POLYNOMIAL APPROXIMATION OF A ONE PARAMETER BRANCH OF (UN)STBLE MANIFOLDS WITH RIGOROUS COMPUTER ASSISTED A-POSTERIORI ERROR ANALYSIS

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Abstract. In this work we develop a calculus for computation of formal series representations of parameter dependent branches, or sheafs, of stable and unstable manifolds in discrete and continuous time dynamical systems. As an essential first step in this process we must develop formal parameter dependent expansions of the fixed point or equilibria, as well as their associated eigenvalues and eigenvectors. Then we use the fact that the family of invariant manifolds satisfies a functional equation to compute formal expansions of some chart maps of the manifold to arbitrary finite order. We also present a-posteriori theorems which allow the error in the finite approximations to be bound rigorously using validated numerics. We present several example computations, as well applications to manifold visualization and computer assisted proof of the existence of a tangency in a family of diffeomorphisms.

1. Introduction.

2. Parameterization of Invariant Manifolds, Regularity With Respect To Parameters, and one Parameter Branches of Invariant Manifolds.

2.1. Parameterization of (Un)stable Manifolds for a Map at a Fixed Parameter Value. Let $p_0 \in \mathbb{R}^d$. Suppose that $f : \mathbb{R}^d \to \mathbb{R}^d$ is real analytic in a neighborhood of p_0 and that p_0 is a fixed point of f. If zero is not an eigenvalue of $Df(p_0)$ then the differential is invertible, and f is a local diffeomorphism about p_0. If in addition $Df(p_0)$ has no eigenvalues on the unit circle, then there are local stable and unstable manifolds $W^{s,u}_{\text{loc}}(p_0)$ tangent at p_0 to the stable and unstable eigenspaces of $Df(p_0)$.

Let d_s and d_u denote the number of stable and unstable eigenvalues respectively, and note that since p_0 is hyperbolic we have that $d_s + d_u = d$. Then there $\nu_s, \nu_u > 0$ and chart maps, or *parameterizations*, $P : B(0, \nu_u) \subset \mathbb{R}^{d_u} \to \mathbb{R}^d$ and $Q : B(0, \nu_s) \subset \mathbb{R}^{d_s} \to \mathbb{R}^d$ so that

$$P[B_{\nu_u}] = W^u_{\text{loc}}(p_0), \qquad Q[B_{\nu_s}] = W^s_{\text{loc}}(p_0).$$

[9, 10, 11] develop a general *Parameterization Method* for studying such chart maps. The method is based on the fact that the chart maps solve certain functional equations. More precisely we will assume that $Df(p_0)$ is diagonalizable and let $\Lambda_s \in \text{GL}(\mathbb{R}^{d_s})$ and $\Lambda_u \in \text{GL}(\mathbb{R}^{d_u})$ denote the diagonal matrices with respectively the stable and unstable eigenvalues of $Df(p_0)$ on the diagonal entries and zeros in all other entries. Further, let A_s and A_u denote the $d \times d_s$ and $d \times d_u$ matrices having the stable and unstable eigenvectors of $Df(p_0)$ as columns. Then the chart maps P and Q are solutions of the following functional initial value problems

$$\begin{aligned} Q(0) &= p_0 \\ DQ(0) &= A_s \\ f[Q(\theta)] &= Q[\Lambda_s \phi] \qquad \text{for all} \qquad \phi \in B_{\nu_s}, \end{aligned} \tag{2.1}$$

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and

$$\begin{aligned} P(0) &= p_0 \\ DP(0) &= A_u \\ f[P(\theta)] &= P[\Lambda_u \theta] \qquad \text{for all} \qquad \theta \in B_{\nu_u}. \end{aligned} \tag{2.2}$$

Since f is analytic it can be shown that under generic non-resonance conditions on the eigenvalues the parameterizations P and Q are analytic functions having power series expansions

$$P(\theta) = \sum_{|\alpha| \geq 0} p_\alpha \theta^\alpha, \qquad Q(\phi) = \sum_{|\beta| \geq 0} q_\beta \phi^\beta$$

where $\alpha \in \mathbb{N}^{d_u}$, $\beta \in \mathbb{N}^{d_s}$, $\phi \in \mathbb{R}^{d_s}$, $\theta \in \mathbb{R}^{d_u}$, and $q_\beta, p_\alpha \in \mathbb{R}^d$. For more detailed discussion of the analyticity of P and Q, see [11] (a non-constructive proof can be given using the Implicit Function Theorem). For a particular map f linear recurrence equations for the unknown power series coefficients can be developed by standard power matching techniques.

Example: Consider the Hénon mapping

$$f(x, y) = \left[\begin{array}{c} y + 1 - ax^2 \\ b\, x \end{array} \right],$$

with $a = 1.4$ and $b = 0.3$ fixed. For these parameters the map has exactly two distinct hyperbolic fixed points, each with a one dimensional stable and unstable manifold. Let p_0, λ and ξ denote respectively a choice of fixed point, eigenvalue, and associated eigenvector. Then the parameterization of the local invariant manifold has power series expansion $P(\theta) \sum_{n=0}^\infty p_n \theta^n$ satisfying

$$\left[\begin{array}{c} P_2(\theta) + 1 - a[P_1(\theta)]^2 \\ b\, P_1(\theta) \end{array} \right] = \left[\begin{array}{c} P_1(\lambda\theta) \\ P_2(\lambda\theta) \end{array} \right].$$

(Here we are not specifying whether λ is stable or unstable so we just mean use P for *parameterization*). Of course the coefficient p_0 is equal the fixed point (making p_n a notation consistent with the notation for both the parameterization and the fixed point) and $p_1 = \xi$. To find the remaining coefficients we exploit the functional equation. The right hand side is

$$\left[\begin{array}{c} P_1(\lambda\theta) \\ P_2(\lambda\theta) \end{array} \right] = \sum_{n=0}^\infty \lambda^n \left[\begin{array}{c} p_n^1 \\ p_n^2 \end{array} \right] \theta^n,$$

while the left hand side is

$$f[P(\theta)] = \left[\begin{array}{c} 1 + \sum_{n=0}^\infty \left[p_n^2 - \sum_{k=0}^n a\, p_{n-k}^1 p_k^1 \right] \theta^n \\ \sum_{n=0}^\infty b\, p_n^1 \theta^n \end{array} \right]$$

Matching like powers of θ and solving for the highest order coefficient p_n in terms of lower order coefficients gives the linear system of equations

$$\begin{pmatrix} -2ap_0^1 - \lambda^n & 1 \\ b & -\lambda^n \end{pmatrix} \begin{bmatrix} p_n^1 \\ p_n^2 \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n-1} ap_{n-k}^1 p_k^1 \\ 0 \end{bmatrix}, \tag{2.3}$$

for $n \geq 2$. Equation (2.3) is referred to as the *homological equation.* Note that the homological equation has the form

$$[Df(p_0) - \lambda^n I]p_n = s_n$$

where s_n depends only on terms of order less than n. Also note that the matrix is the characteristic matrix of $Df(p_0)$. Then since $|\lambda| \neq 1$ we have $\lambda^n \neq \lambda$ for all $n \geq 2$ and matrix is always invertible. Then the coefficients p_n are formally well defined to all orders.

Remarks 2.1.

- Generalization
- Resonances
- Numerics/Radius of Convergence

2.2. Parameterization of (Un)stable Manifolds for a Differential Equation at a Fixed Parameter Value. The parameterization method can also be applied to differential equations, as is also shown in [9, 10, 11]. If $p_0 \in \mathbb{R}^d$ is an equilibria of a vector field $f : \mathbb{R}^d \to \mathbb{R}^d$, we suppose that f is real analytic near p_0, that $Df(p_0)$ is diagonalizable, and that p_0 is a hyperbolic equilibria (all eigenvalues have no-zero real part). Let $\lambda_{s,u}$ again denote the diagonal matrices of stable and unstable eigenvalues and $A_{s,u}$ the matrices whose columns are the associated stable and unstable eigenvectors.

The the parameterizations of the local stable and unstable manifolds solve the initial value functional (in this case partial differential) equations

$$\begin{aligned} Q(0) &= p_0 \\ DQ(0) &= A_s \\ f[Q(\theta)] &= DQ(\phi) \cdot \Lambda_s \cdot \phi \qquad \text{for all} \qquad \phi \in B_{\nu_s}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} P(0) &= p_0 \\ DP(0) &= A_u \\ f[P(\theta)] &= DP(\theta) \cdot \Lambda_u \cdot \theta \qquad \text{for all} \qquad \theta \in B_{\nu_u}. \end{aligned} \tag{2.5}$$

with some $\nu_u, \nu_s > 0$. Here the multiplications are matrix-matrix or matrix-vector as appropriate.

Again, an application of the Implicit Function Theorem gives that there are analytic P and Q solving (see for example [11]). Then P and Q have convergent power series expansions and we can try to formally compute the coefficients by power matching.

Example: Consider the differential equation $\dot{x} = f(x)$ given by the vector field

$$f(x, y, z) = \begin{bmatrix} \sigma(y - x) \\ \rho x - xz - y \\ xy - \beta z \end{bmatrix},$$

Let p_0 denote one of the fixed points, λ_1 and λ_2 denote two eigenvalues of $Df(p_0)$ with like stability (either both stable or both unstable), and ξ_1, ξ_2 be two associated eigenvectors . Let P denote the parameterization of the invariant manifold (whether stable or unstable) Λ denote the matrix with λ_1 and λ_2 as diagonal entries. Then in this case the power series is

$$P(\theta) = \sum_{|\alpha| \geq 0} p_\alpha \theta^\alpha = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} p_{(n_1,n_2)} \theta_1^{n_1} \theta_2^{n_2},$$

with $p_{(n_1,n_2)} \in \mathbb{R}^3$ for each $n_1, n_2 \geq 0$. The linear constraints give that $p_{(0,0)} = p_0$, $p_{(0,1)} = \xi_1$, and $p_{(1,0)} = \xi_2$. The coefficients for $n_1 + n_2 \geq 2$ are worked out by considering the functional equation

$$\begin{bmatrix} \sigma(P_2(\theta) - P_1(\theta)) \\ \rho P_1(\theta) - P_1(\theta)P_3(\theta) - P_2(\theta) \\ P_1(\theta)P_2(\theta) - \beta P_3(\theta) \end{bmatrix} = \begin{bmatrix} \theta_1\lambda_1\partial_{\theta_1}P_1(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_1(\theta) \\ \theta_1\lambda_1\partial_{\theta_1}P_2(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_2(\theta) \\ \theta_1\lambda_1\partial_{\theta_1}P_3(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_3(\theta) \end{bmatrix}$$

The right hand side expands as

$$\begin{bmatrix} \theta_1\lambda_1\partial_{\theta_1}P_1(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_1(\theta) \\ \theta_1\lambda_1\partial_{\theta_1}P_2(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_2(\theta) \\ \theta_1\lambda_1\partial_{\theta_1}P_3(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_3(\theta) \end{bmatrix} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (n_1\lambda_1 + n_2\lambda_2) \begin{bmatrix} p^1_{(n_1,n_2)} \\ p^2_{(n_1,n_2)} \\ p^3_{(n_1,n_2)} \end{bmatrix} \theta_1^{n_1}\theta_2^{n_2}$$

while the left hand side is

$$\begin{bmatrix} \sigma(P_2(\theta) - P_1(\theta)) \\ \rho P_1(\theta) - P_1(\theta)P_3(\theta) - P_2(\theta) \\ P_1(\theta)P_2(\theta) - \beta P_3(\theta) \end{bmatrix} =$$

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \begin{bmatrix} \sigma\left(p^2_{(n_1,n_2)} - p^1_{(n_1,n_2)}\right) \\ \rho\, p^1_{(n_1,n_2)} - p^2_{(n_1,n_2)} - \sum_{k=0}^{n_2}\sum_{j=0}^{n_1} p^1_{(n_1-j,n_2-k)} p^3_{(j,k)} \\ -\beta\, p^3_{(n_1,n_2)} + \sum_{k=0}^{n_2}\sum_{j=0}^{n_1} p^1_{(n_1-j,n_2-k)} p^2_{(j,k)} \end{bmatrix} \theta_1^{n_1}\theta_2^{n_2}.$$

Matching like powers of θ and solving for the highest order terms in terms of the lower order terms gives the homological equation

$$\begin{pmatrix} \sigma - (n_1\lambda_1 + n_2\lambda_2) & \sigma & 0 \\ \rho - p^3_{(0,0)} & -1 - (n_1\lambda_1 + n_2\lambda_2) & -p^1_{(0,0)} \\ p^2_{(0,0)} & p^1_{(0,0)} & -\beta - (n_1\lambda_1 + n_2\lambda_2) \end{pmatrix} \begin{bmatrix} p^1_{(n_1,n_2)} \\ p^2_{(n_1,n_2)} \\ p^3_{(n_1,n_2)} \end{bmatrix}$$

$$= \sum_{k=0}^{n_2} \sum_{j=0}^{n_1} \begin{bmatrix} 0 \\ \bar{p}^1_{(n_1-j,n_2-k)} \bar{p}^3_{(j,k)} \\ -\bar{p}^1_{(n_1-j,n_2-k)} \bar{p}^2_{(j,k)} \end{bmatrix}$$

where

$$\bar{p}_{(j,k)} = \begin{cases} 0 & \text{if either} \quad i = j = 0 \quad \text{or} \quad i = n_1, j = n_2 \\ p_{(i,j)} & \text{otherwise} \end{cases}$$

The homological equation has the form

$$[\,Df(p_0) - (n_1\lambda_1 + n_2\lambda_2)I\,]\, p_{(n_1,n_2)} = s_{(n_1,n_2)},$$

with s depending only on lower order terms. Moreover the matrix is a characteristic matrix for $Df(p_0)$ and is invertible as long as $n_2\lambda_1 + n_2\lambda_2 \neq \lambda_\ell$ for any $n_1 + n_2 \geq 2$ and either of $\ell = 1, 2$. When $\lambda_{1,2}$ are a complex conjugate pair this non-resonance condition holds for all $n_1 + n_2 \geq 2$. If $\lambda_{1,2}$ are real distinct and $\lambda_1 < \lambda_2$ then if

$$n_2\lambda_2 < \lambda_1,$$

then we have

$$n_1\lambda_1 + n_2\lambda_2 < \lambda_\ell \qquad \ell = 1, 2.$$

So there are no resonances for any multi-index (n_1, n_2) with $n_1 + n_2 \geq \lambda_1/\lambda_2$. Once we check that there are no resonances for multi-indices smaller than this then we rule out resonances to all orders.

