USER GUIDE/DOCUMENTATION FOR ONE PARAMETER FAMILY OF (UN)STABLE MANIFOLD COMPUTATION IMPLEMENTATION

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1. Introduction. This note is a supplementary document for the paper [23]:

Polynomial Approximation of One Parameter Families of (Un)Stable Manifolds with Rigorous Computer Assisted Error Bounds

The note fills in technical details for the computations discussed in [23], especially those concerned with rigorous computation of analytic Taylor models. Implementations of all computations discussed in this note and in [23] are found at [24]. The main purpose of the present document is to explain our implementation of analytic Taylor models and operations on them, as discussed in [23]. Taylor models have been implemented by a number of other authors (see especially [18, 11, 10, 9, 4, 5]). However the necessary classes and functions do not exist in the IntLab/MatLab environment and we have implemented what we need for the present work. For thorough discussion of algorithms behind the IntLab library we refer to [28] and the references therein.

One warning is that our analytic Taylor library as implemented is not truly 'object oriented'. The polynomial part of the analytic Taylor model is stored as a vector of interval coefficients. The truncation errors and validated radii of convergence are stored separately. The user is expected to manage the variables correctly. This is usually accomplished via some naming scheme. The library could be substantially improved by combining the polynomial, radii, and truncation errors into a true analytic Taylor data structure under IntLab. Here the excellent and highly optimized IntLab polynomial class could be used. (In fact we sometimes use the IntLab polynomial class for polynomial multiplication and other manipulations, however we usually convert back to matrix containers after the manipulation is performed).

The implementation developed here is meant to serve only as a support package for the invariant manifold computations of [23]. Optimization for general use is beyond the scope of the present work. On the other hand we have designed the library with the philisophy of [23] in mind. In particular, all validated error bounds are obtained via analytic, a-posteriori arguments in function space. No topoligical arguments in phase space or on the graph of the functions are invoked.

The guide is organized as follows. Section 2 describes how to run the main programs which execute the example computations of [23]. In Section 3 we develop a set of tools for solving various kinds of equations using analytic Taylor models. Our aim is to make clear how tail bounds are obtained after the completion of various operations. This information is essential if the numerical implementation of the results in the main body of the paper are to be transparent and reproducible.

Section 3.1 discusses a simple result about products of Taylor models. This provides a baseline against which other analytic Taylor model operations can be compared. In Section 3.2 we show how to obtain an analytic Taylor model for the inverse of a matrix whose coefficients are analytic Taylor models, while in Section 3.3 we see that the composition of a Taylor model with an elementary function can be computed at the cost of a Cauchy Product, as long as the elementary function is given

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by the solution of some first order linear differential equation (powers, roots, exponentials, sines, cosines, etc). In Section 3.4 we discuss using analytic Taylor models to parameterize one parameter families of solutions of finite dimensional nonlinear equations.

In Secction 4 we show how the tools of Section (3) are used in order to obtain analytic Taylor model representations of the linear data needed in [23]. We discuss example calculations for the fixed points, eigenvalues, eigenvectors, diagonalizing transformations, and powers of the eigenvectors for the Hénon family. These are the essential quantities that feed into the algorithms for computing families of invariant manifolds. We also examine numerically the relationship between the order of the Taylor Model, the size of the radius of convergence, the tail error bound, and the computation time in some specific examples.

Some of the bounds required in the hypotheses of Theorems 4.8 and 4.9 of [23] require information about infinitely many terms of some power series of several variables. In Section 5 we discuss how these series are bound in practice using only the finite data available from the analytic Taylor models. We illustrate the computations and derive explicit estimates for the Hénon and Lorenz systems.

2. Starting the Library and Running the Programs (a very brief tutorial). All of the codes for the project [23] can be found in a compressed folder called

`oneParameterFamily_codes.zip`

located at [24]. The top directory contains the a startup file, a read me file, and all the scripts discussed in [23]. These scripts cary out the main computations of the paper. The top directory also contains sub-directories which constitute the analytic Taylor library, and the support programs for the invariant manifold computations.

A warning: absolutely all of the programs require that the IntLab package for interval arithmetic operations in MatLab has been installed. Anyone interested in running our codes should first obtain a copy of IntLab.

Once the file has been unzipped we suggest starting MatLab from the directory `oneParameterFamily_codes` (or starting MatLab and navigating to this directory). Once this has been done the next step is to initialize our library. At the MatLab command prompt simply type

```
>>  startFamilySession
```

This command will initialize all of our library pathways. It will also start IntLab, so if IntLab has not been installed the operation will fail at this juncture.

Once startup executes successfully the example computations are ready to be run. Now one simply types at the MatLab command prompt

```
 >> paperCodeName
```

where `paperCodeName` is one of the following file names:

1. `paperCodeEx1`
2. `paperCodeEx2`
3. `paperCodeEx3`
4. `paperCodeEx4`
5. `paperCode_Ex_lorenzRes`
6. `paperCode_henonBranchProof`
7. `paperCode_henonBranchProof_II`

8. `paperCode_henonBranchProof_III`
9. `paperCode_henonBranchProof_IV`
10. `paperCodePushProofs`
11. `validated2DLorenzBranch`

The programs run “loud”, producing substantial output to the screen. This is messy but gives one a sense of how long various parts of the computation are taking. A general use implementation should produce less output. The programs usually result in a several dozen variables being added to the workspace.

Another warning: all of the programs clear the workspace before execution. So one should not run them with important unsaved data in the work space.

Beyond these simple remarks it is hoped that the programs are relatively self documenting and easy to read. The names of functions are either self explanatory or conform to the notation of [23]. Anyone having questions, comments, or who locates bugs can reach the author at

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3. Operations on Analytic Taylor Models. Many of the convergence results to follow depend on the following standard theorem of nonlinear analysis.
We use the following standard theorem of non-linear analysis.

THEOREM 3.1 (Newton-Kantorovich Theorem). *Let X, Y be Banach spaces and $F : X \to Y$ be a differentiable mapping. Assume that there as an $\hat{x} \in X$ and an $r > 0$ such that*

(i) $DF(\hat{x})$ is boundedly invertible and

(ii) $\|DF(x) - DF(y)\|_{B(X,Y)} \leq \kappa \|x - y\|_X$ for all $x, y \in D_r(\hat{x})$.

If

(I)

$$\epsilon_{NK} \geq \|DF(\hat{x})^{-1} F(\hat{x})\|_Y,$$

(II)

$$\epsilon_{NK} \leq \frac{r}{2},$$

and

(III)

$$4\epsilon_{NK}\, \kappa \, \|DF(\hat{x})^{-1}\|_{B(X,Y)} \leq 1,$$

then the equation

$$F(x) = 0,$$

has a unique solution in $D_r(\hat{x})$.

(See [25] for an exposition of the proof in the language of English).

3.1. Products. We begin with the statement and proof of a simple Lemma which results in an analytic Taylor model of the product of two analytic Taylor models. The Lemma itself is almost trivial, but it will be instructive to compare the cost and accuracy of other operations to the cost and accuracy of a product. The Lemma also illustrates the *a-posteriori* philosophy in an simple setting.

LEMMA 3.2 (Product of Analytic Taylor Models). *Let (f_M, r, δ_f) and (g_M, r, δ_g) be two analytic Taylor models. Then an analytic Taylor model (p_M, r, δ_p) for the product $(f \cdot g)(\omega)$ is given by the M-th order polynomial p_M whose coefficients given by the Cauchy Product formula*

$$p_m = \sum_{k=0}^{M} a_{m-k} b_k, \tag{3.1}$$

(where a_k and b_k are the coefficient of f_M and g_M respectively). Moreover, defining the a-posteriori error

$$E_M(\omega) = f_M(\omega) g_M(\omega) - p_M(\omega),$$

we have the explicit bound

$$\delta_p \le \|E_M\|_r + \|f_M\|_r \delta_g + \|g_M\|_r \delta_f + \delta_f \delta_g. \tag{3.2}$$

Proof: That the coefficients of p_M are given by Equation (3.1) is just the standard Cauchy Product. We note that while we could obtain a bound on the product p simply by bounding $f \cdot g$, this does not provide an explicit truncation estimate for p_M. So we let $\hat{h}$,$\bar{h}$, and h denote the analytic M-tails of f, g, and p respectively. We have that

$$(f_M + \hat{h})(g_M + \bar{h}) = p_M + h.$$

From this we obtain the bound

$$\delta_p = \|h\|_r \le \|f_M g_M - p_M + f_M \bar{h} + g_M \hat{h} + \bar{h}\hat{h}\|_r,$$

from which Equation (3.2) follows.

