POLYNOMIAL APPROXIMATION OF ONE PARAMETER FAMILIES OF (UN)STABLE MANIFOLDS WITH RIGOROUS COMPUTER ASSISTED ERROR BOUNDS

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Abstract. This work describes a method for approximating a branch of stable or unstable manifolds associated with a branch of hyperbolic fixed points or equilibria in a one parameter family of analytic dynamical systems. We approximate the branch of invariant manifolds by polynomials and develop a-posteriori theorems which provide mathematically rigorous bounds on the truncation error. The hypotheses of these theorems are formulated in terms of certain inequalities which are checked via a finite number of calculations on a digital computer. By exploiting the analytic category we are able to obtain mathematically rigorous bounds on the jets of the manifolds, as well as on the derivatives of the manifolds with respect to the parameter. A number of example computations are given.

1. Introduction. The existence and geometry of stable and unstable manifolds plays a central role in the qualitative theory of dynamical systems. The intersection of these manifolds gives rise to connecting orbits, and under some additional assumptions to periodic orbits and chaotic motions as well. For a parameterized family of dynamical systems the study of how the manifolds and their intersections vary with parameters illuminates the transition from regular to chaotic dynamics, bifurcations of connecting orbits, the location of separatrices, and the global structure of attractors. Over the last several decades substantial work has gone into developing numerical methods for studying invariant manifolds. A brief survey of the literature is found in Section 1.1.

The present work is concerned with high order approximation of local stable/unstable manifolds for a one parameter family of analytic dynamical systems. We develop a constructive method for computing the Taylor polynomials of these invariant manifolds to arbitrarily high order in both the dynamical variables and the parameter. We also develop analytical tools which allow us to obtain rigorous computer assisted error bounds on the truncation errors associated with the polynomial approximations. These results are formulated in the analytic category, and rely on functional analytic arguments in some Banach spaces of candidate error functions.

Proceeding somewhat informally, let $f\colon \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ be an analytic family of vector fields. Our approach is based on the *Parameterization Method* of [13, 14, 15], so that we study the family of first order non-linear partial differential equations

$$f[P(\theta, \omega), \omega] = D_1 P(\theta, \omega)\, \Lambda(\omega)\theta, \tag{1.1}$$

subject to the constraints

$$P(0, \omega) = p(\omega), \qquad D_1 P(0, \omega) = A(\omega). \tag{1.2}$$

Here $p(\omega)$ is a family of hyperbolic equilibria, and we assume that for each ω the differential is diagonalizable. Then we take $\Lambda(\omega)$ to be the one parameter family of matrices whose diagonal entries are the stable eigenvalues (and off diagonal entries are zero), and $A(\omega)$ is a family of matrices whose columns are stable eigenvectors. If P is a solution of Equation (1.1), then the image of $P(\cdot, \omega)$ is a local stable manifold

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for $p(\omega)$ for each ω. Moreover, under the assumption of some mild non-resonance relationships between the stable eigenvalues, a unique solution to Equation (1.1) exists (and is unique up to the choice of the scalings of the eigenvectors). See [13, 15] and Section 2.2.

In the present work we develop a formal power series expansion for the parameterization P. Then we are able to compute recursively a polynomial approximation P_{MN} of P to any desired finite order in N in the phase space/dynamical variables and any finite order M in the family parameter. Once the polynomial approximation is fixed we obtain a mathematically rigorous bound on the truncation error. The following is a 'meta' statement of our main result for vector fields.

THEOREM 1.1 (Theorem (4.7) Paraphrased). *Assume that P_{MN} is a "good enough" and "properly constructed" approximate solution of Equation (1.1). Then there is a $\delta > 0$ and a unique true solution P of equation (1.1) so that*

$$\|P - P_{MN}\| \leq \delta.$$

Of course care must be taken in order to make precise the terms "good enough" and "properly constructed", and also to ensure that δ is in fact "small". For the moment we only remark that "good enough" is defined in terms of an *a-posteriori* indicator. More precisely we define the 'defect' associated with the approximation P_{MN} by considering

$$\|f[P_{MN}(\theta,\omega),\omega] - D_1 P_{MN}(\theta,\omega)\Lambda(\omega)\theta\| = \epsilon,$$

in an appropriate norm. We bound this defect rigorously using a computer, and in the full version of Theorem (1.1) the constant δ is made explicitly proportional to the numerical bound on ϵ. In this sense the present work generalizes the constructive, rigorous, a-posteriori numerical methods of [5, 42] to vector fields depending on a parameter.