2.3. Regularity with Respect to a Parameter and Analytic Branches of Parameterizations. Now we consider that $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a one parameter family of maps with $p_0 \in \mathbb{R}^d$ a hyperbolic fixed point of $f(x, 0)$, so that $f(p_0, \omega) = 0$. Moreover we suppose that $f(x, \omega)$ is real analytic jointly in each variable in some neighborhood of $(p_0, 0)$. By the implicit function theorem there is a $\tau > 0$ and a real analytic function $p : (-\tau, \tau) \subset \mathbb{R} \to \mathbb{R}^d$ so that

$$f(p(\omega), \omega) = p(\omega) \qquad \text{for all} \qquad |\omega| < \tau.$$

Suppose in addition that $Df(p_0, 0)$ is diagonalizable and let $\lambda_0^1, \ldots, \lambda_0^{d_u}$, and $\xi_0^1, \ldots, \xi_0^{d_u}$ denote the unstable eigenvalues and associated eigenvectors. Let Λ_0 be the matrix with the λ_0^i as diagonal entries and zeros on the off diagonal entries and A_0^u be the matrix with columns ξ_0^i $(1 \leq i \leq d_u)$.

Since the eigenvalues and eigenvectors solve analytic equations the implicit function theorem gives that there is a $\tau > 0$ and analytic functions $\lambda_1, \ldots, \lambda_{d_s} : (-\tau, \tau) \to \mathbb{C}$, $\xi_1, \ldots, \xi_{d_u} : (-\tau, \tau) \to \mathbb{R}^d$ so that $\lambda_i(\omega)$ is an eigenvalue of $Df(p(\omega), \omega)$ with associated eigenvector $\xi_i(\omega)$ for each $|\omega| < \tau$, and having $\lambda_i(0) = \lambda_0^i$ and $\xi_i(0) = \xi_0^i$ for each $1 \leq i \leq d_u$. Moreover, if $|\xi_0^i| = \hat{K}$ then requiring that $|\xi(\omega)| = \hat{K}$ for each $|\omega| < \tau$ determines the branches uniquely. (Here τ is taken to be the smallest number so that all the implicit function arguments go through simultaneously).

Assume the eigenvalues $\lambda_0^1, \ldots, \lambda_0^{d_u}$ are non-resonant. Then there is a $\nu_u > 0$ and a real analytic chart map $\tilde{P} : B(0, \nu_u) \subset \mathbb{R}^{d_u} \to \mathbb{R}^d$ satisfying Equatoin (2.2) parameterizing the local unstable manifold of p_0. Let

$$\hat{P}(\theta) = \sum_{|\alpha| \geq 0} \hat{p}_\alpha \theta^\alpha$$

Then the power series coefficients satisfy the homological equation

$$[Df(p_0, 0) - \Lambda_0^\alpha]\hat{p}_\alpha = \hat{s}_\alpha, \tag{2.6}$$

and, as per Remark (SOMETHING), the matrix is invertible fo all α by the non-resonance assumption and for each $\alpha \geq 2$, and the right hand side s_α is an analytic function of only the coefficients of order less than α. Moreover, since the eigenvalues are non-resonant there is a $\tau > 0$ so that $\lambda_1(\omega)$, ..., $\lambda_{d_u}(\omega)$ are non-resonant for each $|\omega| < \tau$.

Let $\Lambda(\omega)$ denote the non-constant matrix having $[\Lambda(\omega)]_{ii} = \lambda_i(\omega)$ for each $1 \leq i \leq d_u$ and $[\Lambda(\omega)_{ij} = 0$ if $i \neq j$, and $A_u(\omega)$ be the non-constant matrix having columns $\xi_i(\omega)$ for each $1 \leq i \leq d_u$. It is shown in [10, 11], again using the analytic Implicit Function Theorem, that there is a branch of real analytic functions $P : B(0, \nu_u) \times (-\tau, \tau) \subset \mathbb{R}^{d_u} \times \mathbb{R} \to \mathbb{R}^d$ through $\hat{P}$ (in other words $P(\cdot, 0) = \hat{P}$) solving the functional initial value problems

$$P(0, \omega) = p(\omega) \tag{2.7}$$

$$DP(0, \omega) = A_u(\omega) \tag{2.8}$$

$$f[P(\theta, \omega), \omega] = P[\Lambda(\omega)\theta, \omega] \qquad \text{for all} \qquad \theta \in B_{\nu_u}, \quad |\omega| < \tau. \tag{2.9}$$

Since P is analytic it has a convergent power series representation

$$P(\theta, \omega) = \sum_{|\alpha| \geq 0} \sum_{m=0}^{\infty} p_{(m,\alpha)} \omega^m \theta^\alpha = \sum_{|\alpha| \geq 0} p_\alpha(\omega) \theta^\alpha \qquad \theta \in B_{\nu_s}, \quad |\omega| < \tau, \tag{2.10}$$

where we have defined $p_\alpha(\omega) = \sum_{m=0}^{\infty} p_{(m,\alpha)} \omega^m$. Note that $p_\alpha(\omega)$ is analytic for each α as the series given by Equation (2.10) converges. Because $P(\theta, 0) = \hat{P}(\theta)$ we also have that

$$p_{(0,\alpha)} = p_\alpha(0) = \hat{p}_\alpha \qquad \text{for all} \qquad m, |\alpha| \geq 0. \tag{2.11}$$

We also have that each $\hat{p}_\alpha$ solves the homological equation (Equation 2.6). Since the homological equations are analytic, we apply the Implicit Function Theorem again to obtain real analytic branch functions $\hat{p}_\alpha : (-\tau, \tau) \to \mathbb{R}^d$ so that $\hat{p}(0) = \hat{p}_\alpha$ for each α. But then by the uniqueness of power series coefficients of an analytic function and the uniqueness provided by the Implicit Function Theorem we have

$$\hat{p}_\alpha(\omega) = p_\alpha(\omega) \qquad \text{for all} \qquad |\alpha| \geq 0.$$

In other words the coefficients $p_\alpha(\omega)$ of P solve the ω-dependent homological equations

$$[Df[p(\omega), \omega] - \Lambda(\omega)^\alpha I] p_\alpha(\omega) = s_\alpha(\omega) \tag{2.12}$$

for all $|\omega| < \tau$, where the matrix is invertible for all $|\omega| < \tau$ by the non-resonance assumption.

Consider the $0 \leq |\alpha| \leq 1$ coefficients. We will denote these by letting $\alpha_0 = (0, \ldots, 0)$ be the zero index, and $e_i = (0, \ldots, 1)$ be the index having a one in the i-th component and zeros elsewhere. Let the parameterization of the analytic branch of fixed points have power series expansion $p(\omega) = \sum_{m=0}^{\infty} p_m^0 \omega^m$. Then by Equation (2.7) we have that

$$p_{\alpha_0}(\omega) = p(\omega) \qquad \text{or} \qquad p_{(m,\alpha_0)} = p_m^0 \qquad \text{for all} \qquad m \geq 0. \tag{2.13}$$

So the (m, α_0) terms of P are given by the power series coefficients of the fixed point branch p. Similarly let $\xi_i(\omega) = \sum_{m=0}^{\infty} \xi_m^i \, \omega^m$. Then Equation (2.8) gives that

$$p_{e_i}(\omega) = \xi_i(\omega) \qquad \text{or} \qquad p_{m,e_i} = \xi_m^i \qquad m \geq 0. \tag{2.14}$$

Of course Equaiton (2.11) gives that the $m = 0$ coefficients are determined by the power series expansion of the parameterization $\hat{P}$ of the $\omega = 0$ manifold. The computation of the $m = 1$ coefficients are somewhat more delicate. Considering Equation (2.10) we see that just as the $m = 0$ coefficients are given by the coefficients $p_\alpha(0)$, the $m = 1$ coefficients are given by the coefficients of $\frac{\partial}{\partial \omega} p_\alpha(0)$. Explicitly we have that

$$\frac{\partial}{\partial \omega} P(\theta, 0) = \sum_{|\alpha| \geq 0} \sum_{m=0}^{\infty} p_{(m,\alpha)} \frac{\partial}{\partial \omega}_{|\omega=0} \omega^m = \sum_{|\alpha| \geq 0} p_{1,\alpha} \theta^\alpha$$

On the other hand we have that the coefficients $a_\alpha(\omega)$ solve the parameter dependent homological equation (Equation 2.12). Now since $p_{1,\alpha} = \frac{\partial}{\partial \omega} p_\alpha(0)$ we differentiate both sides of Equatoin (2.12) with respect to ω and evaluate at $\omega = 0$ in order to obtain that $p_{1,\alpha}$ solves the linear equation

$$[Df(p_0, 0) - \Lambda_0^\alpha I] \, p_{(1,\alpha)} = \frac{\partial}{\partial \omega} s_\alpha(0) - \frac{\partial}{\partial \omega}_{|\omega=0} [Df(p(\omega), \omega) - \Lambda^\alpha(\omega) I] \, p_{(0,\alpha)} \tag{2.15}$$

for $|\alpha| \geq 2$. We make no attempt at present to simplify the expressions on the right hand side of Equation (2.15). Rather we will work out the formulas only in the context of specific examples, in which case the expressions may simplify dramatically. The essential fact to note at present is that the matrix on the left hand side of of Equation (2.15) is none other than the characteristic matrix of $Df(p_0, 0)$, so that the coefficients $p_{(1,\alpha)}$ are well defined for all $|\alpha| \geq 2$ due to the non-resonance assumption. In other words, Equation (2.15) introduces no extra constraints.

Finally we must determine the coefficients $p_{m,\alpha}$ when $m + |\alpha| \geq 2$. This could be done by repeatedly differentiating the ω-dependent homological equation (Equatoin 2.12) and evaluation at $\omega = 0$ to obtain homological equations analogous to Equation (2.15) for all $m \geq 2$. Such expressions become both analytically, and computationally cumbersome. It is in fact preferable in the context of specific applications to substitute the power series form of P directly into Equation (2.9) and match like powers in order to develop the homological equations directly. We give examples in Section SOMETHING.