□

The cost of the computation is the cost of a Cauchy Product, plus the cost of the evaluation of $\|E_M\|_r$, $\|f_M\|_r$ and $\|g_M\|_r$. Note that E_M is a $2M$-the order polynomial as this is the order of the product $f_M \cdot g_M$. However, because E_M is obtained by taking p_M from $f_M g_M$ and because the coefficients p_M are determined by the Cauchy Product, E_M will be almost zero to M-th order. (The low order terms of E_M capture the "round off errors" associated with computing the Cauchy Product coefficients). The cost of bounding the sup norms using the sigma norm is the cost of an inner product. Then computing (p_M, r, δ_p) is the cost of a Cauchy product, the cost of a polynomial multiplication, and the cost of three inner products. The bound on the truncation error of the product is the a-posteriori error plus terms proportional to the individual truncation errors of the products.

3.2. Matrix Inversion and Linear Equations. Consider a $K \times K$ matrix of analytic functions

$$B(\omega) = \begin{pmatrix} b_{11}(\omega) & \dots & b_{1K}(\omega) \\ \vdots & \ddots & \vdots \\ b_{K1}(\omega) & \dots & b_{KK}(\omega) \end{pmatrix}.$$

Suppose that each of the $b_{ij}(\omega)$ are analytic on the ball $B_r \subset \mathbb{C}$. Suppose further that associated with each b_{ij} is an analytic Taylor model $(b_{ij}^M, r, \delta_{ij})$. We define the matrix valued polynomial

$$B_M(\omega) = \sum_{m=0}^{M} B_m \omega^m,$$

with coefficients

$$B_m = \begin{pmatrix} b_{11}^m & \dots & b_{1K}^m \\ \vdots & \ddots & \vdots \\ b_{K1}^m & \dots & b_{KK}^m \end{pmatrix},$$

and truncation error with $\delta_B = K\, max_{ij}(\delta_{ij})$. Then we consider the data (B_M, r, δ_B) an analytic Taylor model for the matrix of functions B. Supposing that B is invertible at the origin, we are interested in developing an analytic Taylor model for the matrix inverse of B.

LEMMA 3.3 (Matrix Inversion). *Assume that $B(0) = B_0$ is invertible, that B_0^{-1} is an interval enclosure of its inverse, and that (B_M, r, δ_B) is an analytic Taylor model of B. Moreover assume that there are $M, \tau > 0$ so that*

$$|B_0^{-1}| \left(\tau \sum_{m=1}^{M} |B_m| \tau^{m-1} + \delta_B \right) \leq M < 1. \tag{3.3}$$

Then there is an M-th order analytic Taylor model (C_M, τ, δ_C) for $C(\omega) \equiv B^{-1}(\omega)$, where the coefficients of C_M are defined recursively by

$$C_0 = B_0^{-1}, \qquad \textit{and} \qquad C_m = -B_0^{-1} \sum_{k=0}^{m-1} B_{m-k} C_k \qquad \textit{for} \quad 1 \leq m \leq M. \tag{3.4}$$

Defining the a-posteriori error polynomial

$$E_M(\omega) = Id - B_M(\omega)\, C_M(\omega).$$

we have that the truncation error $\delta_C > 0$ satisfies the explicit bound

$$\delta_C \leq \frac{|B_0^{-1}|}{1 - M} \left(\|E_M\|_\tau + \|C_M\|_\tau \ \delta_B \right). \tag{3.5}$$

Proof: The first part of the proof is formal. We seek

$$C(\omega) = \sum_{m=0}^{\infty} C_m \omega^m,$$

so that

$$B(\omega) C(\omega) = \mathrm{Id}.$$

Expanding as series we have

$$\begin{aligned} B(\omega)C(\omega) &= \left(\sum_{m=0}^{\infty} B_m \omega^m \right) \left(\sum_{m=0}^{\infty} C_m \omega^m \right) \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{m} B_{m-k} C_k \omega^m \\ &= \mathrm{Id} + 0\omega + 0\omega^2 + \dots \end{aligned}$$

Matching like powers of ω we have for $m = 0$ that

$$B_0 C_0 = \mathrm{Id},$$

so that indeed $C_0 = B_0^{-1}$. Similarly when $m \geq 1$ we have that

$$\sum_{k=0}^{m} B_{m-k} C_k = 0,$$

which we solve for C_m in order to obtain that

$$C_m = -B_0^{-1} \sum_{k=0}^{m-1} B_{m-k} C_k,$$

as desired.

This formula is now used in order to compute the M-th order polynomial C_M. Let $G \colon B_r \to GL\left(\mathbb{C}^K\right)$ denote the truncation error associated with B. Then G is a $K \times K$ matrix of analytic M-tails with $\|G\|_r \leq \delta$ so that $B(\omega) = B_M(\omega) + G$. Now we seek a constant $\delta_C > 0$ and a $K \times K$ matrix of analytic M-tails $H \colon B_r \to L\left(\mathbb{C}^K\right)$ so that $\|H\|_\tau \leq \delta_C$ and

$$C(\omega) = C_M(\omega) + H(\omega).$$

Since C is the inverse of B, this last equation is equivalent to the condition

$$(B_M(\omega) + G(\omega))(C_M(\omega) + H(\omega)) = \mathrm{Id},$$

for each $\omega \in B_r$. Then formally we have that

$$H(\omega) = B^{-1}(\omega)\left[E_M(\omega) + G(\omega)\, C_M(\omega)\right]. \tag{3.6}$$

for each ω such that B is invertible. The Neuman series is used in order to obtain in fact that B is invertible on B_τ. Moreover we have the explicit bound

$$\begin{aligned}
\|B^{-1}\|_\tau &= \left\| \left(B_0 + \sum_{m=1}^{M} B_m \omega^m + G(\omega) \right)^{-1} \right\|_\tau \\
&\leq \left\| \left(\mathrm{Id} + B_0^{-1} \omega \sum_{m=1}^{M} B_m \omega^{m-1} + B_0^{-1} G(\omega) \right)^{-1} \right\|_\tau |B_0^{-1}| \\
&\leq \frac{|B_0^{-1}|}{1 - M},
\end{aligned} \tag{3.7}$$

where we use that

$$\left\| B_0^{-1} \omega \sum_{m=1}^{M} B_m \omega^{m-1} + B_0^{-1} G(\omega) \right\|_\tau \leq |B_0^{-1}| \left(\tau \sum_{m=1}^{M} |B_m| \tau^{m-1} + \delta_B \right) \leq M < 1,$$

by the hypothesis that Equation (3.3) holds. Applying the bound of Equation (3.7) to Equation (3.6) gives the bound on δ_C claimed in Equation (3.5)

□

Now if (B_M, r, δ_B) is an analytic Taylor model for an analytic matrix function $B(\omega)$ and (q_M, r, δ_q) is an analytic Taylor modle for a vector of analytic functions q, then we consider the equation $Bp = q$. The following Lemma shows that we can obtain an analytic Taylor model for p without first computing $B^{-1}(\omega)$ directly. The proof of the is almost identical the proof of the previous Lemma and is omitted.

LEMMA 3.4 (Solutions of Linear Equations). *Assume that $B(0) = B_0$ is invertible, that B_0^{-1} is an interval enclosure of its inverse and that (B_M, r, δ_B) is an analytic Taylor model of B. Let (q_M, r, δ_q) be an analytic Taylor model of the analytic function q. Assume in addition that there are $M, \tau > 0$ satisfying the bound given in Equation (3.3). Then there is an M-th order analytic Taylor model (p_M, τ, δ_p) for the analytic function p having*

$$B(\omega)p(\omega) = q(\omega) \qquad \textit{for} \qquad \omega \in B_\tau.$$

The coefficients for the polynomial p_M are defined recursively by

$$p_0 = B_0^{-1} q_0 \qquad \textit{and} \qquad p_m = B_0^{-1}\left(q_m - \sum_{k=0}^{m-1} B_{m-k} p_k\right), \tag{3.8}$$

and that truncation estimate satisfies

$$\delta_p \leq \frac{|B_0^{-1}|}{1-M}\left(\|E_M\|_\tau + \|p_M\|_\tau \ \delta_B + \delta_q\right),$$

where E_M is the a-posteriori error defined by

$$E_M(\omega) = q_M(\omega) - B_M(\omega)p_M(\omega).$$

3.3. Elementary Functions of Analytic Taylor Models. In this section we consider the problem of computing an analytic Taylor model for $F \circ f$ where f is an analytic Taylor model and F is an elementary function which solves a first order ode. The next Lemma addresses the case when F is a power function. Other elementary functions are similar. First an "error tail" lemma for powers.