We also consider $f\colon \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ a family of analytic diffeomorphisms, with $p(\omega)$ a family of hyperbolic fixed points, and $\Lambda(\omega)$, $A(\omega)$ as before. Again we follow [13, 14, 15] and study the nonlinear operator equation

$$f[P(\theta,\omega),\omega] = P[\Lambda(\omega)\theta,\omega], \tag{1.3}$$

subject to the first order constraints

$$P(0,\omega) = p(\omega), \qquad D_1 P(0,\omega) = A(\omega). \tag{1.4}$$

If P is a solution of Equation (1.3) then the image of P is a family of local stable manifolds for the family of fixed points, and again we have that there exists a unique solution to Equation (1.3) assuming only some mild non-resonance relations between the stable eigenvalues (see again [13, 14, 15] and below for fuller discussion). We develop a formalism for approximating solutions of Equation (1.3) as well as a precise version of Meta-Theorem (1.1) for diffeomorphisms. This generalizes the work of [54] to parameter dependent families of discrete time dynamical systems.

REMARK 1.2 (Data Structures for Analytic Functions). In Equations (1.1) and (1.3) the terms $p(\omega)$, $\Lambda(\omega)$, and $A(\omega)$, i.e. the analytic families of fixed points/equilibria, eigenvalues, and eigenvectors appear as known quantities. Before any attempt is made to solve the equations themselves it is essential that we develop appropriate representations of this 'linear data'. Since our goal is to develop polynomial approximations of the families of invariant manifolds with rigorous remainders, we require the same kind of representation of the linear data, i.e. we must be able to compute polynomial expansions of one parameter families of fixed points/equilibria eigenvalues, and eigenvectors.

Throughout this work we discretize functions using neighborhoods in function space about fixed taylor polynomials. We represent such a neighborhood by a polynomial with interval coefficients, a floating point number describing the radius of convergence of the function, and a floating point bound on the truncation error. This data structure is often referred to in the literature as a Taylor Model. In principle (if not in name) the use of Taylor models in computer assisted proofs appears in the literature as early as the works of [22, 23, 26, 45, 46] on universality, renormalization, and the Feigenbaum conjectures. We remark that these works appear to inaugurate the birth of the field of computer assisted proof in dynamical systems. Taylor models were later developed independently (and so named) in the works of [48, 49, 6], leading to the development of the COSY Infinity software for computing and manipulating Taylor models [8]. In the present work we impose some extra regularity conditions on our Taylor Models so that we are able to control derivatives. This leads to the notion of an "analytic Taylor model", and we discuss this notion more formally in Section 2.4.

At present we remark that all the computations discussed in the present work have been implemented numerically for some specific example systems. The computations run under the MatLab/IntLab environment for interval arithmetic. This source code is found at [56]. In order to support the computations discussed in the present work it was convenient to implemented a basic analytic Taylor Model library in the IntLab environment. A "user guide" supporting the analytic Taylor Model library and also describing the applications discussed in the present work is provided by the author and is found at [55]. In the guide we discuss methods for computing mathematically rigorous analytic Taylor models for equilibria/fixed points, eigenvalues, eigenvectors, and several other quantities needed throughout the present work. While similar work has appeared in the literature the guide is provided so that the present work is self-contained, well documented, and reproducible.

Remark 1.3 (Rigorous Computation of Jets). In the work of [42, 5, 54] we find methods for obtaining computer assisted error bounds for polynomial approximations of stable/unstable manifolds of analytic vector fields and diffeomorphism (albeit at fixed parameter values). An important feature of these methods is that in addition to obtaining rigorous C_0 bounds on the truncation error, one actually obtains that the truncation error is an analytic function. Having a representation of the truncation error as a bounded analytic function allows one to bound derivatives of the truncation error using classical estimates of complex analysis in exchange for shrinking the domain of the function. Having some control over the derivatives of the truncation error is essential in certain applications to computer assisted proof of the existence of connecting orbits, chaotic motions, etc. (See also Remark 1.5) This control of derivatives may also be valuable for developing numerical globalization schemes for computing invariant manifolds (see also Section 1.1).