Similar considerations hold in the context of differential equations. For a vector field $f : \mathbb{R}^d \to \mathbb{R}^d$ let $p_0 \in \mathbb{R}^d$ be a hyperbolic equilibria and suppose that f is real analytic in a neighborhood of p_0. By the implicit function theorem there is a $\tau > 0$ and $p : (-\tau, \tau) \to \mathbb{R}^d$ so that

$$f[p(\omega), \omega] = 0 \qquad \text{for all} \qquad |\omega| < \tau.$$

With $\Lambda(\omega)$ and $A(\omega)$ as before we are led to the fact that the parameterization of a branch of invariant manifolds through $p(\omega)$ must satisfy

$$P(0,\omega) = p(\omega) \tag{2.16}$$

$$DP(0,\omega) = A_u(\omega) \tag{2.17}$$

$$f[P(\theta,\omega),\omega] = D_\theta P[\theta,\omega]\cdot\Lambda(\omega)\cdot\theta \qquad \text{for all} \qquad \theta\in B_{\nu_u}, \quad |\omega|<\tau. \tag{2.18}$$

The series for P will be given by Equation (2.10) just as before, however in this case the ω-dependent homological equation must be given by

$$[Df[p(\omega),\omega] - \langle\Lambda(\omega),\alpha\rangle\, I]p_\alpha(\omega) = s_\alpha(\omega) \tag{2.19}$$

Proceeding as above we obtain that Equations (2.13, 2.14, and 2.11) hold exactly as before. However the homological equation for the coefficients $p_{1,\alpha}$ of $\frac{\partial}{\partial\omega}P(\theta,0)$ are given by the homological equation

$$[Df(p_0,0) - \langle\Lambda_0,\alpha\rangle\, I]\, p_{(1,\alpha)} = \frac{\partial}{\partial\omega}s_\alpha(0) - \frac{\partial}{\partial\omega}_{|\omega=0}[Df(p(\omega),\omega) - \langle\Lambda(\omega),\alpha\rangle\, I]\, p_{(0,\alpha)}. \tag{2.20}$$

Again the matrix on the left is just the characteristic matrix of $Df[p_0,0]$ so that the non-resonance assumptions yield that the coefficients are formally well defined. Simplification of Equation (2.20) and the formal computation of $p_{m,\alpha}$ for $m\geq 2$ is carried out only in the context of specific applications, which we consider in Section SOMETHING.

Also: note that all the comments made in this section apply equally well to stable manifolds of maps and flows. We have focused on unstable manifolds in order to minimize the proliferation of sub and superscripts.

Remark 2.1. [$P(\theta,\omega)$**-Algorithm**] The discussion above provides us with a four step meta-algorithm for development of the formal series expansion of a branch of invariant manifolds

Step 1: Compute the parameterization $\hat{P}(\theta)$ of the invariant manifold at $\omega=0$. This determines the coefficients $p_{0,\alpha}$ of P. This step was discussed in Section SOMETHING for both the Hénon map and the Lorenz system.

Step 2: Compute the power series of the analytic branch functions $p(\omega)$, $\lambda_i(\omega)$, and $\xi_i(\omega)$, $1\leq i\leq d_{u,s}$ for the fixed point, eigenvalues, and eigenvectors. The coefficients of $p(\omega)$ determine the coefficients p_{m,α_0}, while the coefficients of $\xi_i(\omega)$ determine the coefficients $p_{(m,e_i)}$. These computations are the subject of Section SOMETHING.

Step 3: Depending on whether f generates a discrete or continuous time dynamical system (maps or differential equations) use either Equation (2.15) or Equation (2.20) along with the specific form of the map f to compute the $p_{1,\alpha}$ coefficients. Examples of this computation are given in Section SOMETHING.

Step 4: Plug the unknown power series given by Equation (2.10) into the either Equation (2.9) if f is a map, or Equation (2.18) if f is a vector field. Expand both sides as power seres, match like powers of $\omega^m\theta^\alpha$, and isolate the highest order coefficients from the lower order coefficients in order to obtain a homological equation for the $p_{m,\alpha}$ coefficients when $m\geq 2$. We illustrate this computation for the Hénon and Lorenz systems in Section (SOMETHING).

3. One Parameter Families of Fixed Points, Equilibria, Eigenvalues, and Eigenvectors. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a one parameter family of real analytic vector fields denoted by $f(x, \omega)$ with $x \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$. Suppose that $x_0 \in \mathbb{R}^n$ is an equilibria for f at $\omega = 0$, so that $f(x_0, 0) = 0$. Then if $D_x f(x_0, 0)$ is non-singular, the implicit function theorem (in the analytic category) gives that the is an analytic branch of equilibria through x_0.

More formally there exists a $\tau > 0$ real analytic function $x : (-\tau, \tau) \to \mathbb{R}^n$ so that $x(0) = x_0$ and

$$f[x(\omega), \omega] = 0 \qquad \text{for all} \qquad \omega \in (-\tau, \tau).$$

We say that x parameterizes the branch of equilibria through x_0. Since x is analytic it has a power series expansion $x(\omega) = \sum_{n=0}^{\infty} x_n \omega^n$ with $x_n \in \mathbb{R}^n$ convergent for $|\omega| < \tau$. In order to exploit this fact in a computational setting we must determine

(I) the coefficients x_n of the power series expansion for the branch of zeros,
(II) the radius of convergence τ of the power series.

Since x solves a (functional) equation, recurrence relations for the coefficients x_n can be computed by the usual power matching schemes. The convergence of the series could be treated in any of several ways. One could for a given problem prove the convergence of the power series directly by the method of majorization. Since we are using these series as inputs into further numerical computations, we pursue a numerical alternative. In any given problem we will compute the coefficients x_n to some finite order $M \in \mathbb{N}$, giving an approximate parameterization $x_M(\omega) = \sum_{n=0}^{M} x_n \omega^n$ of the branch of equilibria. Then we use residual methods based on the Newton-Kantorovich Theorem to prove that the series converge on some finite disk, and to rigorously bound the truncation error of the finite series. The radius of convergence is determined using numerical methods. We discuss the formal computations and the a-posteriori numerical argument in the next two sections respectively.

Finally we note that the comments above apply equally well to parameterizations of fixed points of diffeomorphisms, as well as to parameterizations of eigenvalues and eigenvectors, as in each of these solve functional equations of their own.

3.1. Computation of Formal Series Expansions for Linear Data. We consider the Hénon Family

$$f(x, y, \omega) = \begin{bmatrix} y + 1 - ax^2 \\ (b + \omega)x \end{bmatrix}, \tag{3.1}$$

where we think of a and b as fixed. Let

$$x(\omega) = \sum_{n=0}^{\infty} x_n \omega^n$$

parameterize an analytic branch of the first component of a fixed point of Equation (3.1). Then $x(\omega)$ solves the equaiton

$$a[x(\omega)]^2 + x(\omega)(1 - b - \omega) - 1 = 0. \tag{3.2}$$

Then

$$x_0 = \frac{b - 1 \pm \sqrt{(1-b)^2 + 4a}}{2a},$$

and

$$x_1 = \frac{d}{d\omega} x(0) = \frac{x_0}{2ax_0 - b + 1}.$$

(The expression for x_1 can be obtained by implicit differentiation of Equation 3.2). Matching like powers of ω in equation 3.2 gives that

$$x_n = \frac{1}{2ax_0 - b + 1} \left[x_{n-1} - \sum_{k=1}^{n-1} a\, x_{n-k} x_k \right]. \qquad \text{for} \qquad n \geq 2. \tag{3.3}$$

We note that since the second component of the fixed point is given by $y(\omega) = (b + \omega)x(\omega)$ we also have

$$y_0 = bx_0, \qquad y_1 = bx_1 + x_0 \qquad \text{and} \qquad y_n = bx_n + x_{n-1} \qquad n \geq 2.$$

Similarly, if λ_0 is an eigenvalue of $D_{(x,y)} f(x, y, 0)$ then we let

$$\lambda(\omega) = \sum_{n=0}^{\infty} \lambda_n \omega^n$$

parameterize a branch of eigenvalues passing through λ_0. Then $\lambda(\omega)$ satisfies the equation

$$\lambda(\omega)^2 + 2a\, x(\omega)\, \lambda(\omega) - \omega - b = 0, \tag{3.4}$$

with $\lambda(0) = \lambda_0$. As above we compute that

$$\lambda_0 = ax_0 \pm \sqrt{a^2 x_0^2 + b^2}, \qquad \lambda_1 = \frac{1 - 2ax_1\lambda_0}{2\lambda_0 + 2ax_0}$$

and

$$\lambda_n = \frac{-1}{2\lambda_0 + 2ax_0} \left(\sum_{k=1}^{n-1} \lambda_{n-k}\lambda_k + \sum_{k=0}^{n-1} 2ax_{n-k}\lambda_k \right) \qquad \text{with} \qquad n \geq 2. \tag{3.5}$$

Note that the λ_n are formally well defined as long as $\lambda_0 \neq -ax_0$, i.e. as long as λ_0 is not a repeated eigenvalue, and that the coefficient λ_n depends on the coefficients of $x(\omega)$ recurssively to n-th order.

Now suppose that we choose an eigenvector ξ_0 with $\|\xi\|^2 = \hat{K}$ for some $\hat{K} > 0$, associated with the eigenvalue λ_0. Now denote by

$$\xi(\omega) = \sum_{n=0}^{\infty} \xi_n \omega^n$$

a parameterizatoin of the branch of eigenvectors through ξ_0, where the entire branch is normalized to have length $\sqrt{\hat{K}}$. Then $\xi(\omega)$ satisfies the system of equations

$$\begin{bmatrix} -2ax(\omega) - \lambda(\omega) & 1 \\ b+\omega & -\lambda(\omega) \end{bmatrix} \begin{pmatrix} \xi_1(\omega) \\ \xi_2(\omega) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\xi_1(\omega)^2 + \xi_2(\omega)^2 = \hat{K}$$

but since the rows of the matrix equation are linearly dependent, we have that $\xi(\omega)$ must simultaneously safisfy

$$(b+\omega)\xi_1(\omega) - \lambda(\omega)\xi_2(\omega) = 0 \qquad \text{and} \qquad \xi_1(\omega)^2 + \xi_2(\omega)^2 = \hat{K}.$$

Matching like powers leads to the linear systems

$$\begin{bmatrix} b & -\lambda_0 \\ 2\xi_0^1 & 2\xi_0^2 \end{bmatrix} \begin{pmatrix} \xi_1^1(\omega) \\ \xi_2^1(\omega) \end{pmatrix} = \begin{pmatrix} \lambda_1\xi_0^2 - \xi_0^1 \\ 0 \end{pmatrix}$$

for the coefficient ξ_1 and

$$\begin{bmatrix} b & -\lambda_0 \\ 2\xi_0^1 & 2\xi_0^2 \end{bmatrix} \begin{pmatrix} \xi_n^1(\omega) \\ \xi_n^2(\omega) \end{pmatrix} = \begin{pmatrix} -\xi_{n-1}^1 + \sum_{k=0}^{n-1} \lambda_{n-k}\xi_k^2 \\ -\sum_{k=1}^{n-1} \xi_{n-k}^1\xi_k^1 + \xi_{n-k}^2\xi_k^2 \end{pmatrix}$$

for ξ_n when $n \geq 2$. The coefficient ξ_n depends recursively on the coefficients of $\lambda(\omega)$ to n-th order.

We consider also the Lorenz System, which is given by the flow of the vector field

$$f(x,y,z,\omega) = \begin{bmatrix} \sigma(y-x) \\ (\rho+\omega)x - xz - y \\ xy - \beta z \end{bmatrix}, \tag{3.6}$$

where we think of σ, ρ, and β as fixed. When $\omega = 0$ the system has equilibria at $p_0 = (0,0,0)^T$ and

$$p_{1,2} = \begin{bmatrix} \pm\sqrt{\beta(\rho-1)} \\ \pm\sqrt{\beta(\rho-1)} \\ \rho - 1 \end{bmatrix}$$

In fact p_0 is fixed for all ω. On the other hand, if we let $p(\omega) = (x(\omega), y(\omega), z(\omega))^T$ be a branch of either $p_{1,2}$, then we can work out that $x_0 = y_0 = \pm\sqrt{\beta(\rho-1)}$, $z_0 = \rho - 1$,

$$x_1 = y_1 = \frac{\pm\beta}{2\sqrt{\beta(\rho-1)}}, \qquad z_1 = \frac{2x_0x_1}{\beta}$$

and

$$x_n = y_n = \frac{-1}{2x_0} \sum_{k=1}^{n-1} x_{n-k} x_k, \qquad z_n = \frac{1}{\beta} \sum_{k=0}^{n} x_{n-k} x_k \qquad n \geq 2.$$

In the applications section we will be more interested in the fixed point at the origin, so we develop the expansions for the eigenvalues and eigenvectors only at p_0. We note that at the origin $-\beta$ is an eigenvalue for all ω. The remaining two eigenvalues do depend on ω and solve the equation

$$\lambda(\omega)^2 + (1+\sigma)\lambda(\omega) - \sigma(\rho + \omega - 1) = 0. \tag{3.7}$$

Denote a parameterization of a branch of these by

$$\lambda(\omega) = \sum_{n=0}^{\infty} \lambda_0 \omega^n.$$

Then

$$\lambda_0 = \frac{-(1+\sigma) \pm \sqrt{(1+\sigma)^2 - 4(\rho - 1)}}{2}, \qquad \lambda_1 = \frac{\sigma}{2\lambda_0 + \sigma + 1}$$

and

$$\lambda_n = \frac{-1}{2\lambda_0 + 1 + \sigma} \sum_{k=1}^{n-1} \lambda_{n-k} \lambda_k \qquad n \geq 2.$$

An eigenvector associated with the eigenvalue $-\beta$ is $\xi = (0,0,1)^T$ for all ω. The eigenvectors associated with the solutions of 3.7 lie in the xy-plane for all ω. A computation similar to as in the Hénon case shows that the coefficients of parameterizations of the these planar eigenvectors are given by the solutions of the linear systems

$$\begin{bmatrix} -(\sigma + \lambda_0) & \sigma \\ 2\xi_0^1 & 2\xi_0^2 \end{bmatrix} \begin{pmatrix} \xi_1^1(\omega) \\ \xi_2^1(\omega) \end{pmatrix} = \begin{pmatrix} \lambda_1 \xi_0^1 \\ 0 \end{pmatrix}$$

for the coefficient ξ_1 and

$$\begin{bmatrix} -(\sigma + \lambda_0) & \sigma \\ 2\xi_0^1 & 2\xi_0^2 \end{bmatrix} \begin{pmatrix} \xi_n^1(\omega) \\ \xi_n^2(\omega) \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{n-1} \lambda_{n-k} \xi_k^1 \\ -\sum_{k=1}^{n-1} \xi_{n-k}^1 \xi_k^1 + \xi_{n-k}^2 \xi_k^2 \end{pmatrix}$$

for ξ_n when $n \geq 2$. Moreover $\xi_n^3 = 0$ for all n.