LEMMA 3.5. *Suppose that $E\colon B_R \subset \mathbb{C} \to \mathbb{C}$ is an analytic M-tail, $K \in \mathbb{N}$ with $K > 1$, and that $f\colon B_R \to \mathbb{C}$ is an analytic function. Suppose in addition that $\|f\|_R \leq C$ and that*

$$|f(\omega)| \geq M \qquad \textit{for all} \qquad \omega \in B_R.$$

Then the differential equation

$$f(\omega)h'(\omega) - Kf'(\omega)h(\omega) = E(\omega),$$

has a unique solution $h\colon B_R \to \mathbb{C}$. Moreover h is an analytic M-tail with

$$\|h\|_R \leq R\left(\frac{C}{M}\right)^K \|E\|_R. \tag{3.9}$$

Proof: We note that since f^K is analytic and nonzero on $B_R(0)$, f^{-K} is analytic and bounded on B_R. Multiplying both sides of the equation by $f^{-K}(z)$ we obtain the equivalent equation

$$f^{-K}h' - Kf^{K-1}f'h = f^{-K}E,$$

or

$$\frac{d}{dz}\left(f^{-K}h\right) = f^{-K}E, \tag{3.10}$$

for any $z \in B_R(0)$. Since $B_R(0)$ is a convex neighborhood about the origin we have that the line segment between the origin and z is contained in z. We parameterize this line by $\gamma\colon [0,1] \to B_R(0)$ by the formula

$$\gamma(t) = tz.$$

Taking the line integral over γ on both sides of Equation (3.10) we have

$$\int_0^1 \frac{d}{dz}\left(f^{-K}[\gamma(t)]h[\gamma(t)]\right)\gamma'(t)\ dt = \int_0^1 f^{-K}[\gamma(t)]E[\gamma(t)]\gamma'(t)\ dt.$$

Since $B_R(0)$ is simply connected we have that the left hand side is

$$\begin{aligned}\int_0^1 \frac{d}{dz}\left(f^{-K}[\gamma(t)]h[\gamma(t)]\right)\gamma'(t)\ dt &= f^{-K}[\gamma(1)]h[\gamma(1)] - f^{-K}[\gamma(0)]h[\gamma(0)]\\ &= f^{-K}(z)h(z),\end{aligned}$$

as $\gamma(0) = 0$ and h is an analytic N-tail. Then

$$h(z) = f^K(z)\int_0^1 f^{-K}[\gamma(t)]\, E[\gamma(t)]\, \gamma'(t)\ dt,$$

and we note that h is an analytic N-tail due to the fact that E is. Now we bound

$$\begin{aligned}\sup_{|z|\le R}|h(z)| &\le \sup_{|z|\le R}\left|f^K(z)\right|\left|\int_0^1 f^{-K}[\gamma(t)]E[\gamma(t)]z\ dt,\right|\\ &\le C^K\frac{1}{M^K}\ \|E\|_R\ R,\end{aligned}$$

as desired.

$\square$

Lemma 3.6. *Suppose that (f_M, r, δ_f) is an analytic Taylor model for the analytic function f and $K \in \mathbb{R}$, $K \neq 0$. Denote the coefficients of the polynomial f_M by a_m for $0 \le m \le M$. Assume also that $f(0) = a_0 \neq 0$ and in fact that there are $M, \tau > 0$ so that*

$$|a_0| - \tau\sum_{m=1}^{M}|a_m|\tau^{m-1} - \delta_f \ge M > 0. \tag{3.11}$$

Then for any $0 < \sigma \leq 1$ an analytic Taylor model for $p(\omega) \equiv f^K(\omega)$ is given by (p_M, R, δ_p) where

$$R = \min\left(\tau, re^{-\sigma}\right),$$

the coefficients of p_M are defined recursively by

$$p_0 = a_0^K \qquad \text{and} \qquad p_m = \frac{1}{ma_0}\sum_{k=0}^{m-1}(mK - k(K+1))a_{m-k}p_k \quad \text{for } 1 \leq m \leq M, \tag{3.12}$$

and moreover we have an explicit bound on the truncation error of $f^K = p$ given by

$$\delta_p \leq \left(\frac{\|f_M\|_r + \delta_f}{M}\right)^K R\left(\|E_M\|_R + K\|p_M\|_R \frac{2\pi}{r\sigma}\delta_f + \|p'_M\|_R\delta_f\right), \tag{3.13}$$

where the a-posteriori error is defined by

$$E_M(\omega) = Kf'_M(\omega)p_M(\omega) - f_M(\omega)p'_M(\omega). \tag{3.14}$$

Proof: Of course we actually know that $p = f^K$ is analytic on the same disk B_r as f regardless of the magnitude of a_0. The additional constraints are imposed in order to obtain explicit bounds on the truncation error associated with the M-th order approximation of p while obtaining a computational cost proportional to the cost of a product.

The coefficients of p_M are computed formally as follows. Let p' denote the derivative of f^K. Then we have

$$p' = Kf^{K-1}f'.$$

Multiplying both sides by f gives

$$fp' = Kpf'. \tag{3.15}$$

Expanding f, f', p, and p' as power series (with the coefficients of p unknown), exploiting the Cauchy Product formula, matching like powers, and isolating the m-th coefficient of p leads to the recursion relations given in Equation (3.12).

The functional relation given by Equation (3.15) also leads to an effective a-posteriori analysis scheme for p. Let g denote the analytic M-tail so that $f(\omega) = f_M(\omega) + g(\omega)$ on B_r and $\|g\|_r \leq \delta_f$. We seek an analytic M-tail h defined on B_R so that $f^K(\omega) = p_M(\omega) + h(\omega)$ on B_R and $\|h\|_R \leq \delta_p$. Expanding Equation (3.15) gives the first order linear differential equation for h defined by

$$f(\omega)h'(\omega) - Kf'(\omega)h(\omega) = E(\omega), \tag{3.16}$$

where

$$E(\omega) = E_M(\omega) + K\,p_M(\omega)g'(\omega) - p'_M(\omega)g(\omega),$$

and $E_M(\omega)$ is the a-posteriori error given by Equation (3.14). The right hand side has the bound

$$\|E\|_R \leq \|E_M\|_R + \frac{2K\,\pi\|p_M\|_R}{r\sigma}\delta_f + \|p'_M\|_R\delta_f,$$

and the bound in Equation (3.13) is obtainde once we realize that Equation (3.16) has the form discussed in Lemma (3.5) so that the Estimate given by Equation (3.9) provides the needed bound on h.

$\square$

Remark 3.7. Similar Lemmas can be obtained for other elementary functions by utilizing that such functions can be expressed as solutions of linear differential equations. For example if we want to compute sin and cos of an analytic Taylor model then we define p by $p(\omega) \equiv e^{if(\omega)} = \sin(f(\omega)) + i\cos(f(\omega))$ and note that

$$p' = ipf'.$$

Again the coefficients can be computed for the cost of a Cauchy Product. Taking real and imaginary parts gives the sine and cosine series. The a-posteriori analysis of the truncations errors can be done by exploiting the differential equation.

3.4. One Parameter Branches of Zeros for Finite Dimensional Non-Linear Problems. For many applications we need to be able to compute an analytic Taylor model of a function p which parameterizes a branch of solutions of a nonlinear system of equations. Then suppose that $f\colon \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is a one parameter family of analytic maps, that $p_0 \in \mathbb{C}^n$ has $f(p_0, 0) = 0$, and that $D_1 f(p_0, 0)$ is invertible. (Here D_1 applied to $f(x, \omega)$ means the differential with respect to the 'first' variable, namely the variable x. Since $x \in \mathbb{C}^n$ $D_1 f$ is an $n \times n$ matrix of analytic functions. The entries of $D_1 f$ are functions in the variables $x \in \mathbb{C}^n$ and $\omega \in \mathbb{C}$). In addition we assume the existence of the following data.

