Another Estimate for A-Posteriori Validation of the Invariant Manifolds

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1 Linear Operator (Revisited)

The troubling estimate comes from the following Lemma:

Lemma 1 (Lemma 4.3 GS paper)**.** *The the linear operator $\mathfrak{L} : X_0 \to X_0$ defined by*

$$\mathfrak{L}(h) = Dh \circ \Lambda_s - Dg[P_N] \cdot h \tag{1}$$

is well defined and boundedly invertible as long as

$$N + 1 > \frac{C_1}{\mu}.$$

Supposing this is so, we have the bound

$$\|\mathfrak{L}^{-1}\|_{\mathcal{X}_0} \leq \frac{1}{(N+1)\mu - C_1}$$

The condition $N + 1 > C_1/\mu$ is essentially requiring us to take the order so high that there can be no possible resonance (as J.B. pointed out). I will call this an *a-priori condition* on the parameterization order, because it is something we have to check in order to begin the error analysis of the manifolds. The a-priori condition seems to be completely independent of the a-posteriori error ϵ_{tol}.

For the revised estimate I will assume that the vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ is polynomial. Recall the setup of the proof of the lemma. We define

$$A(t) = Dg[P_N(e^{\Lambda t}\theta)]$$

$$\bar{p}(\theta, t) = \bar{p}(t) = p(e^{\Lambda t}\theta)$$

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and

$$C(\theta, t) = e^{-\int_0^t A(\theta,s)\ ds}$$

The critical estimate is the estimate of $C(t)$. In the GS paper we obtain an elementary bound of the form

$$\|C(t)\|_{M,\nu} \leq e^{C_1 t}$$

where C_1 is a bound on $Dg[P_N(\theta)]$ over the poly disk of radius ν. This is the estimate we will try to improve.

Suppose that g is an M-th order polynomial. Then the entries of Dg are $M-1$-th order polynomials. Since P_N is an N-th order polynomial, we have that $Dg \circ P_N$ is an $\bar{N} = N(M+1)$-th order polynomial. Then let's write

$$Dg[P_N(\theta)] = \sum_{0 \leq |\alpha| \leq \bar{N}} A_\alpha \theta^\alpha$$

where each coefficient A_α is an $n \times n$ matrix. In fact,

$$A_\alpha^{ij} = \left[\partial_j g_i(z)|_{z=P_N(\theta)} \right]_\alpha$$

where g_i is the i-th component of the vector field and $[\cdot]_\alpha$ is the α-th coefficient of the power series $[\cdot]$. We will use that this is a finite sum.

Using this notation we have that

$$\begin{aligned}
A(t) = -\int_0^t \sum_{0 \leq |\alpha| \leq \bar{N}} A_\alpha (e^{\Lambda s}\theta)^\alpha \ ds &= -\int_0^t \sum_{0 \leq |\alpha| \leq \bar{N}} A_\alpha e^{<\Lambda,\alpha>s} \theta^\alpha \ ds \\
&= -\int_0^t A_{(0,\dots,0)} + \sum_{1 \leq |\alpha| \leq \bar{N}} A_\alpha e^{<\Lambda,\alpha>s} \theta^\alpha \ ds \\
&= -\int_0^t A_{(0,\dots,0)} \ ds - \int_0^t \sum_{1 \leq |\alpha| \leq \bar{N}} A_\alpha e^{<\Lambda,\alpha>s} \theta^\alpha \ ds \\
&= -A_0 t - \sum_{1 \leq |\alpha| \leq \bar{N}} A_\alpha \theta^\alpha \int_0^t e^{<\Lambda,\alpha>s} \ ds \\
&-A_0 t - \sum_{1 \leq |\alpha| \leq \bar{N}} \frac{A_\alpha}{|<\Lambda,\alpha>|} \left(1 - e^{<\Lambda,\alpha>t}\right) \theta^\alpha.
\end{aligned}$$

Note that the coefficients of Λ have negative real part, hence the reversal of the sign in the $e^{\cdot} - 1$ term and the absolute value in the denominator. Then

$$\begin{aligned}
|C(\theta,t)| &\leq |e^{A(\theta,t)}| \\
&= |\exp\left(-A_0 t - \sum_{1 \leq |\alpha| \leq \bar{N}} \frac{A_\alpha}{|<\Lambda,\alpha>|} \left(1 - e^{<\Lambda,\alpha>t}\right) \theta^\alpha \right)| \\
&\leq \|e^{-A_0 t}\| \exp\left(\sum_{1 \leq |\alpha| \leq \bar{N}} \frac{|A_\alpha|}{|<\Lambda,\alpha>|} |\nu|^{|\alpha|} \right)
\end{aligned}$$

as $|1 - e^{<\Lambda,\alpha>t}| \leq 1$ uniformly in α and t, due to the fact that Λ have negative real part. If we let C_3 be any number with

$$\exp\left(\sum_{1\leq|\alpha|\leq\bar{N}} \frac{|A_\alpha|}{|<\Lambda,\alpha>|}|\nu|^{|\alpha|}\right) \leq C_3$$

and C_1 be any number so that

$$\|e^{-A_0 t}\| \leq e^{C_1 t}$$

then we have

$$|C(\theta,t)| \leq C_3 e^{C_1 t}.$$

Then C_3 is a finite sum of known quantities, and can be computed numerically via interval arithmetic as long as there are no resonances up to order $\bar{N}$. Since we are using interval arithmetic the condition is self checking. To obtain C_1, note that A_0 is the leading coefficient of $Dg[P_N(\theta)]$, i.e.

$$A_0 = Dg[P_N(0)] = Dg(p).$$

To obtain and explicit expression for C_1 note that

$$e^{-A_0 t} = e^{-Dg(p)t} = Q\ e^{-\Omega t}\ Q^{-1},$$

where Ω is the diagonal matrix of eigenvalues (stable and unstable) of $Dg(p) = A_0$, and Q is the matrix of eigenvectors. Now if

$$\mu_+ = |\min(\mathrm{real}(\lambda))|,$$

where λ is an eigenvalue of A_0, then we have

$$\|e^{-A_0 t}\| \leq \|Q\|\,\|Q^{-1}\| e^{\mu_+ t},$$

by taking norms in the above equation.

Then the bound on the linear operator becomes becomes

$$|\mathfrak{L}^{-1}[p](\theta)| \leq \frac{C_3}{(N+1)\mu - \mu_+}$$

where we absorb the Q terms into C_3. In other words, C_3 is now any constant so that

$$\|Q\|\,\|Q^{-1}\|\ \exp\left(\sum_{1\leq|\alpha|\leq\bar{N}} \frac{|A_\alpha|}{|<\Lambda,\alpha>|}|\nu|^{|\alpha|}\right) \leq C_3,$$

(in the code I define $\|Q\|\|Q^{-1}\| = C_4$ and leave C_3 as defined several paragraphs back. Then there are C_3C_4 terms in the estimates). Then we will only have to insure that

$$N + 1 > \frac{\mu_+}{\mu},$$

in order that the operator $\mathfrak{L}$ is well defined. Let's call this the *a-priori* condition for the validation of the manifolds. Note that this is a spectral condition; i.e. $N+1$ has to be larger than the ratio of the largest to the smallest stable eigenvalue. This condition is natural, in the sense that it rules out any resonances in the coefficients of h. In other words, it does not makes any sense to try to define the tail for the approximation P_N unless we are sure that N is large enough insure that h is defined. The only thing that can prevent the existence of h is if the homological equation is not invertible at some order. Then the *a-posteriori* condition is exactly the one that insures there can be no resonances hiding in h.