Remark 1.4 (Other Validated Computations for One Parameter Branches of Local Stable/Unstable Manifolds). Part of the present work (the portion pertaining to on parameter families of vector fields) is closely related to the work of *[2]*. There the authors develop polynomial approximations to one parameter families of stable/unstable manifolds for the purpose of proving the existence of a homoclinic tangency in a certain model of cardiac muscle. A difference between the present work and the work of [2] is in the formulation of the fixed point problem which determines the truncation error. The remark is technical but important as each method has its strengths. In [2] the second order Taylor remainder of the vector field f *about the equilibria* is exploited in order to obtain a contraction mapping. This simplifies the error analysis while imposing the constraint that the image of the polynomial approximation lie inside the neighborhood of the origin where the remainder is a good approximation. In the present work we follow [5, 54] and formulate the contraction mapping problem for the truncation error in terms of the second order Taylor remainder of the vector field (or diffeomorphism) f *about the image of the polynomial approximation* P_{MN} *itself.* This results in a-posteriori theorems which we are able to apply for high order polynomial approximations in larger neighborhoods of the origin, however it also complicates both the computations and analysis.

Remark 1.5 (Computer Assisted Proofs for Connecting Orbits). A classical method for numerical computation of connecting orbits in discrete and continuous time dynamical systems is the method of projected boundary conditions [9, 11, 12, 24, 25]. The idea here is to reformulate the connecting orbits as the solution of a (finite time) boundary value problem. So instead of looking for an orbit with prescribed asymptotics we look instead for an orbit which begins on an unstable manifold and ends on a stable manifold. Recently some authors have developed validated numerical schemes for these boundary value problems which lead to computer assisted proof of the existence of connecting orbits. See for example [5, 54, 2, 29] and especially the references therein. It is worth mentioning that in the references just mentioned the boundary conditions are formulated in terms of chart maps for the stable and unstable manifolds. These chart maps must be computed rigorously and the Parameterization Method proves to be an excellent tool for carrying out this analysis.

A natural extension of the methods just mentioned it to combine them with an infinite dimensional continuation method such as [61, 28]. By combining the methods of [5, 54, 2, 29] with the methods of [61, 28] it should be possible to study rigorously one parameter branches of connecting orbits. However in order to cary out this analysis it will be essential to to control the boundary conditions, and even derivatives of the boundary conditions, with respect to parameter. Since the boundary conditions are formulated in terms of one parameter families of local stable/unstable manifolds, this reduces to the problem solved by the present work, and is in fact one of our primary motivations. We mention again the work of [2], which proves the existence of a homoclinic tangency for a differential equation. This can be seen as a bifurcation in the connecting orbit structure of the family of vector fields.

The remainder of the paper is organized as follows. In the next subsection we make some brief remarks about the literature. In Section (2) we establish some notation and recall some elementary notions of the theory of analytic functions of several complex variables. In Section (2.2) we formalize the problem studied in the remainder of the present work, establish the notation used throughout the remainder of the paper, and describe the overall solution strategy. In Section (2.3) we define a certain family of analytic function which we call 'one parameter families of analytic N-tails.' These comprise the main technical tools of our error analysis. In Section (2.4) we define the data structure which we use throughout the paper in order to model analytic functions on the computer.

Section (3) belongs to the study of certain operator equations on the Banach Space of one parameter families of analytic N-tails. In Section (3.1) we study a pair of linear operators on the space of one parameter families of analytic N-tails. These linear operators play a central role in our analysis of invariant manifolds in the sequel. Section (3.2) is devoted to an abstract non-linear equation on the space of one parameter families of analytic N-tails, and we prove an existence theorem. We also examine a concrete instantiation of this nonlinear equation which unifies our a-posteriori error analysis later in the paper.