Definition 3.1. [Validation Values for a One Parameter Branch of Zeros]

(1) Assume that B_0^{-1} is an interval inclosure of $D_1 f(p_0, 0)^{-1}$, and suppose that (B_M, r, δ_B) is an analytic Taylor model for

$$B(\omega) = D_1 f(p_M(\omega), \omega).$$

(2) Assume that there exist $M, \tau > 0$ having that $0 < \tau < r$ and

$$|B_0^{-1}| \left(\tau \sum_{m=1}^{M} |B_m| \tau^{m-1} + \delta_B \right) \leq M < 1.$$

Then Lemma (3.3) allows us to construct an analytic Taylor model (C_M, τ, δ_C) so that $C(\omega) = B^{-1}(\omega)$. In particular we have that

$$\|C\|_\tau = \sup_{\omega \in B_\tau} \left| [D_1 f(p_M(\omega), \omega)]^{-1} \right| \leq \|C_M\|_\tau + \delta_C.$$

(3) Assume that there is an $\epsilon > 0$ and an M-th order polynomial $p_M\colon B_\tau \subset \mathbb{C} \to \mathbb{C}^N$ having

$$|f\left(p_M(\omega), \omega\right)| < \epsilon \qquad \text{for all} \qquad \omega \in B_\tau.$$

Lemma 3.8 (A-Posteriori Validation of a Branch of Zeros). *Suppose that f, p_0, B_0^{-1}, B_M, τ, δ_B, ϵ, p_M, M, C_M, and δ_C are as in Definition (3.1). Let $\epsilon_{NK} > 0$ be any constant with*

$$(\|C_M\|_\tau + \delta_C)\, \epsilon \leq \epsilon_{NK}.$$

Define

$$R = 2\epsilon_{NK}.$$

Let $C = \sum_{m=1}^{M} |p_m|\tau^m + R$ *and define the set*

$$U = \{z \in \mathbb{C}^n \,:\, |z - p_0| \leq C\}.$$

Now let $\kappa > 0$ *have that*

$$n^2 \sup_{x \in U} \max_{1 \leq i \leq j \leq n} \|\partial_{ij} f(x)\| \leq \kappa.$$

Suppose that

$$4\epsilon_{NK}\kappa(\|C_M\|_\tau + \delta_C) < 1. \tag{3.17}$$

Then there is a unique analytic M*-tail* $h\colon B_\tau \to \mathbb{C}^N$ *with*

$$\|h\|_\tau \leq R,$$

so that $p(\omega) = p_M(\omega) + h(\omega)$ *is a one parameter analytic branch of zeros of* f*. In other words* (p_M, τ, R) *is an analytic Taylor model for the analytic function* $p\colon B_\tau \to \mathbb{C}^n$ *having*

$$f\left[p(\omega), \omega)\right] = 0 \quad \text{for all} \quad \omega \in B_\tau.$$

Proof: Let $\mathcal{X} = C^\omega(B_\tau \subset \mathbb{C}, \mathbb{C}^n)$ and define the operator $\Phi\colon \mathcal{X} \to \mathcal{X}$ by

$$\Phi[q](\omega) = f[q(\omega), \omega].$$

Note that the domain of Φ is a Banach Space under the C_0 norm. The Frechette Derivative of Φ has $D\Phi \in \mathfrak{L}(\mathcal{X}, \mathcal{X})$ and is given by

$$D\Phi[q](\omega) = D_1 f[q(\omega), \omega].$$

$B(\omega) = D_1 f[p_M(\omega), \omega] \equiv D\Phi[p_M]$ is invertible for all $\omega \in B_\tau$ by hypothesis. Moreover we have that

$$\|[D\Phi(p_M)]^{-1}\|_{B(\mathcal{X})} \leq \|C_M\|_\tau + \delta_C.$$

Then

$$\|[D\Phi(p_M)]^{-1}\Phi[p_M]\|_\tau \leq (\|C_M\|_\tau + \delta_C)\epsilon \leq \epsilon_{\mathrm{NK}}.$$

Let $V = \{v \in \mathcal{X} \,:\, \|v\|_\tau \leq R\}$, where we recall that $R = 2\epsilon_{\mathrm{NK}}$. Then for any $q \in p_M + U$ and $\omega \in B_\tau$ we have that

$$\|p_0 - (p_M(\omega) + q(\omega))\| \leq \sum_{m=1}^{M} |p_m|\tau^m + R = C.$$

From this we see that image$(p_M+q) \subset p_0+U \subset \mathbb{C}^n$. It follows that for any $q_1, q_2 \in V$ we have that

$$
\begin{aligned}
\|D\Phi[p_M+q_1] - D\Phi[p_M+q_2]\| &\leq \left(\sup_{q \in p_M+V} \|D^2\Phi[q]\|_\tau \right) \|q_1-q_2\|_\tau \\
&= \left(\sup_{q \in p_M+V,\ \omega \in B_\tau} \|D^2 f[p_M(\omega)+q(\omega),\omega]\right) \|q_1-q_2\|_\tau \\
&\leq \left(\sup_{x \in U,\ \omega \in B_\tau} \|D^2 f[x,\omega]\| \right) \|q_1-q_2\|_\tau \\
&\leq \kappa \|q_1-q_2\|_\tau,
\end{aligned}
$$

by the Mean-Value Theorem and the definition of κ. Recalling Equation (3.17), the Newton-Kantorovich Theorem applied to $\Phi[p_M](\omega)$ provides a unique $h \in V$ so that $\Phi[p_M+h](\omega) = f[p_M(\omega)+h(\omega),\omega] = 0$ for all $\omega \in B_\tau$.

□

4. Using Analytic Taylor Models to Satisfy A1-A3-maps-flows. Lemma (3.8) can be applied directly in order to validate analytic Taylor models for one parameter families of equilibria. Since fixed points of diffeomorphisms can be expressed as zeros of some equations, Lemma (3.8) can also be used to validate analytic Taylor models for families of fixed points. Similarily for one parameter branches eigenvalues and eigenvectors, so that Lemma (3.8) can be used in order to validate polynomial expnasions of all the linear data. We consider several examples.

4.1. A One Parameter Branch of Fixed Points for the Hénon Map. Consider the one parameter family of Hénon mappings defined by

$$f(x,y,\omega) = \begin{bmatrix} y+1-ax^2 \\ (b+\omega)x \end{bmatrix}, \tag{4.1}$$

where we think of a and b as fixed. We begin by developing a formal expansion for a branch of fixed points for the family. Let

$$x(\omega) = \sum_{n=0}^{\infty} x_n \omega^n,$$

parameterize an analytic branch of the first component of a fixed point of Equation (4.1). Then $x(\omega)$ solves

$$a[x(\omega)]^2 + x(\omega)(1-b-\omega) - 1 = 0. \tag{4.2}$$

From this we see that

$$x_0 = \frac{b-1 \pm \sqrt{(1-b)^2+4a}}{2a}, \qquad \text{and} \qquad x_1 = \frac{d}{d\omega}x(0) = \frac{x_0}{2ax_0-b+1}. \tag{4.3}$$

Matching like powers of ω in equation 4.2 gives that

$$x_n = \frac{1}{2ax_0-b+1}\left[x_{n-1} - \sum_{k=1}^{n-1} a\, x_{n-k}x_k\right]. \quad \text{for} \quad n \geq 2. \tag{4.4}$$

M	τ	δ_{p_1}	δ_{p_2}	r_1	r_2	t
2	10^{-4}	1.66×10^{-13}	1.75×10^{-13}	6.91×10^{-14}	6.92×10^{-15}	0.31(sec)
2	10^{-2}	1.67×10^{-7}	1.73×10^{-7}	6.89×10^{-8}	6.90×10^{-8}	0.31(sec)
2	0.23	0.0024	0.0028	8.57×10^{-4}	8.57×10^{-4}	0.3(sec)
5	0.23	3.47×10^{-7}	2.85×10^{-7}	1.13×10^{-7}	1.13×10^{-7}	0.71(sec)
10	0.23	8.51×10^{-13}	5.89×10^{-13}	2.45×10^{-13}	2.45×10^{-13}	1.7(sec)
15	0.23	4.73×10^{-15}	7.33×10^{-15}	7.77×10^{-16}	8.88×10^{-16}	3.1(sec)

