Estimate

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We used the following theorem from Schultz Spline Analysis:

Theorem 0.1 *Let $\Psi \in P^{2,\infty}(0,1)$. Define $h = \max_{i=0,\dots,m-1} \|t_{i+1} - t_i\|$. Then with*

$$M = \max_{i=0,\dots,m-1} \sup_{t \in (t_i, t_{i+1})} |\Psi^{(2)}(t)|$$

for $i = 0, \dots, m-1$ the following inequality holds:

$$\|\Psi - \Pi_h(\Psi)\| \leq \frac{h^2}{8} M$$

In our case this amounts to applying it to

$$g(t) = \int_0^t f(u_h(s))ds$$

Calculating the second derivative leads in the 1D case to the following expression for M

$$M = \max_{i=0,\dots,m-1} \frac{|x_{i+1} - x_i|}{t_{i+1} - t_i} \sup_{t \in (t_i, t_{i+1})} |\partial_x f(u_h(t))|$$

Now we introduced the following trick to estimate $|x_{i+1} - x_i|$:

$$\begin{aligned}
|x_{i+1} - x_i| = |u_h(t_{i+1}) - u_h(t_i)| = \\
|u_h(t_{i+1}) - P^u(\theta) - \int_0^{t_{i+1}} f(u_h(s))ds + P^u(\theta) + \int_0^{t_{i+1}} f(u_h(s))ds - u_h(t_i)| = \\
|F_1(u_h(t_{i+1})) - F_1(u_h(t_i)) - \int_{t_i}^{t_{i+1}} f(u_h(s))ds| \leq \\
|F_1(u_h(t_{i+1}))| + |F_1(u_h(t_i))| + h\|f\|
\end{aligned}$$

where F_1 denotes the second component of our operator.

Finally we get:

$$\frac{|x_{i+1} - x_i|}{t_{i+1} - t_i} \leq \frac{2\epsilon}{h} + \|f\|$$

where ϵ is the computed bound on $|F_1(u_h(t_j))|$. Then imposing

$$\frac{2\epsilon}{h} \leq Ch$$

leads to a quadratic error estimate.