4. Formal Computation of $P(\theta, \omega)$; Branch of Parameterized Manifolds. In this section we illustrate steps 3-4 of the algorithm stated in Remark (2.1) of Section (SOMETHING). We discuss separately the case of maps and flows.

4.1. Formal Expansion of $P(\theta,\omega)$ for the Hénon Map. Consider again the Hénon Family given by Equation (3.1). At $\omega = 0$ choose p_0 one of the maps two fixed points and λ_0 and ξ_0 an eigenvalue and associated eigenvector of $Df(p_0, 0)$. Using the expansions developed in Section (SOMETHING) we have series

$$\lambda(\omega) = \sum_{m=0}^{\infty} \lambda_m \omega^m \qquad \text{and} \qquad \xi(\omega) = \sum_{m=0}^{\infty} \xi_m \omega^m,$$

where we can compute the coefficients λ_m and ξ_m to any desired order using Equations (SOMETHING) and (SOMETHING). Let $\hat{P}$ be the parameterization of the invariant manifold at $\omega = 0$ associated with λ_0 and having $DP(0) = \xi_0$. Then the coefficients $\hat{p}_n$ of $\hat{P}$ can be computed to any desired order using the homological equation (Equation SOMETHING) given in Section (SOMETHING).

By the discussion in Section (SOMETHING) we know that there is a branch of parameterizations given by

$$P(\theta,\omega) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{(mn)}\theta^n\omega^m,$$

satisfying the functional equation

$$f[P(\theta,\omega),\omega] = P[\lambda(\omega)\theta,\omega]. \tag{4.1}$$

Then $p_{m0} = p_m$, $p_{m1} = \xi_m$ and $p_{0n} = \hat{p}_n$ as discussed in Section SOMETHING. This completes Steps $1-2$ of the algorithm given in Remark (2.1). Now we turn to steps $3-4$.

Step 3: Now we compute the $m = 1$ coefficients for the case of the Hénon map. Recall that the $m = 1$ coefficients solve the homological equation given by Equation (2.15), and we want to simplify the right hand side into a computable form for the specific case of the Hénon map. Then consider that

$$\frac{\partial}{\partial\omega}s_n(\omega) - \frac{\partial}{\partial\omega}\left[Df[p(\omega),\omega] - \Lambda^n(\omega)I\right]p_n(\omega) =$$

$$\frac{\partial}{\partial\omega}\sum_{k=1}^{n-1}\begin{bmatrix} ap^1_{n-k}(\omega)p^1_k(\omega) \\ 0 \end{bmatrix} - \frac{\partial}{\partial\omega}\begin{pmatrix} -2p^1(\omega) - \lambda(\omega)^n & 1 \\ b+\omega & -\lambda(\omega)^n \end{pmatrix}\begin{bmatrix} p^1_n(\omega) \\ p^2_n(\omega) \end{bmatrix}$$

$$= \sum_{k=1}^{n-1}\begin{bmatrix} a\left(p^1_k(\omega)\frac{\partial}{\partial\omega}p^1_{n-k}(\omega) + p^1_{n-k}(\omega)\frac{\partial}{\partial\omega}p^1_k(\omega)\right) \\ 0 \end{bmatrix}$$

$$-\begin{pmatrix} -2\frac{\partial}{\partial\omega}p^1(\omega) - n\lambda(\omega)^{n-1}\frac{\partial}{\partial\omega}\lambda(\omega) & 0 \\ 1 & -n\lambda(\omega)^{n-1}\frac{\partial}{\partial\omega}\lambda(\omega) \end{pmatrix}\begin{bmatrix} p^1_n(\omega) \\ p^2_n(\omega) \end{bmatrix}.$$

Evaluating at $\omega = 0$ gives

$$\frac{\partial}{\partial \omega} s_n(0) - \frac{\partial}{\partial \omega}\left[Df[p(\omega),\omega] - \Lambda^n(\omega) I\right]|_{\omega=0} \left[\begin{array}{c} p_n^1(0) \\ p_n^2(0) \end{array}\right] =$$

$$\sum_{k=1}^{n-1} \left[\begin{array}{c} a\left(p^1_{(1,n-k)} p^1_{(0,k)} + p^1_{(0,n-k)} p^1_{(1,k)}\right) \\ 0 \end{array}\right]$$

$$- \begin{pmatrix} -2p^1_{(1,0)} - n\lambda_0^{n-1}\lambda_1 & 0 \\ 1 & -n\lambda_0^{n-1}\lambda_1 \end{pmatrix} \left[\begin{array}{c} p^1_{(0,n)} \\ p^2_{(0,n)} \end{array}\right]. \tag{4.2}$$

Taking Equation (4.2) as the right hand side of Equation (2.15) gives the homological equation for the coefficients $p_{(1,n)}$ in the case of the Hénon map.

Step 4: Finally we obtain the equations for the coefficients $p_{(mn)}$ when $m + n \geq 2$. First we define the coefficients $\lambda_{(m,n)}$ be the series expansion of $\lambda(\omega)^n$. So

$$\lambda(\omega)^n = \sum_{m=0}^{\infty} \lambda_{(m,n)} \omega^m.$$

We expand the right hand side of Equation (4.1) and obtain

$$\begin{aligned} P[\lambda(\omega)\theta, \omega] &= \sum_{n=0}^{\infty} p_n(\omega)[\lambda(\omega)]^n \theta^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} p_{(m,n)}\omega^m\right)\left(\sum_{m=0}^{\infty} \lambda_{(m,n)}\omega^m\right)\theta^n \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m} \lambda_{(m-k,n)} \left[\begin{array}{c} p^1_{(k,n)} \\ p^2_{(k,n)} \end{array}\right] \omega^m \theta^n, \end{aligned} \tag{4.3}$$

Expanding the left hand side of Equation (4.1) as a power series gives

$$f[P(\theta,\omega),\omega] = \left[\begin{array}{c} 1 + P_2(\theta,\omega) - a[P_1(\theta,\omega)]^2 \\ (b+\omega)P_2(\theta,\omega) \end{array}\right]$$

which we expand componentwise to obtain

$$f[P(\theta,\omega),\omega]_1 = 1 + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p^2_{mn}\theta^n\omega^m$$

$$- \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{m} a\; p^1_{(m-j,n-k)} p^1_{(j,k)} \theta^n \omega^m, \tag{4.4}$$

and

$$f[P(\theta,\omega),\omega]_2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b p^1_{mn}\omega^m\theta^n + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} p^1_{(m-1,n)}\omega^m\theta^n \tag{4.5}$$

Now we equate the power series expressions for the left and right hand sides, match like powers, and isolate the highest order terms to obtain the homological equation

$$\begin{bmatrix} -2ap^1_{(00)} - \lambda_0^n & 1 \\ b & \lambda_0^n \end{bmatrix} \begin{bmatrix} p^1_{(m,n)} \\ p^2_{(m,n)} \end{bmatrix} = \begin{bmatrix} s^1_{(m,n)} \\ s^2_{(m,n)} \end{bmatrix} \tag{4.6}$$

where

$$s^1_{(m,n)} = \sum_{j=0}^{m-1} \lambda_{(m-j,n)} p^1_{(j,n)} + \sum_{k=0}^{n} \sum_{j=0}^{m} a\ \bar{p}^1_{(m-j,n-k)} \bar{p}^1_{(j,k)}$$

and

$$s^2_{(m,n)} = -p^1_{(m-1,n)} + \sum_{j=0}^{m-1} \lambda_{(m-j,n)} p^2_{(j,n)}$$

for $n + m \geq 2$. Again note that the matrix on the left hand side of the homological equation is just the characteristic matrix of $Df(p_0, 0)$, so that no new constraints are introduced for computing the branch expansions. The formal series is well defined to all orders by the non-resonance assumption on the eigenvalues at $\omega = 0$.

4.2. Formal Expansion of $P(\theta, \omega)$ for the Lorenz System. Let f be given by Equation (3.6). We consider the equilibria $p_0 = (0, 0, 0)$ at the origin, and fix σ, ρ, and β. Let λ_0 denote the stable eigenvalue of $Df(0, 0)$ and ξ_0 be the associated eigenvector. Then the branch of one dimensional manifolds through $\hat{P}$ is given by

$$P(\theta, \omega) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_{(mn)} \omega^m \theta^n.$$

In this case the computations are similar to the case of the Hénon map, except that P satisfies the functional equation

$$f[P(\theta, \omega), \omega] = \lambda(\omega)\theta \frac{\partial}{\partial \theta} P(\theta, \omega) \tag{4.7}$$

where

$$\lambda(\omega) = \sum_{m=0}^{\infty} \lambda_m \omega^m$$

parameterizes the branch of stable eigenvalues of $Df(0, \omega)$ through λ_0. Let

$$\xi(\omega) = \sum_{m=0}^{\infty} \xi_m \omega^m \qquad \text{and} \qquad \hat{P}(\theta) = \sum_{n=0}^{\infty} \hat{p}_n \theta^n,$$

be the series expansions for the stable eigenvector and the parameterization of the stable manifolds associated with λ_0. Explicit formulas for λ_m andξ_m are developed in

Section (SOMETHING). The homological equation for the $\hat{p}_n$ is Equation (SOMETHING) developed in Section (SOMETHING).

The remaining computations for Steps $3-4$ are similar to the one dimensional case for maps already studied in detail in Section (SOMETHING). We simply report that when $m=1$ coefficients of P satisfy the homological equation given by

$$[Df(p_0,0)-n\lambda_0]p_{(1,n)}=s_{(1,n)}$$

where

$$s_{(1,n)}=\left[\begin{array}{c} n\lambda_1 p^1_{(0,n)} \\ n\lambda_1 p^2_{(0,n)}-p^1_{(0,n)}+\sum_{k=1}^{n-1}\left[p^1_{(1,n-k)}p^3_{(0,k)}+p^1_{(0,n-k)}p^3_{(1,k)}\right] \\ n\lambda_1 p^3_{(0,n)}-\sum_{k=1}^{n-1}\left[p^1_{(1,n-k)}p^2_{(0,k)}+p^1_{(0,n-k)}p^2_{(1,k)}\right]\end{array}\right],$$

and for $m\geq 2$ we have the homological equation

$$\left[\begin{array}{ccc} -\sigma-n\lambda_0 & \sigma & 0 \\ \rho-p^3_{(00)} & -1-n\lambda_0 & -p^1_{(00)} \\ p^2_{(00)} & p^1_{(00)} & -\beta-n\lambda_0\end{array}\right]\left(\begin{array}{c} p^1_{(mn)} \\ p^2_{(mn)} \\ p^3_{(mn)}\end{array}\right)=\left(\begin{array}{c} s^1_{mn} \\ s^2_{mn} \\ s^3_{mn}\end{array}\right), \qquad (4.8)$$

for $n+m\geq 2$, where

$$s^1_{mn}=$$

$$s^2_{mn}=$$

and

$$s^3_{mn}=$$

We illustrate the computation for the branch of two dimensional stable manifolds at the origin in detail. Let

$$P(\theta,\omega)=P(\theta_1,\theta_2,\omega)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{m=0}^{\infty}p_{(m,n_1,n_2)}\omega^m\theta_1^{n_1}\theta_2^{n_2}$$

denote the parameterization of the one parameter branch of two dimensional stable manifolds through the origin. Then P satisfies the functional equation

$$f[P(\theta_1,\theta_2,\omega),\omega]=[D_\theta P(\theta,\omega)]\Lambda(\omega)\theta,$$

where we let $\lambda^1(\omega)$, and $\lambda^2(\omega)$ denote the parameterizations of the stable eigenvalues and define

$$\Lambda(\omega)=\left[\begin{array}{cc}\lambda^1(\omega) & 0 \\ 0 & \lambda^2(\omega)\end{array}\right]=\sum_{m=0}^{\infty}\left[\begin{array}{cc}\lambda^1_m & 0 \\ 0 & \lambda^2_m\end{array}\right]\omega^m.$$

Since the origin is a fixed point for all ω the series expansion of $p(\omega)$ is trivial to all orders. Moreover since we take $\beta > 0$, we have that $\lambda^1(\omega) = -\beta$ and $\xi^1(\omega) = (0, 0, 1)$ are a stable eigenvalue/eigenvector pair for all ω. The remaining unstable eigenvalue/eigenvector pair $\lambda^2(\omega)$ and $\xi_2(\omega)$ do depend on ω and are computed as in Section (SOMETHING). In addition, let

$$\hat{P}(\theta_1, \theta_2) = \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} \hat{p}_{(n_1,n_2)} \; \theta_1^{n_1} \theta_2^{n_2}$$

be the parameterization of the two dimensional unstable $\omega = 0$ manifold through the origin, where the coefficients $\hat{p}_{(n_1,n_2)}$ solve the homological equation given by Equation (SOMETHING). Then we have that $p_{(0,n_1,n_2)} = \hat{p}_{(n_1,n_2)}$ for all $n_1, n_2 \geq 0$, $p_{(m,0,0)} = 0$ for all $m \geq 0$, $p_{(m,1,0)} = \xi_m$ for all $m \geq 0$, $p_{0,0,1} = (0, 01)$, and $p_{m,0,2} = 0$ for all $m \geq 1$, and can consider Steps $1-2$ of the P-Algorithm complete (see Remark SOMETHING).