Table 4.1

Fixed Point Branch Performance Data for the Hénon Family: *M is the parameterization order, τ is the radius of the domain of the analytic Taylor model, i.e. each model is validated for the real interval $\omega \in [-\tau, \tau]$. We compute models of a branch of fixed points for both p_1 and p_2. The associated truncation errors (the δ values) are given for each model. The columns labeled r_1 and r_2 are qualitative assessments of the error. For each branch we evaluate the polynomial at $\omega = \pm\tau$. We include the truncation errors into the interval results. We compare this to values of the fixed points given by the explicit formulas. r_1 is the maximum error over $\pm\tau$ for the first fixed point and similarly for r_2. Then r_1 and r_2 represent the observed error, while the δ's give theoretical bounds on the error. Note that the r's are always smaller than the δ's. The computation time for each fixed point branch is given as well. We note that the proof fails for $\tau = 0.23$ due to loss of control of the bounds on the norm of the inverse of the differential. For $\tau = 0.23$ the accuracy is not noticeably increased by computing to higher order than fifteen.*

We note that since the second component of the fixed point is given by $y(\omega) = (b+\omega)x(\omega)$ we now have

$$y_0 = bx_0, \qquad y_1 = bx_1 + x_0 \qquad \text{and} \qquad y_n = bx_n + x_{n-1} \qquad n \geq 2. \tag{4.5}$$

We write $p_0(\omega)$ to denote the branch of fixed points where we take the positive sign in Equation (4.3) and $p_1(\omega)$ to denote the branch where the minus is chosen.

These recursion relations can be used to define a polynomial approximation

$$p_M(\omega) = \sum_{m=0}^{M} \begin{bmatrix} x_m \\ y_m \end{bmatrix} \omega^m,$$

of a branch of fixed points for this Hénon family to any desired finite order M. Then Lemma (3.8) can be applied in order to validate a branch of zeros of the map

$$F(p_M(\omega), \omega) = f[p_M(\omega), \omega] - p_M(\omega).$$

An analytic Taylor model for a branch of zeros of F is a model of a branch of fixed points of the Hénon family. This calculation is carried out by the program `paperCodeEx1.m` which can be found at [24]. Performance results for several program parameters at the classic values of $a = 1.4$, $b = 0.3$ are given in Table (4.1).

4.2. One Parameter Family of Eigenvalues and Eigenvectors for a Fixed Point of the Hénon Map. We now consider the eigenvalue problem at a fixed point of the Hénon family. If λ_0 is an eigenvalue of $D_{(x,y)}f(x, y, 0)$ then we seek

$$\lambda(\omega) = \sum_{n=0}^{\infty} \lambda_n \omega^n,$$

a branch of eigenvalues passing through λ_0. Then $\lambda(\omega)$ satisfies the equation

$$\lambda(\omega)^2 + 2a\, x(\omega)\, \lambda(\omega) - \omega - b = 0, \tag{4.6}$$

with $\lambda(0) = \lambda_0$. We have that

$$\lambda_0 = -ax_0 \pm \sqrt{a^2x_0^2 + b}, \qquad \lambda_1 = \frac{1 - 2ax_1\lambda_0}{2\lambda_0 + 2ax_0}, \tag{4.7}$$

and

$$\lambda_n = \frac{-1}{2\lambda_0 + 2ax_0}\left(\sum_{k=1}^{n-1}\lambda_{n-k}\lambda_k + \sum_{k=0}^{n-1}2ax_{n-k}\lambda_k\right), \qquad \text{with} \qquad n \geq 2. \tag{4.8}$$

Then the λ_n are formally well defined as long as $\lambda_0 \neq -ax_0$, i.e. as long as λ_0 is not a repeated eigenvalue. Also note that the coefficient λ_n depends on the coefficients of x_i of $x(\omega)$ but only for $0 \leq i \leq n$. Then if we want to compute $\lambda(\omega)$ to order M we need only compute $x(\omega)$ up to order M. Now we choose an eigenvector ξ_0 with $\|\xi\|^2 = \hat{K}$ for some $\hat{K} > 0$, associated with the eigenvalue λ_0. Denote by

$$\xi(\omega) = \sum_{n=0}^{\infty}\xi_n\omega^n,$$

a parameterizatoin of the branch of eigenvectors through ξ_0, where the entire branch is normalized to have length $\sqrt{\hat{K}}$. Then $\xi(\omega)$ satisfies the system of nonlinear equations

$$\begin{bmatrix} -2ax(\omega) - \lambda(\omega) & 1 \\ b + \omega & -\lambda(\omega) \end{bmatrix}\begin{pmatrix} \xi_1(\omega) \\ \xi_2(\omega) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\xi_1(\omega)^2 + \xi_2(\omega)^2 = \hat{K},$$

but since the matrix has one dimensional kernel, we drop the first row of the matrix and have that $\xi(\omega)$ solves

$$\begin{pmatrix} (b+\omega)\xi_1(\omega) - \lambda(\omega)\xi_2(\omega) \\ \xi_1(\omega)^2 + \xi_2(\omega)^2 - \hat{K} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.9}$$

Matching like powers leads to

$$\begin{bmatrix} b & -\lambda_0 \\ 2\xi_0^1 & 2\xi_0^2 \end{bmatrix}\begin{pmatrix} \xi_1^1(\omega) \\ \xi_1^2(\omega) \end{pmatrix} = \begin{pmatrix} \lambda_1\xi_0^2 - \xi_0^1 \\ 0 \end{pmatrix},$$

for the coefficient ξ_1 and

$$\begin{bmatrix} b & -\lambda_0 \\ 2\xi_0^1 & 2\xi_0^2 \end{bmatrix}\begin{pmatrix} \xi_n^1(\omega) \\ \xi_n^2(\omega) \end{pmatrix} = \begin{pmatrix} -\xi_{n-1}^1 + \sum_{k=0}^{n-1}\lambda_{n-k}\xi_k^2 \\ -\sum_{k=1}^{n-1}\xi_{n-k}^1\xi_k^1 + \xi_{n-k}^2\xi_k^2 \end{pmatrix}, \tag{4.10}$$

for ξ_n when $n \geq 2$. The coefficient ξ_n depends recursively on the coefficients of $\lambda(\omega)$ to n-th order.

Now suppose that we use the recursion relations just described in order to compute M-th order polynomial approximations $\lambda_M(\omega)$ and $\xi_M(\omega)$ for a branch of eigenvalues and eigenvectors for Hénon. We need to approximate the truncation error associated with these polynomial approximations in order to obtain rigorous analytic Taylor models. To do this we simply define the maps $F_{\text{eigenvalue}}\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $F_{\text{eigenvector}}\colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$ by

$$F_{\text{eigenvalue}}(\lambda, \omega) = \lambda^2 + 2ax(\omega)\lambda - \omega - b,$$

and

$$F_{\text{eigenvector}} = \begin{pmatrix} (b+\omega)\xi_1 - \lambda(\omega)\xi_2 \\ \xi_1^2 + \xi_2^2 - \hat{K} \end{pmatrix}.$$

Since $F_{\text{eigenvalue}}(\lambda_M(\omega), \omega)$ and $F_{\text{eigenvector}}(\xi_M(\omega), \omega)$ are approximately zero we again use Lemma (3.8) in order to obtain rigorous rigorous bounds of the truncation errors for the eigendata.

Note the first component of the branch of fixed points $x(\omega)$ in the definition of $F_{\text{eigenvalue}}$ and the branch of eigenvalues $\lambda(\omega)$ in the definition of $F_{\text{eigenvectors}}$ are only known up to analytic Taylor approximation. More precisely let (x_M, τ, δ_x) be the analytic Taylor model for the first component of the fixed point branch, and $(\lambda_M, \tau, \delta_\lambda)$ be the analytic Taylor model for the branch of eigenvalues through λ_0. The Newton-Kanrorivich a-posteriori errors have

$$\|F_{\text{eigenvalue}}(\lambda_M(\omega), \omega)\|_\tau \leq \|\lambda_M(\omega)^2 + 2a\, x_M(\omega)\lambda_M(\omega) - \omega - b\|_\tau + 2a\|\lambda_M\|_\tau \delta_x,$$

and

$$\|F_{\text{eigenvector}}(\xi_M(\omega), \omega)\|_\tau \leq \left\| \begin{pmatrix} b\xi_M^1(\omega) + \omega\xi_M^1(\omega) - \lambda_M(\omega)\xi_M^2(\omega) \\ \left[\xi_M^1(\omega)\right]^2 + \left[\xi_M^2(\omega)\right]^2 - \hat{K} \end{pmatrix} \right\|_\tau + \|\xi_M^2\|_\tau \delta_\lambda,$$

where in each case the first term on the right depends only on multiplication of known M-th order polynomials.

