with a 3.7 GHz Quad-Core Intel Xeon E5 processor and 64 GB 1866 MHz of RAM. Each of the connecting orbit

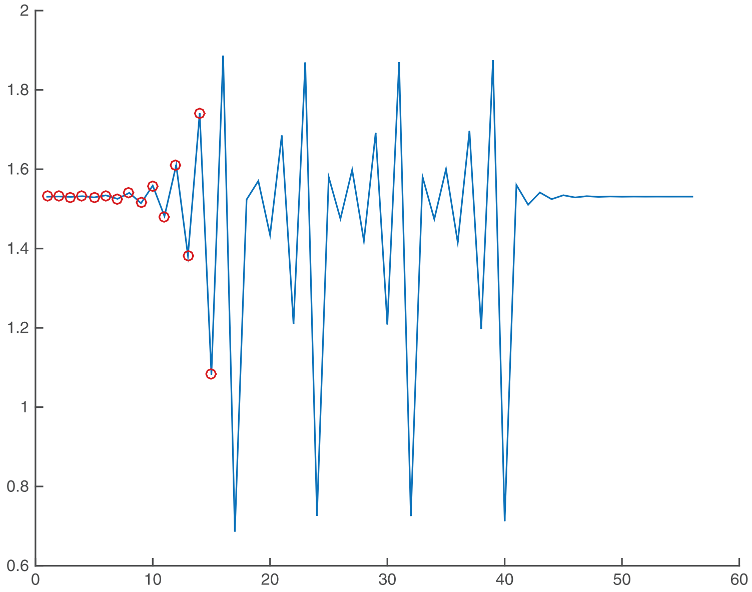

Figure 7. *The figure illustrates the same orbit shown in Figure 6. However in this figure the vertical axis gives the averages of the functions (instead of the ℓ^1_ν norms), and the red dots indicate points on the orbit which are on the parameterization of the unstable manifold. This gives an indication of the role the high order parameterization is playing, as these points do not enter explicitly into the a posteriori analysis of the connecting orbit operator.*

proofs runs in less than 4 minutes, and this includes the time needed to validate the fixed points, the rigorous linear analysis, and the validation of the stable/unstable manifolds. The reader can examine the details of the proofs by running the programs discussed above.

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