Step 3: For the Lorenz equations we consider the right hand side of Equation (2.20), first for ω free, and see that this simplifies to

$$\frac{\partial}{\partial \omega} s_\alpha(\omega) - \frac{\partial}{\partial \omega} \left[Df(p(\omega), \omega) - \langle \Lambda(\omega), \alpha \rangle \, I \right] p_\alpha(\omega) =$$

$$\sum_{k=0}^{n_2} \sum_{j=0}^{n_1} \begin{bmatrix} 0 \\ \bar{p}^3_{(j,k)}(\omega) \partial_\omega \bar{p}^1_{(n_1-j,n_2-k)}(\omega) + \bar{p}^1_{(n_1-j,n_2-k)}(\omega) \partial_\omega \bar{p}^3_{(j,k)}(\omega) \\ -\bar{p}^2_{(j,k)}(\omega) \partial_\omega \bar{p}^1_{(n_1-j,n_2-k)}(\omega) - \bar{p}^1_{(n_1-j,n_2-k)}(\omega) \partial_\omega \bar{p}^2_{(j,k)}(\omega) \end{bmatrix}$$

$$- \begin{pmatrix} -(n_1 \partial_\omega \lambda_1(\omega) + n_2 \partial_\omega \lambda_2(\omega)) & 0 & 0 \\ 1 & -(n_1 \partial_\omega \lambda_1(\omega) + n_2 \partial_\omega \lambda_2(\omega)) & 0 \\ 0 & 0 & -(n_1 \partial_\omega \lambda_1(\omega) + n_2 \partial_\omega \lambda_2(\omega)) \end{pmatrix} \begin{bmatrix} p^1_{(n_1,n_2)}(\omega) \\ p^2_{(n_1,n_2)}(\omega) \\ p^3_{(n_1,n_2)}(\omega) \end{bmatrix}.$$

Evaluating at $\omega = 0$ gives and setting equal to the left hand side of Equation (SOMETHING) gives the homological equation

$$[Df(0,0) - (n_1 \lambda_1^1 + n_2 \lambda_1^2) I] p_{(1,n_1,n_2)} = s_{(1,n_1,n_2)}$$

(where we recall that $\lambda_1^2 = 0$ for the Lorenz System) and where

$$s^1_{(1,n_1,n_2)} = (n_1 \lambda_1 - n_2 \lambda_2) p^1_{(0,n_1,n_2)}$$

$$s^2_{(1,n_1,n_2)} = -p^1_{(0,n_1,n_2)} + (n_1 \lambda_1 - n_2 \lambda_2) p^2_{(0,n_1,n_2)} + \sum_{k=0}^{n_2} \sum_{j=0}^{n_1} \bar{p}^1_{(1,n_1-j,n_2-k)} \bar{p}^3_{(0,j,k)} + \bar{p}^1_{(0,n_1-j,n_2-k)} \bar{p}^3_{(1,j,k)}$$

$$s^3_{(1,n_1,n_2)} = (n_1 \lambda_1 - n_2 \lambda_2) p^3_{(0,n_1,n_2)} - \sum_{k=0}^{n_2} \sum_{j=0}^{n_1} \bar{p}^1_{(1,n_1-j,n_2-k)} \bar{p}^2_{(0,j,k)} + \bar{p}^1_{(0,n_1-j,n_2-k)} \bar{p}^2_{(1,j,k)}$$

Step 4: Since

$$\vdots$$

Then we can work out that for $n_1+n_2+m\geq 2$ the coefficients of P are solutions of the homological equation

$$\begin{bmatrix} -\sigma-(n_1\lambda_0^1+n_2\lambda_0^2) & \sigma & 0 \\ \rho-a^3_{(00)} & -1-(n_1\lambda_0^1+n_2\lambda_0^2) & -a^1_{(00)} \\ a^2_{(00)} & a^1_{(00)} & -\beta-(n_1\lambda_0^1+n_2\lambda_0^2) \end{bmatrix}\begin{pmatrix} p^1_{(m,n_1,n_2)} \\ p^2_{(m,n_1,n_2)} \\ p^3_{(m,n_1,n_2)} \end{pmatrix} \tag{4.9}$$

$$= \begin{pmatrix} s^1_{(m,n_1,n_2)} \\ s^2_{(m,n_1,n_2)} \\ s^3_{(m,n_1,n_2)} \end{pmatrix},$$

where

$$s^1_{(m,n_1,n_2)} = \sum_{k=0}^{m-1}\left[n_1\lambda^1_{m-k}+n_2\lambda^2_{m-k}\right]p^1_{(k,n_1,n_2)}$$

$$s^2_{(m,n_1,n_2)} = -p_{(m-1,n_1,m_2)} + \sum_{k=0}^{m-1}\left[n_1\lambda^1_{m-k}+n_2\lambda^2_{m-k}\right]\ p^2_{(k,n_1,n_2)}$$

$$+\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{m}\bar{p}^1_{(m-k,n_1-i,n_2-j)}\bar{p}^3_{(kij)}$$

and

$$s^3_{(m,n_1,n_2)} = \sum_{k=0}^{m-1}\left[n_1\lambda^1_{m-k}+n_2\lambda^2_{m-k}\right]\ p^3_{(k,n_1,n_2)} - \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{m}\bar{p}^1_{(m-k,n_1-i,n_2-j)}\bar{p}^2_{(kij)}.$$

4.3. A Formalism for Polynomial Mappings f.

5. A-Posteriori Validation for the Formal Expansion of a Branch of Invariant Manifolds.

5.1. Background and Notation. THEOREM 5.1 (Newton-Kantorovich Method). *Let X, Y be Bancah spaces and $F : X \to Y$ be a differentiable mapping. Assume that there as an $\hat{x}\in X$ and an $r>0$ such that*

(i) $DF(\hat{x})$ has bounded inverse, and

(ii) $\|DF(x)-DF(y)\|_{B(X,Y)}\leq\kappa\|x-y\|$ for all $x,y\in B_r(\hat{x})$.

If

(I)

$$\epsilon_{NK}\geq\|DF(\hat{x})^{-1}\,F(\hat{x})\|_X,$$

(II)

$$\epsilon_{NK}\leq\frac{r}{2},$$

and

(III)

$$4\epsilon_{NK}\, \kappa\, \|DF(\hat{x})^{-1}\|_{B(X,Y)} \leq 1,$$

then the equation

$$F(x) = 0$$

has a unique solution in $B(r, \hat{x})$.

Co-Homological Equations: Let $\mathcal{X}$ be a Banach Space and $\Phi : \mathcal{X} \to \mathcal{Y}$ be a differentiable mapping. Let $D\Phi : \mathcal{X} \to \mathcal{Y}$ denote the Fréchet derivative of Φ. Let $x \in \mathcal{X}$, $b \in \mathcal{Y}$ and consider the linear equation $D\Phi(x)a = b$ with unknown $a \in \mathcal{X}$. Such linear operator equations are often referred to as *co-homological equations*, and play a key role in Contraction mapping and/or Newton-Kantorovich arguments involving Φ. In the sequel we are especially interested in two different co-homological equations associated with Parameterization problems; one which arises in the context of discrete time dynamical systems and another for continuous time dynamical systems.

Let p, q be bounded analytic MN-tails. When we study the a-posteriori equations associated with a one parameter branch of parameterizations for maps, we will be interested in the linear equation

$$\mathcal{L}[q](\theta, \omega) = p(\theta, \omega)$$

where $\mathcal{L}$ is defined by

$$\mathcal{L}[q](\theta, \omega) = q[\Lambda(\omega)\theta, \omega] - Df[P_{MN}(\theta, \omega), \omega].$$

Let Q_0 be the matrix of eigenvalues of $Df[p_0, 0]$ and Q_0^{-1} denote its inverse. Let C_1, C_2, μ^*, and μ_* be positive constants such that

$$0 < \mu_* \leq \min_{1 \leq i \leq d_s} \inf_{\tau \in (-\tau, \tau)} |\lambda_i^s(\omega)| \leq \max_{1 \leq i \leq d_s} \sup_{\tau \in (-\tau, \tau)} |\lambda_i^s(\omega)| \leq \mu^* < 1,$$

$$C_1 \geq \|Q_0\|_M \|Q_0^{-1}\|M,$$

and

$$\sup_{\tau \in (-\tau, \tau)} \sup_{|\theta| < \nu} \|Df[P_{MN}(\theta, \omega), \omega]^{-1}\|_M \leq C_1 \mu_*^{-1} + C_2.$$

Then we have the following lemma, which is a parameter dependent version of Lemma 4.4 in (CITE TANGLE PAPER).

Lemma 5.1. *Suppose that*

$$N + 1 > \frac{\ln(\mu_*) - \ln(C_1 + \mu_* C_2)}{\ln(\mu^*)}.$$

Then the linear operator $\mathcal{L}$ is boundedly invertible, and we have that

$$\|\mathcal{L}^{-1}\| \leq \frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}}.$$

The proof is identical to the proof of Lemma 4.4 in CITE TANGLE PAPER, once the parameter dependent definitions of C_1, C_2, μ^* and μ_* are taken into account, and the parameter dependent versions of the norms are used. We omit the details.

On the other hand, when we study the a-posteriori equations associated with a one parameter branch of parameterizations for flows the definition of the linear operator becomes

$$\mathcal{L}[q](\theta,\omega) = Dq(\theta,\omega)\Lambda(\omega)\theta - Df[P_{MN}(\theta,\omega),\omega]q(\theta,\omega).$$

This is the parameter dependent version of the linear operator studied in (CITE GREY SCOTT) (See specifically Lemma 4.3). However by examining the $Df[P_{MN}]^{-1}$ term somewhat more carefully than was done in (CITE GS) we can obtain a sharper estimate (even in the single parameter case).