Finally, in both cases we must provide analytic Taylor models for the differentials as these are functions of models themselves. For example if $h(\omega)$ is the truncation error associated with the analytic Taylor model (x_M, τ, δ_x) then we have

$$D_1 F_{\text{eigenvalue}}(\lambda_M(\omega), \omega) = 2\lambda_M(\omega) + 2a\, x(\omega) + 2a\delta_x.$$

So that $(2\lambda_M + 2a\xi_M^1, \tau, 2|a|\delta_x)$ is an analytic Taylor model for the differential. Similarly if $h_\lambda(\omega)$ is the truncation error associated with the analytic Taylor model $(\lambda_M, \tau, \delta_\lambda)$ we have that

$$D_1 F_{\text{eigenvector}}(\xi_M(\omega), \omega) = \sum_{m=0}^{M} B_m \omega^m \quad + \quad \begin{pmatrix} 0 & -h_\lambda(\omega) \\ 0 & 0 \end{pmatrix},$$

where

$$B_0 = \begin{pmatrix} b & -\lambda_0 \\ 2\xi_0^1 & 2\xi_0^2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & -\lambda_1 \\ 2\xi_1^1 & 2\xi_1^2 \end{pmatrix}, \text{and} \quad B_m = \begin{pmatrix} 0 & -\lambda_m \\ 2\xi_m^1 & 2\xi_m^2 \end{pmatrix},$$

so that $(B_M, \tau, \delta_\lambda)$ is an analytic Taylor model for $D_1 F_{\text{eigenvectors}}(\xi_M(\omega), \omega)$. Then we can compute analytic Taylor models for

$$[D_1 F_{\text{eigenvalues}}(\lambda_M(\omega), \omega)]^{-1}, \quad \text{and} \quad [D_1 F_{\text{eigenvectors}}(\xi_M(\omega), \omega)]^{-1},$$

using Lemma (3.3). Once this is done we have all the ingredients needed to apply Lemma(3.8). An implementation of these computations can be found in `paperCodeEx2.m` (see [24]). Some performance data is recorded in Table (4.2).

M	τ_1	τ_2	$\delta_{\lambda_1,\lambda_2}$	$\delta_{\lambda_3,\lambda_4}$	δ_{ξ_1,ξ_2}	δ_{ξ_3,ξ_4}	time
5	0.001	0.001	2.5×10^{-14}	3.7×10^{-14}	1.3×10^{-14}	7.9×10^{-15}	3.4 (sec)
5	0.13	0.14	1.5×10^{-6}	1.4×10^{-7}	1.9×10^{-5}	3.5×10^{-7}	3.4 (sec)
10	0.13	0.14	8.92×10^{-11}	9.7×10^{-14}	1.2×10^{-9}	2.9×10^{-14}	8.8 (sec)
25	0.13	0.14	4.4×10^{-14}	6.4×10^{-14}	5.3×10^{-14}	3.8×10^{-14}	43 (sec)

Table 4.2

Branch of Eigenvalues/vectors Performance Data for the Hénon Family: *M is the parameterization order, τ_1 and τ_2 are the parameterization domains for the branches associated with fixed points one and two respectively. The eigenvalues and eigenvectors associated with fixed point one are subscripted one and two, while the eigenvalues and eigenvectors associated with fixed point two are subscripted three and four. $\delta_{\lambda_1,\lambda_2}$ is the maximum truncation error over the two eigenvalues and similarly for the remaining deltas.*

4.3. One Parameter Families of Powers of the Eigenvalues for Hénon. We now consider powers of the analytic Taylor models of the one parameter expansions of the eigenvalues computed in the previous section. Consider for example the stable eigenvalue associated with p_1 for the Hénon map with $a = 1.4$ and $b = 0.3$. Recall that the stable eigenvalue associated with the fixed point p_1 has

$$\lambda_s \in [0.155946322302793, 0.155946322302794].$$

We begin by computing an analytic Taylor model for the one parameter branch of eigenvalues through λ_s. We take a model $(\lambda_s^M(\omega), \tau, \delta)$ for the eigenvalue branch with $M = 10$, $\tau = 0.1$, and $\delta = 4.8 \times 10^{-12}$. Using Lemma (3.6) we compute an analytic Taylor model for the fifth power of $\lambda_s(\omega)$ with $M = 10$, $\bar{\tau} = 0.995$, and validated error $\delta_5 = 1.6 \times 10^{-10}$. Here we choose, somewhat arbitrarily, a loss of domain parameter $\sigma = 0.005$ in order to apply Lemma (3.6). An analytic Taylor model for the twelfth power of $\lambda_s(\omega)$ with the same loss of domain parameter has $\delta_{12} = 7.4 \times 10^{-13}$ while the analytic Taylor model for the twentieth and thirtith powers have $\delta_{20} = 6.2 \times 10^{-15}$ and $\delta_{30} = \times 1.9 \times 10^{-16}$. This decay in the truncation error is due to the fact that we are working with the expansion of an eigenvalue whose norm is less than one. These computations are implemented in the program `paperCodeEx3.m` found at [24].

Suppose instead we work with the unstable eigenvalue associated with p_1, which we recall has

$$\lambda_u \in [-1.923738858153409, -1.923738858153407],$$

and compute an analytic Taylor model for $\lambda_u(\omega)$ with $M = 10$, $\tau = 0.1$, and $\delta = 4.8 \times 10^{-12}$. Now we compute analytic Taylor models for say the second, fifth, tenth and twentieth powers of $\lambda_u(\omega)$ and obtain $\delta_2 = 5.1 \times 10^{-8}$, $\delta_5 = 1.1 \times 10^{-6}$, $\delta_1 0 = 7.5 \times 10^{-5}$, and $\delta_{20} = 0.299$. We see that the truncation errors don't decay, but rather grow when the constant term of the original analytic Taylor model is greater than one. This is not surprising when we observe that the same "wrapping" phenomenon occurs when simply computing powers of intervals.

Fortunately when we validate the errors of stable/unstable manifolds for maps using the techniques of [23] we never need to compute validated bounds on powers of analytic functions whose constant terms have absolute value greater than one. The reason for this is that we validate a polynomial expansion for the unstable manifold of a diffeomorphism f by treating it as the stable manifold of the inverse map f^{-1}. For example when we validate the stable manifold of the inverse of the Hénon (i.e. the unstable manifold of Hénon) then instead of computing analytic Taylor models for the powers $\lambda_u^n(\omega)$ we compute analytic Taylor models for the function $\lambda_u^{-1}(\omega)$ (which of course has constant term with absolute value less than one) and consider powers of this. We recall that an analytic Taylor model can be computed for $\lambda_u^{-1}(\omega)$ by utilizing Lemma (3.3) and the fact that a number can be thought of as a 1×1 matrix and the reciprocal as the matrix inverse. Using this scheme we obtain results for the powers of the reciprocal of the unstable eigenvalues which are as good as the results above for powers of the stable eigenvalues, i.e. we have the the truncation errors decay as a function of the powers of n.

Note also that for differential equations we can have stable or unstable eigenvalues with absolute value greater than one because for differential equations stability is determined instead by the sign of the real part of the eigenvalue. However for differential equations we are do not have to consider powers of eigenvalues at all (rather we must only contend with linear combinations of eigenvalues) so this particular problem does not arise at all.