In Section (4) we treat the rigorous computation of one parameter families of stable/unstable manifolds. We begin by illustrating the formal computation of the coefficients for the polynomial approximation of the family of invariant manifolds. We discuss conditions which guarantee that the coefficients are formally well defined to all orders, and illustrate the computations for specific families of diffeomorphisms and differential equations. We focus on the examples of the classical Hénon map and the Lorenz differential equation. In Section (4.4) we provide a method which allows us to compute explicitly a parameter interval on which the formal solution converges. We think of the parameterization of the invariant manifold as a power series in the dynamical variables, whose coefficients are power series in the parameter. In Section (4.5) we show how to bound the truncation errors of a finite number of these coefficient power series. The remaining truncation error is now a one parameter family of analytic N-tails, and in Section (4.6) we apply the theory of Section (2) in order obtain the desired bound. The cases of maps and flows are studied separately.

Section (5) presents example computations with rigorous error bounds for the Hénon map and the Lorenz system. Specifically we compute one parameter branches of all four stable and unstable manifolds of the two fixed points of the Hénon map. Since the phase space of the map is two dimensional and all the (un)stable manifolds are one dimensional we can represent the resulting one parameter families of invariant manifolds graphically. We also discuss computations of the one parameter family of two dimensional stable manifolds at the origin of the Lorenz system. Since this results in polynomials of three variables we present only tabular results.

1.1. Computing Invariant Manifolds: A Brief Overview. The literature on numerical methods for computing stable/unstable manifolds is rich and a thorough review of the literature quite beyond the scope of the present work. The much more modest goal of this section is simply to point the interested reader in the direction of more complete coverage. With this in mind it seems reasonable to partition the computational literature based on two distinct but related concerns. The first concern is the computation of local stable/unstable manifolds, while the second concern is the development of methods for globalizing these local invariant objects.

The more classical and in many ways more difficult question is the second of these. Explicitly stated it is: given a good local approximation of the stable/unstable manifold how can we obtain good numerical approximations of the global manifold? The question is difficult as the global manifold is in general neither compact nor neatly embedded. Moreover the growth of the manifold depends in a highly nonlinear way on it's embedding in phase space and blindly iterating the local manifold often leads to very poor results. There are a number of sophisticated approaches to this problem and we refer the interested reader to the excellent review given in [47]. In addition to describing the authors own methods for globalizing (un)stable manifolds [47] also serves as an overview of the the entire field. We remark that since the methods discussed in [47] focus on globalization they often begin with the simplest representation of the local manifold possible, namely the linear approximation given by the eigenvectors.

The other end of the spectrum is occupied by methods for computing high order approximations of the local stable/unstable manifold itself. These methods are usually based on the fact that there are chart maps for the local manifold which solve an invariance equation, and solutions of this invariance equation are usually approximated well by polynomials (the polynomial approximations are easily generated either by some formal power matching scheme or by an iterative procedure such as used in the graph transform method). Some of the earliest numerical implementations of methods for computing local stable/unstable manifolds using high order polynomial approximation are found in [27, 10, 31]. In the mean time computational methods based on "automatic differentiation" have found wide application in the dynamical systems community and even a cursory overview is beyond the scope of the present work. We refer for example to [57, 60] for more systematic discussion of this topic. We also remark that the idea of exploiting an invariance equation in order to compute series approximations of the stable/unstable manifold has a history predating the digital computer by more than half a century, appearing as early as the seminal works of Poincaré, Lyapunov, and Darboux. We refer the reader to Appendix B of [15] for a historical overview.

Some more recent work has focused on the middle ground between the two extremes just mentioned. We mention for example the work of [34, 35], where the authors treat one and two dimensional invariant manifolds of maps and develop an adaptive globalization scheme which exploits Bézier curves and triangles. A feature of this work is that the authors pre-condition their globalization scheme with a high order approximation of the local invariant manifold based on the Parameterization Method.

This hybrid idea of combining sophisticated globalization schemes with high order representations of the local invariant manifold is a promising direction of future research. We remark that the the information about derivatives one can obtain using the Parameterization Method seems not to have been fully exploited in a globalization scheme. We also remark that some less sophisticated globalization schemes which also exploit the high order preconditioning given by the Parameterization Method were used in [53, 52] in order to compute some one dimensional invariant heteroclinic sets as well as to visualize the vortex dynamics and tangle generated by a certain family of volume preserving diffeomorphisms. These computations suggest that even a naive approach to globalizing the local manifold can produce quite good results when preconditioned with a good enough representation of the local invariant manifold, and make the idea of combining sophisticated globalization methods with high order preconditioning all the more enticing.