Let μ_0 be the absolute value of the real part of the most negative eigenvalue of $Df(p_0, 0)$. Then define the positive constants

$$0 < \mu_* \leq \left| \max_{1\leq i \leq d_s} \sup_{\omega\in(-\tau,\tau)} \mathrm{real}(\lambda_i^s(\omega))\right| \leq \left| \min_{1\leq i \leq d_s} \inf_{\omega\in(-\tau,\tau)} \mathrm{real}(\lambda_i^s(\omega))\right| \leq \mu^* < \infty,$$

$$C_4 \geq \|Q_0\|_M \|Q_0^{-1}\|_M$$

where Q_0 and Q_0^{-1} are as before. Suppose that

$$Df[P_{MN}(\theta,\omega),\omega] = \sum_{0\leq|\alpha|\leq\bar{N}} \sum_{m=0}^{\bar{M}} A_{(m,\alpha)}\omega^m\theta^\alpha.$$

and

$$C_3 \geq \exp\left[\sum_{1\leq|\alpha|\leq\bar{M}} \sum_{m=0}^{\bar{M}} \frac{\|A_{(m,\alpha)}\|_M}{\mu_*|\alpha|}\tau^m \nu^{|\alpha|}\right]$$

and

$$C_2 \geq \sum_{m=1}^{\bar{M}} \|A_{(m,\alpha_0)}\|_M \tau^m.$$

Lemma 5.2. *Suppose that*

$$(N+1) \geq \frac{\mu_0 + C_2}{\mu_*}.$$

Then $\mathcal{L}$ is boundedly invertible and

$$\|\mathcal{L}^{-1}\| \leq \frac{C_3 C_4}{(N+1)\mu_* - (\mu_0 + C_2)}.$$

The proof is an integrating factor argument. We define

$$x(t) = q[e^{\Lambda(\omega)t}\theta, \omega],$$

$$A(t) = Df[P_{MN}(e^{\Lambda(\omega)t}\theta, \omega), \omega]$$

$$\bar{p}(\theta, \omega) = p(e^{\Lambda(\omega)t}\theta, \omega),$$

and

$$C(t) = e^{-\int_0^t A(t)ds},$$

so that the equation

$$\mathcal{L}[q] = p,$$

is equivalent to the ordinary differential equation

$$\frac{d}{dt}x - A(t)x = \bar{p},$$

(for each ω). The solution is given by

$$x(t) = -C^{-1}(t)\int_t^\infty C(s)\bar{p}(s)ds,$$

so that, taking $t \to 0$ we have

$$q(\theta, \omega) = x(0) = -\int_0^\infty C(s)\bar{p}(s)ds.$$

The observation that lets us improve the results of CITE GS is that

$$A(t) = -\int_0^t \sum_{0\le|\alpha|\le\bar{N}} \sum_{m=0}^{\bar{M}} A_{(m,\alpha)}\omega^m e^{<\Lambda,\alpha>s}\theta^\alpha \, ds$$

$$= -\int_0^t \left(A_{(0,\alpha_0)} + \sum_{m=1}^{\bar{M}} A_{(m,\alpha_0)}\omega^m + \sum_{1\le|\alpha|\le\bar{N}} \sum_{m=0}^{\bar{M}} A_{(m,\alpha)}\omega^m e^{<\Lambda,\alpha>s}\theta^\alpha \right) ds$$

Then since

$$\left\|e^{-A_{(0,\alpha_0)}t}\right\|_M = \left\|e^{-Df(p_0,0)t}\right\|_M = \left\|Q_0 e^{-\Omega_0 t}\right\|_M \le \|Q_0\|_M \, \|Q_0^{-1}\|_M e^{\mu_0 t},$$

where Ω_0 is the full diagonal $d \times d$ matrix of eigenvalues of $Df(p_0, 0)$, we have that

$$\|C(t)\|_M \le \|e^{A(t)}\|_M$$

$$\le \left\|e^{-A_{(0,\alpha_0)}t}\right\| \left\|\exp\left[-t\sum_{m=1}^{\bar{M}} A_{(m,\alpha_0)}\omega^m\right]\right\| \left\|\exp\left[\sum_{1\le|\alpha|\le\bar{N}} \sum_{m=0}^{\bar{M}} A_{(m,\alpha)}\omega^m e^{<\Lambda,\alpha>s}\theta^\alpha \, ds\right]\right\|$$

$$C_3 C_4 e^{\mu_0 + C_2}$$

Then using estimate (SOMETHING) we have that

$$\|C(t)\bar{p}(t)\| \leq C_3 C_4 e^{(N+1)\mu_* - (\mu_0 + C_2)}$$

as long as N satisfies the Inequality (SOMETHING). Then the indefinite integral in Equation (SOMETHING) converges and we have theorem.

REMARK 5.3. Note that if the theorem is applied with $\omega = 0$ then $C_2 = 0$ and the invertibility condition becomes

$$N + 1 > \frac{\mu_+}{\mu_-}$$

where μ_+ and μ_- are the largest and smallest real parts of the stable eigenvalues. Then the condition is just that $N+1$ be larger than the stable spectral gap fraction. In this case the estimate given by Equation (SOMETHING) is an improvement over the estimate in Lemma 4.3 CITE GREY SCOTT even for $\omega = 0$.

5.2. A-Posteriori Validation for a Formal Expansion of the Linear Data. Observe that the problem of computing a-posteriori bounds on the truncation error associated with a polynomial expansion of a branch of fixed points, equilibria, eigenvalues, or eigenvectors can always be put in the following framework. Suppose that $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a one parameter family of smooth maps and that $p_0 \in \mathbb{R}^d$ has $F(p_0, 0) = 0$ with $DF(p_0, 0)$ invertible. We assume also that $F(x, \omega)$ is real analytic in a neighborhood of $(p_0, 0) \in \mathbb{R}^{d+1}$. Then there is a $\tau > 0$ and an analytic function $p : (-\tau, \tau) \subset \mathbb{R} \to \mathbb{R}^d$ so that $p(0) = p_0$ and $F[p(\omega), \omega] = 0$, i.e. p parameterizes an ω dependent analytic arc of zeros through p_0. Let

$$p_M(\omega) = \sum_{m=0}^{M} p_m \omega^m$$

be the polynomial whose coefficients are determined by power matching. Then $p_M = p$ exactly to M-th order and there is an analytic M-tail $h : (-\tau, \tau) \subset \mathbb{R} \to \mathbb{R}^d$ so that $p = p_M + h$. The goal of this section is to determine, given a candidate for $\tau > 0$, a bound on $\|h\|_\tau$.

The idea is that we first choose, based on numerical experimentation, a 'good' τ. Here 'good' means that the C^0 norm of the composition of F and p_M is small on $(-\tau, \tau)$. Then apply the Newton-Kantorovich argument to prove the there is a true solution p near p_M. Since p_M equals p to M-th order, the difference $p - p_M$ is the desired analytic M-tail.

DEFINITION 5.4. [Validation Values for a Branch of Zeros of a Finite Dimensional Map] We say that the polynomial p_M and positive constants R, κ, ε, K and τ are *validation values* for the 'branch of zeros problem' if the following conditions are met:

1. The constant R has

$$\sum_{m=1}^{M} |p_m| \tau^m \leq R,$$

so that $p_M[(-\tau, \tau)] \subset B(p_0, R)$.

2. Let $\beta \in \mathbb{N}^d$ be a d dimensional multi-index N_F be the number of non-zero second partial derivatives of F. We require that M bounds the second derivative of F in the sense that

$$N_F \sup_{x \in B(p_0,R)} \sup_{|\omega|<\tau} \max_{1 \le i \le d} \max_{|\beta|=2} \left| \partial^\beta f_i(x,\omega) \right| \le M.$$

3. ϵ bounds the *a-posteriori error* associated with p_M in the sense that

$$\sup_{|\omega| \le \tau} |F[p_M(\omega), \omega]| \le \epsilon.$$

4. K bounds the inverse of the derivative of F along the branch, in the sense that

$$\sup_{|\omega|<\tau} \left\| DF[p_m(\omega), \omega]^{-1} \right\| \le K$$

Theorem 5.2 (A-Posteriori Existence of a Branch Funciton). *Given validation values p_M, R, κ, ε, K and τ we define a "Newton-Kantorovich Epsilon" δ_F by*

$$2K\varepsilon \le \delta_F$$

Assume that

$$4\kappa\, K\, \delta_F \le 1$$

Then there is a unique analytic N-tail h with

$$\|h\|_\tau \le \delta_F$$

so that

$$F\,[\,p_M(\omega) + h(\omega), \omega\,] = 0$$

for all $\omega \in (-\tau, \tau)$.

Proof: Let $\mathcal{X}$ be the Banach Space of real analytic functions from $(-\tau, \tau)$ into $\mathbb{R}^d$ endowed with the supremum norm. Then define the map $\Phi : \mathcal{X} \to \mathcal{X}$ by

$$\Phi[p](\omega) = F[p(\omega), \omega],$$

and note that Φ is differentiable with

$$D\Phi[p](\omega) = DF[p(\omega), \omega].$$

Taking into account the definitions of the validation values in Definition (5.4), the the proof that there is an analytic p so that $\Phi[p](\omega) = 0$ on $(-\tau, \tau)$ and $\|p - p_M\|_\tau < 2\varepsilon_{\text{NK}}$ is just a straight forward application of the Newton-Kantorovich Theorem applied to the map Φ, with $\varepsilon_{\text{NK}} = \sigma_F/2$. Then define $h = p - p_M$, and note that h is an analytic M tail by the assumption that p_M equals p to M-th order (i.e. that the coefficients of p_M solve a homological equation for F) and the uniqueness of the power series coefficients of an analytic function.

$\square$

In practice computing the validation values is a problem dependent exercies.

Examples: Consider for example the case of an analytic branch $\lambda(\omega)$ of eigenvalue for the Lorenz system. The branch solves the equation

$$F(\lambda(\omega),\omega) = [\lambda(\omega)]^2 + (1+\sigma)\lambda(\omega) - \sigma(\rho+\omega-1) = 0.$$

Let $\lambda_M = \sum_{m=0}^{M} \lambda_m \omega^m$ be the polynomial with coefficients defined by Equations (SOMETHING, SOMETHING) and (SOMETHING). Then

$$E(\omega) = F(\lambda_M(\omega),\omega)$$

is an $2M$-th order polynomial and we easily compute $\varepsilon = \sum_{m=0}^{2M} |E_m|\tau^m$ by interval arethmetic (this involves only one Cauchy Product). Moreover we compute directly that

$$\left\| DF[\lambda_M(\omega),\omega]^{-1} \right\| = \left\| \frac{1}{(\lambda_0+\sigma+1)\left[1+\frac{2}{\lambda_0+\sigma+1}\sum_{n=1}^{M}\lambda_m\omega^m\right]} \right\| \leq \frac{2}{|\lambda_0+(1+\sigma)|(1-M)}$$

so long as we require that

$$\frac{2}{|\lambda_0+(\sigma+1)|}\sum_{n=1}^{M}|\lambda_m|\tau^m = M < 1.$$

Note also that there is only one non-zero partial derivative of F and that it is uniformly bounded (in fact identically equal to) two. Then Theorem (SOMETHING) provides the conditions which must be satisfied in order to bound the truncation error on $(-\tau,\tau)$.

The case of an eigenvector of the Lorenz systems is only slightly more complicated. Suppose that λ_0 is one of ω dependent eigenvalues at the origin and that λ_M is the M-th order polynomial approximation of a branch as computed discussed in (SOMETHING). Suppose also that, following the discussion above, we have the existence of an analytic M-tail h so that $\|h\|_{(-\tau,\tau)} \leq \delta_\lambda$. Then we can define $\lambda(\omega) = \lambda_M(\omega) + h(\omega)$. Moreover let $\xi_M(\omega) = \sum_{m=0}^{M}\xi_m\omega^m$ have coefficients given by Equations (SOMETHING), (SOMETHING), and (SOMETHING). The true branch of eigenvectors $\xi(\omega)$ solves simultaneously the equations

$$-[\sigma+\lambda(\omega)]\xi_1(\omega) + \sigma\xi_2(\omega) = 0 \qquad \text{and} \qquad [\xi_1(\omega)]^2 + [\xi_2(\omega)]^2 = \hat{K}^2$$

where $\hat{K} = \|\xi_0\|$ is the length of the eigenvalue at $\omega = 0$. Then in this case we seek a branch of solution of the non-linear equation $F(\xi(\omega),\omega) = 0$ with

$$F(\xi,\omega) = \begin{pmatrix} -[\sigma+\lambda(\omega)]\xi_1 + \sigma\xi_2 \\ \xi_1^2 + \xi_2^2 - \hat{K}^2 \end{pmatrix}.$$

Now the a-posteriori error depends on $\xi_M(\omega)$, and $\lambda_M(\omega)$, as well as on the unknown function $h(\omega)$. Here the rigorous bounds on h are necessary in order to compute the a-posteriori error associated with $\xi_M(\omega)$. We define the $2M$-th order polynomial

$$E_M(\omega) = \begin{pmatrix} -\sigma\xi_M^1(\omega) - \lambda_M(\omega)\xi_M^1(\omega) + \sigma\xi_M^2(\omega) \\ (\xi_M^1)^2(\omega) + (\xi_M^2)^2(\omega) - \hat{K}^2 \end{pmatrix}$$

by taking Cauchy Products where necessary. Then

$$E(\omega) = F(\xi_M(\omega), \omega) = E_M(\omega) + \begin{pmatrix} -h(\omega)\xi_M^1(\omega) \\ 0 \end{pmatrix}$$

so that

$$\|E\|_\tau \leq \sum_{m=0}^{2M} |E_m|\tau^m + \delta_\lambda \sum_{m=0}^{M} |\xi_m|\tau^m.$$

We evaluate the sum on the right using interval arithmetic and use the result to define ε for the eigenvector problem.