4.4. Resonance Conditions/Bounds for the Lorenz System. Consider the Lorenz System with parameter values $\sigma = 10$, $\beta = 8/3$, and $\rho = 13.9265$ (parameters close to the classical homoclinic tangency). Using IntLab we compute the eigenvalue enclosures

$$\begin{aligned}
\lambda_1 &\in B\left(-18.12992478204046, 3.56 \times 10^{-15}\right), \\
\lambda_2 &\in B\left(-2.66666666666666, 4.45 \times 10^{-16}\right), \\
\lambda_3 &\in B\left(7.12992478204047, 6.22 \times 10^{-15}\right),
\end{aligned}$$

which are clearly real and distinct. Considering only the stable eigenvalues we take

$$\mu_* = 2.6 < \min_{1 \le i \le 2} |\mathrm{real}(\lambda_i)| \qquad \text{and} \qquad \mu^* = 18.13 > \max_{1 \le i \le 2} |\mathrm{real}(\lambda_i)|.$$

We can check that $\lceil \mu^*/\mu_* \rceil = 7$. Then if $n_1 + n_2 > 7$, we have that

$$n_1\lambda_1 + n_2\lambda_2 < -n_1\mu_* - n_2\mu_* < -7\mu_* < -\mu^* < \lambda_1 < \lambda_2,$$

which shows explicitly that $n_1 + n_2 > 7$ implies that $n_1\lambda_1 + n_2\lambda_2 \neq \lambda_i$ for $1 = 1, 2$ and we conclude there are no possible resonances with for multi-indices of order greater or equal to 7. What remains is to check the 33 remaining non-resonance conditions of the form

$$b_{(n_1,n_2)} = \min_{1 \le i \le 2} |n_1\lambda_1 + n_2\lambda_2 - \lambda_i| > 0.$$

with $2 \le n_1 + n_2 \le 7$. We compute $b_{(n_1,n_2)}$ using interval arithmetic and check that the resulting interval does not contain zero. We tabulate the results and find the the closest the system ever comes to resonance is when $(n_1, n_2) = (0, 7)$, in which case

$$|0\lambda_1 + 7\lambda_2 - \lambda_1| \in [0.53674188462619, 0.53674188462621].$$

This tells us that when we compute the analytic Taylor models which parameterize the branches of stable eigenvalues, we have to take care with the equation

$$7\lambda_2(\omega) - \lambda_1(\omega) = 0.$$

We compute the analytic Taylor models for λ_i, $i = 1, 2$ to order 12, and check the resonance bounds for all $2 \leq n_1 + n_2 \leq 7$. For the multi-indices with $(n_1, n_2) \neq (0, 7)$ we find that for τ as large as 1.3 we have that $b_{(n_1,n_2)}(\tau) > 1$. The difficult multi-index is $(0, 7)$ where we only have $b_{(0,7)}(1.3) > 0.01$. However, we also report that when $\tau = 1.4$ we cannot guarantee that $b_{(0,7)}(1.4) > 0$ using interval arithmetic and the bounds given by Equation (**??**). On the other hand if we take $\tau \leq 0.5$ we have $b_{(n_1,n_2)}(0.5) > 1$ for all $2 \leq n_1 + n_2 \leq 7$.

Remark 4.1 (Range Bounding Using Interval Arithmetic). Since the resonance conditions always involve bounding functions of one variable away from zero we could replace the arguments above with more sophisticated range bounding methods. This could be especially useful for any multi-indices where there is a near resonance in the $\omega = 0$ equation, such as the multi-index $(0, 7)$ for the homoclinic tangency parameters in Lorenz discussed above. See [26] for a more sophisticated treatment of techniques for obtaining range bounds using interval arithmetic. For the present work Equation (**??**) is sufficient.

5. Analytic Taylor Model Bounds for the Error Term E_N and Differential Term $Df[P_N]$ for Hénon and Lorenz. We will illustrate how the a-posteriori error estimates and the estimates on the differential terms required by the hypotheses of Theorems 4.8 and 4.9 [23] are bound in practice. In particular we develop explicit formulas for the Hénon and Lorenz systems. We note that even though both the example systems are low dimensional we attempt to proceed in a general manner. The fact that the Hénon and Lorenz systems are quadratic will hold the technical difficulties to a minimum. Nevertheless the following examples illustrate the general procedure.

5.1. Example: The Henon Map. We will focus on the stable manifold computation. The unstable is the stable manifold of f^{-1}, and the formulas and bounds are similar. Suppose that

$$\begin{aligned} P_N(\theta, \omega) &= \sum_{n=0}^{N} a_n(\omega)\theta^n \\ &= P_{MN}(\theta, \omega) + H_N(\theta, \omega) \\ &= \sum_{n=0}^{N} \sum_{m=0}^{M} a_{(n,m)}\omega^m \theta^n + \sum_{n=0}^{N} h_n(\omega)\theta^n \end{aligned}$$

with $\|h_n\|_\tau \leq \delta_n$ for $0 \leq n \leq N$ is the validated N-th order formal approximation to a one parameter family of stable manifolds for the Hénon Maps as discussed in Section 4.5 of [23]. In order to apply Theorem 4.8 to P_N we have to choose a validation domain $D_\nu \subset \mathbb{C}$ and obtain bounds on both the a-posteriori error term

$$E_N(\theta, \omega) = f[P_N(\theta, \omega)] - P_N[\lambda(\omega)\theta, \omega]$$

and the inverse of the differential term

$$A(\theta, \omega) = Df[P_N(\theta, \omega), \omega]^{-1}.$$

The complication is that we know the one parameter branch for the stable eigenvalue $\lambda(\omega)$, and the one parameter branches of coefficients $a_n(\omega)$ only up to analytic Taylor Model approximation. That is, the data that we know explicitly are the analytic

Taylor models

$$\lambda(\omega) = (\lambda_M(\omega), \tau, \delta_\lambda)\,,$$

$$\lambda^n(\omega) = (\lambda^n_M(\omega), \tau, \delta_{\lambda^n})\,, \qquad \text{for} \quad 2 \le n \le N,$$

and

$$a_n(\omega) = \left(a^M_n(\omega), \tau, \delta_n\right) \qquad \text{for} \quad 0 \le n \le N.$$

Given a $\nu > 0$, our goal is to obtain computable bounds on $\|E_N\|_{\nu,\tau}$ and $\|A\|_{\nu,\tau}$ in terms of the known analytic Taylor models. Define

$$\delta_N = \sum_{n=0}^{N} \delta_n \nu^n.$$

We note that for Hénon, the matrix $Df[x, y, \omega]$ is a 2×2 matrix of functions of three variables. We could develop an algorithm and validation theorem similar to those of Section (3.2) for inverting matrices of functions of several variables, however we find that this is usually unnecessary for problems where explicit formulas for f, f^{-1}, their differentials and the inverses of their differential are known.

For Hénon we have that

$$Df(x, y, \omega)^{-1} = \frac{1}{b+\omega}\begin{pmatrix} 0 & 1 \\ b+\omega & 2ax \end{pmatrix}.$$

Then

$$\|Df[P_N]^{-1}\|_{\nu,\tau} \le 1 + \frac{2|a|}{b-\tau}\|P_N\|_{\nu,\tau},$$

where

$$\|P_N\|_{\nu,\tau} \le \sum_{n=0}^{N}\sum_{m=0}^{M} |a_{(n,m)}|\nu^n\tau^m + \delta_N,$$

a term which can be computed numerically. We must also require that $0 < \tau < b$. This gives a bound on the inverse of the differential in terms of know quantities.

For the a-posteriori error consider

$$E_N(\theta, \omega) = f[P_{MN} + H_N](\theta, \omega) - [P_{MN} + H_N](\lambda(\omega)\theta, \omega).$$

The first term on the right hand side can be expressed explicitly in terms of the known formula for the Hénon mapping. We see that

$$\begin{aligned} f[P_{MN} + H_N](\theta, \omega) &= \left[\begin{array}{c} 1 + P^2_{MN} + H^2_N - a(P^1_{MN})^2 - 2aP^1_{MN}H^1_N - a(H^1_N)^2 \\ bP^1_{MN} + bH^1_N \end{array}\right](\theta, \omega) \\ &= f[P_{MN}(\theta, \omega)] + \left[\begin{array}{c} H^2_N - 2aP^1_{MN}H^1_N - a(H^1_N)^2 \\ bH^1_N \end{array}\right](\theta, \omega). \end{aligned}$$

For the second term on the right we proceed more generally. Consider

$$[P_{MN} + H_N](\lambda(\omega)\theta, \omega) = P_{MN}\left[(\lambda(\omega)\theta, \omega)\right] + H_N\left[\lambda(\omega)\theta, \omega\right]$$

$$= \sum_{n=0}^{N} \sum_{m=0}^{M} a_{(m,n)} \omega^m \lambda^n(\omega)\ \theta^n + \sum_{n=0}^{N} \sum_{m=M+1}^{\infty} a_{(m,n)} \omega^m \lambda^n(\omega)\ \theta^n.$$