In order to bound the differential we consider

$$DF[\xi_M(\omega), \omega] = \begin{pmatrix} -\sigma - \lambda(\omega) & \sigma \\ 2\xi_M^1(\omega) & 2\xi_M^2(\omega) \end{pmatrix} = A_0 + \sum_{m=1}^{M} A_m \omega^m + A_\infty,$$

where

$$A_0 = \begin{pmatrix} -\sigma - \lambda_0 & \sigma \\ 2\xi_0^1 & 2\xi_0^2 \end{pmatrix}, \quad A_m = \begin{pmatrix} -\lambda_m & 0 \\ 2\xi_m^1 & 2\xi_m^2 \end{pmatrix}, \quad \text{and} \quad A_\infty = \begin{pmatrix} -h(\omega) & 0 \\ 0 & 0 \end{pmatrix}.$$

The inverse becomes

$$DF[\xi_M(\omega), \omega]^{-1} = \left[I + A_0^{-1} \left(\sum_{m=1}^{M} A_m \omega^m + A_\infty \right) \right]^{-1} A_0^{-1}.$$

so that by requiring

$$\|A_0^{-1}\| \left(\sum_{m=1}^{M} \max\{|\lambda_m|, 2(|\xi_m^1| + |\xi_m^2|)\} \ \tau^m + \sigma_\lambda \right) = M < 1,$$

the Neumann Series gives that

$$\|DF[\xi_M(\omega), \omega]^{-1}\|_\tau \leq \frac{\|A_0^{-1}\|}{1 - M}.$$

Note that M can always be made less than one by taking $\tau > 0$ small enough (assuming that σ_λ is small), but in practice this is often mitigated by the decay rate of the coefficients $\|A_m\|$. For example, at the parameters given in (SOMEWHERE), the $\|A_0^{-1}\| < 1$ and the coefficients A_m actually decay so fast that we can take $\tau > 1$.

5.3. A-Posteriori Validation for a Formal Expansion of $P(\theta, 0)$. In Something we show that...

5.4. A-Posteriori Validation for a Formal Expansion of $\frac{\partial}{\partial \omega} P(\theta, 0)$. For maps we have that the branch parameterization solves the invariance equaton given by Equatoin (SOMETHING). Differentiating both sides of Equation (SOMETHING) with respect to ω gives that

$$Df[P(\theta,\omega),\omega]\frac{\partial}{\partial\omega}P(\theta,\omega) - \frac{\partial}{\partial\omega}P[\Lambda(\omega)\theta,\omega] = DP[\Lambda(\omega)\theta,\omega]\left[\frac{\partial}{\partial\omega}\Lambda(\omega)\right]\theta - \frac{\partial}{\partial\omega}f[P(\theta,\omega),\omega].$$

We denote by $\frac{\partial}{\partial\omega}\Lambda(0) = \Lambda_0'$ and $\frac{\partial}{\partial\omega}P(\theta, 0) = \hat{K}(\theta)$ and evaluate at $\omega = 0$ to obtain that $\hat{K}$ solves the functional equation

$$Df[\hat{P}(\theta)]\hat{K}(\theta) - \hat{K}[\Lambda_0\theta] = D\hat{P}[\Lambda_0\theta]\Lambda_0'\theta - \frac{\partial}{\partial\omega}f[\hat{P}(\theta)]. \tag{5.1}$$

On the other hand we know (by a different argument) that the $\hat{K}$ exists and is analytic and that the power series coefficients of $\hat{K}$ solve the homological equation given by Equation (SOMETHING).

Then assume that

$$\hat{K}_N(\theta) = \sum_{0\le|\alpha|\le N} \hat{k}_\alpha \theta^\alpha$$

has coefficients as defined in (SOME SECTION), i.e, the $\hat{h}_\alpha$ solve the homological equaton (SOMETHING) to N-th order. Then $\hat{K}_N = K$ exactly to N-th order and we seek $\nu > 0$ and an analytic N-tail $\hat{H}$ so that

$$K(\theta) = \hat{K}_N(\theta) + \hat{H}(\theta) \qquad \text{for all} \qquad |\theta| < \nu.$$

We define the defect, or a-posteriori error function

$$\hat{E}(\theta) = Df[\hat{P}(\theta)]\hat{K}_N(\theta) - \hat{K}_N[\Lambda_0\theta] - D\hat{P}[\Lambda_0\theta]\Lambda_0'\theta + \frac{\partial}{\partial\omega}f[\hat{P}(\theta)],$$

and plug $\hat{K}_N + \hat{H}$ into Equation 5.1 to obtain that the truncation error $\hat{H}$ solves the linear functional equation

$$Df[\hat{P}(\theta)]\hat{H}(\theta) - \hat{H}(\Lambda_0\,\theta) = \hat{E}(\theta). \tag{5.2}$$

Note that this is exactly the equation treated by Lemma 4.4 of (CITE TANGLE PAPER). Then assuming that ... we have that

$$\hat{H} = \mathfrak{L}^{-1}[\hat{E}]$$

with

$$\|\hat{H}\|_\nu \le \frac{C_1 + \mu_* C_2}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^N}\|E\|_\nu.$$

A similar computation for differential equations (differentiating Equation SOMETHING with respect to ω and evaluating at $\omega = 0$) shows that the first partial of the parameterization function with respect to ω satisfies the functional equation

$$D\hat{K}(\theta)\Lambda_0\theta - Df[\hat{P}(\theta)]\hat{K}(\theta) = \frac{\partial}{\partial\omega}f[\hat{P}(\theta)] - D\hat{P}(\theta)\Lambda_0'\theta. \tag{5.3}$$

Now suppose that $\hat{K}_N(\theta) = \sum_{0\le|\alpha|\le N}\hat{k}_\alpha\theta^\alpha$ has coefficients solving the homological equation (Equation SOMETHING) then $\hat{K}_N = \hat{K}$ exactly to N-th order. Then the truncation error $\hat{H}$ is an analytic N-tail solving the linear functional equation

$$D\hat{H}(\theta)\Lambda_0\theta - Df[\hat{P}(\theta)]\hat{H}(\theta) = \hat{E}(\theta), \tag{5.4}$$

where the a-posteriori error $\hat{E}$ is defined by

$$\hat{E}(\theta) = \frac{\partial}{\partial\omega} f[\hat{P}(\theta)] - D\hat{P}(\theta)\Lambda_0'\theta + D\hat{K}_N(\theta)\Lambda_0\theta - Df[\hat{P}(\theta)]\hat{K}_N(\theta)$$

Again, this equation was studied in (CITE GS PAPER). So if ... then there is a unique H so that $\hat{K}(\theta) = \hat{K}_N(\theta) + \hat{H}(\theta)$ for all $|\theta| < \nu$ and

$$\|\hat{H}\|_\nu \leq \frac{C_2}{(N+1)\mu_* - \mu^*}\|\hat{E}\|_\nu$$

Remarks 5.3.

- P_N versus P in the linear operator equation.
- P_N versus P in the error bound $\|\hat{E}\|_\nu$

5.5. Validation Theorem for $P(\theta, \omega)$; the Case of Diffeomorphisms. The validation theorem is almost identical to the Theorem Theorem 4.1 of (CITE TANGLE PAPER). The only difference is that several of the constants must in this case be uniform in ω. We state explicitly the assumptions, the definitions, and the Theorem in order to show exactly what conditions must be checked in the computer assisted proof of the a-posteriori error bounds.

A1: Let $p : (-\tau, \tau) \to \mathbb{R}^d$ be an analytic branch of fixed points of f and be given by $p = p_M + h_p$ where p_M is an M-th order polynomial and $\|h_p\|_\tau \leq \delta_p$.

A2: Assume that $Df[p(\omega), \omega]$ is non-singular, diagonalizble, and hyperbolic. Let $\{\lambda_1^s(\omega), \ldots, \lambda_{n_s}^s(\omega)\}$ denote analytic branches of the stable eigenvalues (which are distinct as Df is diagonalizable). Suppose that each $\lambda_i^s(\omega) = \lambda_{i,M}^s(\omega) + h_{\lambda_i}(\theta)$ where $\lambda_{i,M}^s$ is an M-th order polynomial and $\|h_{\lambda_i}\|_\tau < \delta_i$. Define $\Lambda_M(\omega)$ to be the diagonal matrix whose non-zero elements are the $\lambda_{i,M}^s$ and $\Lambda(\omega)$ to be the diagonal matrix containing the λ_i^s. Define $\delta_\Lambda = \max(\delta_i)$.

A3: Assume that $P_{MN} : (-\tau, \tau) \times B(0, \nu) \subset \mathbb{R} \times \mathbb{C}^{n_s} \to \mathbb{C}^n$ is a finite formal series of the form

$$P_{MN}(\theta, \omega) = \sum_{0 \leq |\alpha| \leq N} \sum_{m=0}^{M} p_{(m,\alpha)}\omega^m\theta^\alpha$$

which solves the equation

$$f[P_{MN}(\theta, \omega), \omega] = P_{MN}(\Lambda(\omega)\theta, \omega)$$

exactly to N-th order in α and M-th order on m (in the sense that the power series coefficients of the function on the left are equal to the power series coefficients of the function on the right to the specified order). By the discussion in Section (SOMETHING) the $m = 0$ coefficients of P_{MN} solve the homological equation (SOMETHING), the $m = 1$ coefficients solve homological equation (SOMETHING), the $|\alpha| = 0$ are the coefficients of p_M, the $|\alpha| = 1$ coefficients are given by (SOMETHING). For $2 \leq |\alpha| \leq N$ the coefficients solve a homological equation of the form of (SOMETHING).

A4: Assume that $\rho > 0$ has that

$$\delta_p + \sum_{m=1}^{M} |p_m|\tau^m + \sum_{1\leq|\alpha|\leq N}\sum_{m=0}^{M} |p_{(m,\alpha)}|\tau^m\nu^{|\alpha|} < \rho$$

This guarantees that

$$|P_{MN}(\theta,\omega) - p(\omega)| = |P_{MN}(\theta,\omega) - p_M(\omega) - h_p(\omega)| \leq |P_{MN} - p_M| + |h_p| < \rho$$

for all $|\theta| < \nu, |\omega| < \tau$. Since $|p_0 - p(\omega)| \leq \sum_{m=1}^{M} |p_m|\tau^m$ we have that for all $\omega \in (-\tau,\tau)$ we have that $\text{image}[P_{MN}(\cdot,\omega)] \subset B(p_0,\rho)$.

We make the following definition.

Definition 5.5. [Validation values for discrete dynamical systems] The collection of positive constants ν, ϵ_{tol}, C_1, C_2, K_1, ρ, ρ', μ_* and μ^* are validation values for P_N if

1. $$\sup_{|\theta|\leq\nu}\sup_{|\omega|\leq\tau} |f[P_{MN}(\theta,\omega),\omega] - P_{MN}[\Lambda(\omega)\theta,\omega]| \leq \epsilon_{\text{tol}};$$

2. $$\sup_{|\theta|\leq\nu}\sup_{|\omega|\leq\tau} |P_{MN}(\theta,\omega)| \leq \rho' < \rho;$$

3. $$0 < \mu_* \leq \min_{1\leq i\leq n_s}\inf_{\omega\in(-\tau,\tau)} |\lambda_i^s(\omega)| \leq \max_{1\leq i\leq n_s}\sup_{\omega\in(-\tau,\tau)} |\lambda_i^s(\omega)| \leq \mu^* < 1;$$

4. Let Q_0 denote the matrix of eigenvectors of $Df[p(0),0]$ and Q_0^{-1} denote it's inverse (so these are constant matrices). We require of C_1 and C_2 that

$$\sup_{|\theta|\leq\nu}\sup_{|\omega|\leq\tau} \|Df[P_{MN}(\theta,\omega),\omega]^{-1}\|_M \leq C_1\mu_*^{-1} + C_2(\nu,\tau);$$

where we take C_1 to be any constant with

$$\|Q_0\|\|Q_0^{-1}\| \leq C_1,$$

so that C_2 is any bound on the theta-omega dependent terms of $Df[P_{MN}(\theta,\omega),\omega]^{-1}$.

5. K_1 is any number with

$$\max_{1\leq j\leq d}\max_{|\beta|=2}\sup_{|x-p_0|<\rho}\sup_{|\omega|\leq\tau} |\partial^\beta f_j(x,\omega)| \leq K_1.$$

We define a number N_f which counts the number of non-zero second partials of f with respect to the phase space variables.

$$N_f = \max_{1\leq j\leq n} \#\{\beta \in \mathbb{Z}^n : |\beta| = 2 \;\text{ and }\; \partial^\beta f_j \neq 0\}, \tag{5.5}$$

and have $N_f \leq n^2$. Here a second partial is considered to vanish only if it is identically zero for all $\omega \in (-\tau,\tau)$. Now we state out a-posteriori validation theorem for maps.