Using the analytic Taylor models of the powers of $\lambda(\omega)$ gives that

$$\sum_{n=0}^{N} \sum_{m=0}^{M} a_{(m,n)} \omega^m \lambda^n(\omega)\ \theta^n = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{(m,n)} \omega^m \left[\lambda_M^n(\omega) + h_{\lambda^n}(\omega)\right]\ \theta^n$$

$$= \sum_{n=0}^{N} \left(\sum_{m=0}^{M} \lambda_m^n \omega^m \right) \left(\sum_{m=0}^{M} a_{(m,n)} \omega^m \right) \theta^n + \sum_{n=0}^{N} \sum_{m=0}^{M} h_{\lambda^n}(\omega) a_{(m,n)} \omega^m \theta^n.$$

Define

$$(P \circ \lambda)_{MN}(\omega, \theta) \equiv \sum_{n=0}^{N} \left(\sum_{m=0}^{M} \lambda_m^n \omega^m \right) \left(\sum_{m=0}^{M} a_{(m,n)} \omega^m \right) \theta^n,$$

and note that this is $2M$-th order polynomial in ω with explicitly known coefficients. On the other hand, the error bounds on the coefficient functions give that

$$\sum_{n=0}^{N} \sum_{m=M+1}^{\infty} a_{(m,n)} \omega^m \lambda^n(\omega)\ \theta^n = \sum_{n=0}^{N} \lambda^n(\omega) h_n(\omega) \theta^n.$$

Let

$$E_{MN}(\theta, \omega) = f[P_{MN}(\theta, \omega), \omega] - (P \circ \lambda)_{MN}[\theta, \omega].$$

We note that this is a composition of only known polynomials and we can numerically bound the quantity $\|E_{MN}\|_{\nu,\tau}$ using the usual sigma norms (the resulting sums are finite). Let ϵ_{MN} be any numerical bound so obtained. We have proven the following Lemma.

Lemma 5.1 (Total A-Posteriori Error for Hénon). *The the validation value a-posteriori error E_N for the Hénon mapping satisfies the following bound;*

$$\|E_N\|_{\nu,\tau} \leq \epsilon_{NM} + \max(\delta_N + 2|a|\|P_{MN}\|_{\nu,\tau}\delta_N + |a|\delta_N^2, |b|\delta_N) + \sum_{n=0}^{N} \delta_{\lambda^n} \sum_{m=0}^{M} |a_{(m,n)}| \tau^m \nu^n + \sum_{n=0}^{N} \delta_n (\mu^*)^n \nu^n.$$

5.2. Example: The Lorenz System. This time suppose that

$$\begin{aligned} P_N(\theta, \omega) &= \sum_{|\alpha|=0}^{N} a_\alpha(\omega) \theta^\alpha \\ &= P_{MN}(\theta, \omega) + H_N(\theta, \omega) \\ &= \sum_{|\alpha|=0}^{N} \sum_{m=0}^{M} a_{(\alpha,m)} \omega^m \theta^\alpha + \sum_{|\alpha|=0}^{N} h_\alpha(\omega) \theta^\alpha \end{aligned}$$

with $\|h_\alpha\|_\tau \le \delta_\alpha$ for $0 \le |\alpha| \le N$ is the validated N-th order formal approximation to a one parameter family of two dimensional stable manifolds at the origin of the Lorenz system as discussed in Section 4.5 of [23]. We have analytic Taylor models of the stable eigenvalues which we denote by

$$\Lambda(\omega) = \left(\begin{bmatrix} \lambda_M^1(\omega) & 0 \\ 0 & \lambda_M^2(\omega) \end{bmatrix}, \tau, \delta_\Lambda \right),$$

and for the unstable eigenvalue

$$\lambda(\omega) = (\lambda_M(\omega), \tau, \delta_\lambda).$$

Let $\Sigma(\omega) = \mathrm{diag}(\lambda_1(\omega), \lambda_2(\omega), \lambda(\omega))$. We also have analytic Taylor models

$$\xi_i(\omega) = (\xi_M^i(\omega), \tau, \delta_{\xi^i}) \quad \text{for} \quad i = 1, 2, 3,$$

for the associated eigenvectors.

An analytic Taylor model for Q is given by

$$Q(\omega) = ([\xi_M^1 | \xi_M^2 | \xi_M^3], \tau, \delta_Q),$$

where $\delta_Q = \max(\delta_\Lambda, \delta_\lambda)$. We assume that we also have a validated branch of Q^{-1}, represented by the analytic Taylor model

$$Q^{-1}(\omega) = (Q_M^{-1}, \tau, \delta_{Q^{-1}}),$$

obtained using Lemma (3.3).

Now consider that

$$Df(x, y, z, \omega) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho + \omega - z & -1 & -x \\ y & x & -\beta \end{pmatrix}.$$

Then

$$Df[P_N(\theta, \omega), \omega] = Q(\omega)\Sigma(\omega)Q^{-1}(\omega) + \sum_{|\alpha|=1}^{N} A_\alpha(\omega)\theta^\alpha,$$

where

$$A_\alpha(\omega) = \sum_{m=0}^{M} A_{(\alpha,m)}\omega^m + H_\alpha(\omega),$$

with

$$A_{(\alpha,m)} = \begin{pmatrix} 0 & 0 & 0 \\ a^3_{(\alpha,m)} & 0 & -a^1_{(\alpha,m)} \\ a^2_{(\alpha,m)} & a^1_{(\alpha,m)} & 0 \end{pmatrix},$$

for $1 \le |\alpha| \le N$, $0 \le m \le M$, and

$$\|H_\alpha\|_\tau \le 2\delta_\alpha.$$

Then take C_1 to be a numerically computed constant having

$$\|Q\|_\tau \|Q^{-1}\| \leq (\|Q_M\|_\tau + \delta_Q)\left(\|Q_M^{-1}\|_\tau + \delta_{Q^{-1}}\right) \leq C_1,$$

and C_2 to be any numerically computed constant having

$$\sum_{|\alpha|=1}^{N} \sum_{m=0}^{M} \frac{|A_{(\alpha,m)}|}{\mu_* |\alpha|} \tau^m \nu^{|\alpha|} + \sum_{|\alpha|=1}^{N} \frac{\delta_\alpha}{|\alpha| \mu_*} \nu^{|\alpha|} \leq C_2.$$

Now consider the a-posteriori error

$$E_N(\theta, \omega) = f[P_{MN}(\theta, \omega) + H_N(\theta, \omega), \omega] - D_1 P_{MN}(\theta, \omega)\Lambda(\omega)\theta - D_1 H_N(\omega)\Lambda(\omega)\theta.$$

Again for the first term on the right hand side it is advantageous to exploit the explicit formula for the Lorenz field and obtain that

$$f[P_{MN}(\theta, \omega) + H_N(\theta, \omega), \omega] = f[P_{MN}(\theta, \omega), \omega] + Df[P_{MN}(\theta, \omega), \omega] H_N(\theta, \omega) + \left[\begin{array}{c} 0 \\ H_N^1 H_N^3 \\ H_N^1 H_N^2 \end{array}\right].$$

For the second term on the left we have that

$$D_1 P_{MN}(\theta, \omega)\Lambda(\omega)\theta = D_1 P_{MN}(\theta, \omega)\Lambda_M(\omega)\theta + D_1 P_{MN}(\theta, \omega) h_\Lambda(\omega)\theta,$$

while the third term on the right is

$$D_1 H_N(\omega)\Lambda_\omega \theta = \left(\sum_{|\alpha|=0}^{N} h_\alpha(\omega) D_1 \theta^\alpha\right) \Lambda(\omega)\theta.$$

Again we define the term

$$E_{MN}(\theta, \omega) = f[P_{MN}(\theta, \omega), \omega] - D_1 P_{MN}(\theta, \omega)\Lambda_M(\omega)\theta,$$

which is the explicitly polynomial part of the a-posteriori error, and which can be easily bound numerically. Let ϵ_{MN} be any numerically computed constant with $\|E_{MN}\|_{\nu,\tau} \leq \epsilon_{MN}$. We now have that

Lemma 5.2 (Total A-Posteriori Error for Lorenz). *The validation value a-posteriori error for the Lorenz system satisfies*

$$\|E_N\|_{\nu,\tau} \leq \epsilon_{NM} + \|Df[P_{MN}]\|_{\nu,\tau}\delta_N + \delta_N^2 + \|D_1 P_{MN}\|_{\nu,\tau}\delta_\Lambda \nu + \sum_{|\alpha|=0}^{N} |\alpha| \delta_\alpha \mu^* \nu^{|\alpha|}.$$

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