THEOREM 5.4 (A-posteriori manifold validation). *Given validation values ν, ϵ_{tol} , K_1, C_1, C_2, ρ, ρ', μ_* and μ^*, assume that N and δ satisfy the three inequalities*

$$N+1 > \frac{\ln(\mu_*) - \ln(C_1 + \mu_* C_2)}{\ln(\mu^*)}; \tag{5.6}$$

$$\delta < \min\left(\frac{[\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}]}{2ne\pi N_f (C_1 + \mu_* C_2) K_1}, (\rho - \rho')e^{-1} \right) \tag{5.7}$$

$$\delta > \frac{2(C_1 + \mu_* C_2)\epsilon_{tol}}{\mu_* - (C_1 + \mu_* C_2)(\mu^*)^{N+1}} \tag{5.8}$$

Then there is a unique parameterization function $P : (-\tau, \tau) \times B(0, \nu) \subset \mathbb{R} \times \mathbb{C}^{n_s} \to \mathbb{C}^n$ solving Equation (SOMETHING). Additionally, the truncation error is bounded by

$$\sup_{|\omega|<\tau} \sup_{|\theta|<\nu} |P(\theta, \omega) - P_{MN}(\theta, \omega)| \leq \delta$$

and the parameterization coefficients $p_{(m,\alpha)} \in \mathbb{C}^n$ of the true solution P decay as

$$|p_{(m,\alpha)}| \leq \frac{\delta}{\tau^m \nu^{|\alpha|}} \qquad \textit{for} \ \ |\alpha| > N, m > M.$$

The proof is almost identical to the Proof of Theorem 4.1 in (CITE TANGLE PAPER), with the exception that .... The main complication in applying Theorem (5.4) is actually the evaluation of the a-posteriori error ε_{tol}. The subtly arises due to the fact that the expression for the a-posteriori error contains the term Λ which we only know explicitly up to order M equals the polynomial Λ_M. Then the evaluation of ε_{tol} will depend also on the a-posteriori error given by δ_Λ. In fact we have that

$$\begin{aligned} E(\theta, \omega) &= f[P_{MN}(\theta, \omega), \omega] - P_{MN}[\Lambda(\omega)\theta, \omega] \\ &= f[P_{MN}(\theta, \omega), \omega] - P_{MN}[\Lambda_M(\omega)\theta + h_\Lambda(\omega)\theta, \omega]. \end{aligned} \tag{5.9}$$

We expand the right hand side to obtain

$$P_{MN}[\Lambda_M(\omega)\theta + h_\Lambda(\omega)\theta, \omega] = P_{MN}[\Lambda_M(\omega)\theta, \omega] + DP_{MN}[\Lambda_M(\omega)\theta, \omega] h_\Lambda(\omega)\theta + T_{(\Lambda_M(\omega)\theta, \omega)}[h_\Lambda(\omega)],$$

where T is the second order Taylor remainder for P_{MN}. (Note that this is the Taylor expansion in the phase space variable, uniformly in the parameter ω). Let

$$E_{MN}(\theta, \omega) = f[P_{MN}(\theta, \omega), \omega] - P_{MN}[\Lambda_M(\omega)\theta, \omega],$$

and note that we can compute the coefficients of E_{MN} explicitly by composing the polynomial representations. Then in practice we take $\varepsilon_{\text{tol}} > 0$ to be any number having

$$\begin{aligned}\|E(\theta,\omega)\|_{(\tau,\nu)} &\leq \|E_{MN}\|_{\Sigma,(\tau,\nu)} + \|DP\|_{(\mu^*\nu,\tau)}\delta_\Lambda \nu + \max_{1\leq j\leq d_s} \sum_{|\alpha|=2} \frac{2\delta_\Lambda^2}{\alpha!}\|\partial^\alpha P_j\|_{(\mu^*\nu,\tau)} \\ &\leq \varepsilon_{\text{tol}}, \end{aligned} \tag{5.10}$$

where we have used the Lagrange form of the remainder. Note that all three terms can be evaluated using interval arithmetic. We expect that $\|E_{MN}\|_{\Sigma,(\tau,\nu)}$ will be small because it is essentially a weighted sum of of the round off error accumulated in the computations of the $p_{m,\alpha}$, for $0 \leq m \leq M$ and $0 \leq |\alpha| \leq N$. We expect the remaining terms to be small so long as we have arranged that the a-posteriori error δ_Λ is small.

5.6. Validation Theorem for $P(\theta,\omega)$; the Case of Vector Fields. The validation theorem is similar to the Theorem Theorem 4.2 of (CITE GS PAPER), except we incorporate the Cauchy estimates used in CITE TANGLE PAPER, and the improved estimates on the co-homology equation given by Lemma SOMETHING.

B1: Let $p : (-\tau,\tau) \to \mathbb{R}^d$ be an analytic branch of equilibria of f and be given by $p = p_M + h_p$ where p_M is an M-th order polynomial and $\|h_p\|_\tau \leq \delta_p$.

B2: Assume that $Df[p(\omega),\omega]$ is non-singular, diagonalizble, and hyperbolic. Let $\{\lambda_1^s(\omega),\ldots,\lambda_{n_s}^s(\omega)\}$ denote analytic branches of the stable eigenvalues (which are distinct as Df is diagonalizable). Suppose that each $\lambda_i^s(\omega) = \lambda_{i,M}^s(\omega) + h_{\lambda_i}(\theta)$ where $\lambda_{i,M}^s$ is an M-th order polynomial and $\|h_{\lambda_i}\|_\tau < \delta_i$. Define $\Lambda_M(\omega)$ to be the diagonal matrix whose non-zero elements are the $\lambda_{i,M}^s$ and $\Lambda(\omega)$ to be the diagonal matrix containing the λ_i^s. Define $\delta_\Lambda = \max(\delta_i)$.

B3: Assume that $P_{MN} : (-\tau,\tau) \times B(0,\nu) \subset \mathbb{R}\times\mathbb{C}^{n_s} \to \mathbb{C}^n$ is a finite formal series of the form

$$P_{MN}(\theta,\omega) = \sum_{0\leq|\alpha|\leq N}\sum_{m=0}^{M} p_{(m,\alpha)}\omega^m\theta^\alpha$$

which solves the equation

$$f[P_{MN}(\theta,\omega),\omega] = DP_{MN}(\theta,\omega)\Lambda(\omega)\theta$$

exactly to N-th order in α and M-th order on m (in the sense that the power series coefficients of the function on the left are equal to the power series coefficients of the function on the right to the specified order). By the discussion in Section (SOMETHING) the $m = 0$ coefficients of P_{MN} solve the homological equation (SOMETHING), the $m = 1$ coefficients solve homological equation (SOMETHING), the $|\alpha| = 0$ are the coefficients of p_M, the $|\alpha| = 1$ coefficients are given by (SOMETHING). For $2 \leq |\alpha| \leq N$ the coefficients solve a homological equation of the form of (SOMETHING).

B4: Assume that $\rho > 0$ has that

$$\delta_p + \sum_{m=1}^{M}|p_m|\tau^m + \sum_{1\leq|\alpha|\leq N}\sum_{m=0}^{M}|p_{(m,\alpha)}|\tau^m\nu^{|\alpha|} < \rho$$

This guarantees that

$$|P_{MN}(\theta,\omega) - p(\omega)| = |P_{MN}(\theta,\omega) - p_M(\omega) - h_p(\omega)| \leq |P_{MN} - p_M| + |h_p| < \rho$$

for all $|\theta| < \nu, |\omega| < \tau$. Since $|p_0 - p(\omega)| \leq \sum_{m=1}^{M}|p_m|\tau^m$ we have that for all $\omega \in (-\tau,\tau)$ we have that image$[P_{MN}(\cdot,\omega)] \subset B(p_0,\rho)$.

We make the following definition.

Definition 5.6. [Validation values for continuous dynamical systems] The collection of positive constants ν, ϵ_{tol}, C_1, C_2, K_1, ρ, ρ', μ_* and μ^* are validation values for P_N if

1.
$$\sup_{|\theta|\leq\nu}\sup_{|\omega|\leq\tau}|f[P_{MN}(\theta,\omega),\omega]-DP_{MN}[\theta,\omega]\Lambda(\omega)\theta|\leq\epsilon_{\text{tol}};$$

2.
$$\sup_{|\theta|\leq\nu}\sup_{|\omega|\leq\tau}|P_{MN}(\theta,\omega)|\leq\rho'<\rho;$$

3.
$$0<\mu_*\leq\left|\max_{1\leq i\leq n_s}\sup_{\omega\in(-\tau,\tau)}\text{real}(\lambda_i^s(\omega))\right|\leq\left|\min_{1\leq i\leq n_s}\inf_{\omega\in(-\tau,\tau)}\text{real}(\lambda_i^s(\omega))\right|\leq\mu^*<\infty;$$

4. Let Q_0 denote the matrix of eigenvectors of $Df[p(0),0]$ and Q_0^{-1} denote it's inverse (so these are constant matrices). We require of C_2, C_3, and C_4 are as in the assumptions of Lemma (SOMETHING).
5. K_1 is any number with
$$\max_{1\leq j\leq d}\max_{|\beta|=2}\sup_{|x-p_0|<\rho}\sup_{|\omega|\leq\tau}|\partial^\beta f_j(x,\omega)|\leq K_1.$$

We also define a number N_f which counts the number of non-zero second partials of f with respect to the phase space variables.

$$N_f=\max_{1\leq j\leq n}\#\{\beta\in\mathbb{Z}^n:|\beta|=2\ \text{ and }\ \partial^\beta f_j\neq 0\},\tag{5.11}$$

and have $N_f\leq n^2$. Here a second partial is considered to vanish only if it is identically zero for all $\omega\in(-\tau,\tau)$. Now we state out a-posteriori validation theorem for differential equaitons.

Theorem 5.5 (A-posteriori manifold validation). *Given validation values ν, ϵ_{tol}, K_1, C_2, C_3, C_4, ρ, ρ', μ_* and μ^*, assume that N and δ satisfy the three inequalities*

$$N+1>\frac{\mu_0+C_2}{\mu_*};\tag{5.12}$$

$$\delta<\min\left(\frac{(N+1)\mu_*-(\mu_0+C_2)}{2ne\pi N_fC_3C_4K_1},(\rho-\rho')e^{-1}\right)\tag{5.13}$$

$$\delta>\frac{2C_2C_3\epsilon_{tol}}{(N+1)\mu_*-(\mu_0+C_2)}\tag{5.14}$$

Then there is a unique parameterization function $P : (-\tau, \tau) \times B(0, \nu) \subset \mathbb{R} \times \mathbb{C}^{n_s} \to \mathbb{C}^n$ solving Equation (SOMETHING). Additionally, the truncation error is bounded by

$$\sup_{|\omega|<\tau} \sup_{|\theta|<\nu} |P(\theta, \omega) - P_{MN}(\theta, \omega)| \leq \delta$$

and the parameterization coefficients $p_{(m,\alpha)} \in \mathbb{C}^n$ of the true solution P decay as

$$|p_{(m,\alpha)}| \leq \frac{\delta}{\tau^m \nu^{|\alpha|}} \qquad \textit{for } |\alpha| > N, m > M.$$

The proof is similar to the Proof of Theorem 4.2 in (CITE GS), except that we use Lemma (SOMETHING) in order to solve the co-homological equation, and follow the argument given in Section SOMETHING of CITE TANGLE PAPER to obtain the contraction. The main difference between the contraction mapping arguments given in CITE GS and the one in CITE TANGLE PAPER is the use of the Cauchy Bound of Theorem (TP SOMETHING) in stead of the weaker bound of THEOREM (GS SOMETHING).

Again, the complication in applying Theorem (5.4) is the numerical evaluation of the a-posteriori error ε_{tol}. We define

$$\begin{aligned} E(\theta, \omega) &= f[P_{MN}(\theta, \omega), \omega] - DP_{MN}[\theta, \omega]\Lambda(\omega)\theta \\ &= f[P_{MN}(\theta, \omega), \omega] - DP_{MN}[\theta, \omega][\Lambda_M(\omega) + h_\Lambda(\omega)]\theta. \end{aligned} \tag{5.15}$$

and note that the problem is much simpler than in the case of maps (the dependence of a-posteriori error E on the truncation error h_Λ is linear). Defining the numerical a-posteriori error function to be

$$E_{MN}(\theta, \omega) = f[P_{MN}(\theta, \omega), \omega] - DP_{MN}[\theta, \omega]\Lambda_M(\omega)\theta.$$

Then in practice we take ε_{tol} be any positive constant with

$$\|E\|_{(\tau,\nu)} \leq \|E_{MN}\|_{\Sigma,(\tau,\nu)} + \|DP_{MN}\|_{\Sigma,(\tau,\nu)}\delta_\Lambda \nu \leq \varepsilon_{\text{tol}}.$$

Again, such a ε_{tol} is easily computed via interval arithmetic.

6. Applications.

6.1. Visualization of Validated Sheafs of Invariant Manifolds.

6.2. Computer Assisted Proof of the Existence of Tangencies for Families of Diffeomorphisms.

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