We also remark that application of the Parameterization Method is not limited to the study of stable/unstable manifolds. The method has been used to study hyperbolic invariant tori and their "whiskers" [38, 39, 37], stable and unstable manifolds of periodic orbits for differential equations [15], and to develop KAM arguments without action angle coordinates [51, 50]. Moreover this list is by no means exhaustive and the interested reader should consult the references just mentioned for a much more complete picture of the literature. In fact the key to the Parameterization Method is the existence of an invariance equation which one can study via analytic tools, and conjugacy and semi-conjugacy relations are fundamental in the description of qualitative dynamics. Seen in this light the growth of this literature in recent years is not surprising. Examples of numerical computations of invariant objects using generalizations of the Parameterization Method are found in [36, 16, 17]. Again the literature is substantial and this short discussion can only scratch the surface.

We conclude with some comments closer the the topic of the present work, namely the literature on computing mathematically rigorous enclosures of stable/unstable manifolds. Again the literature is rich, and begins as early as [58]. Even in the case of validated computation for stable/unstable manifolds we make no attempt at an exhaustive survey of the literature. The interested reader will find by consulting the references below many additional techniques, ideas, and applications.

We refer to the work of [18, 19, 20, 21, 33, 67] for a general theory of validated computation for stable/unstable (and other types of normally hyperbolic) invariant manifolds based on the topological notion of covering relations and cone conditions. The computer assisted topological arguments are carried out in phase space using polygonal elements. Efficient implementations of these methods are provided in the CAPD C++ library [7] for rigorous computation in dynamical systems. These tools have been used in order to rigorously study many problems in celestial mechanics and dynamical systems theory, and in addition to the references just cited we refer to [63, 32, 3, 4, 43, 44, 64] for more applications and discussion. We point out that these comments provide only the briefest of introductions to this active area of research.

Another general theory for computing rigorous enclosures of stable/unstable manifolds is found in the work of [66, 65, 48, 49, 41, 6]. These authors use the theory of Taylor Models in order to approximate the invariant manifolds, and then apply computer assisted arguments in order to obtain validated error bounds for these approximations. Error bounds for the manifold enclosure are obtained by making a high order covering argument in phase space (nonlinear polygonal coverings with cone conditions). The ideas are implemented in the high level COSY programing language [8] for validated Taylor Model computations with interval arithmetic, which has been applied to a number of applications in physics and engineering. Once again these are meant as introductory remarks and the interested reader should consult the works just cited for a proper discussion of this field.

Finally we refer briefly to the work of [42, 2, 5, 54] which are based on applying functional analytic arguments to the invariance equation for the local invariant manifold. These methods are most closely related to [13, 14, 15] and also to the present work. These works are discussed in some detail in the introduction above and we devote no more time to them here.

2. Background.

2.1. Spaces, Norms, and Theorems of Analysis. We endow $\mathbb{C}$ with the usual Euclidean norm $|z| = |x+iy| = \sqrt{x^2+y^2}$, and $\mathbb{C}^n$ with the max norm

$$|(z_1, \ldots, z_n)|_\infty = \max_{1 \leq i \leq n} |z_i|.$$

We sometime suppress the infinity subscript when it is clear from context weather z is a point or a vector. These norms induce the balls $B_r(z) = \{w \in \mathbb{C} : |w - z| < r\}$ in the complex plane, and the *poly-disks*

$$D_r(z) = \{w \in \mathbb{C}^n \ : \ |w - z|_\infty < r\},$$

in the complex vector space $\mathbb{C}^n$. We often write $B_r = B_r(0)$ and $D_r = D_r(0)$ to denote respectively balls and poly-disks centered at the origin.

Let $z \in \mathbb{C}^m$ and $f\colon D_r(z) \subset \mathbb{C}^m \to \mathbb{C}^n$ be an analytic function. Then we let

$$\|f\|_{D_r(z)} \equiv \sup_{|w-z|_\infty < r} |f(w)|_\infty,$$

denote the (componentwise) C^0 norm of f on $D_r(z)$. When it is clear from context what the domain is we often write simply $\|f\|_r$. This norm induces the usual Banach Space structure on the collection of bounded analytic functions on a given poly-disk. We denote this Banach Space by $C^\omega(D_r(z), \mathbb{C}^n)$.

Let $\tau > 0$ and B_τ denote the ball of radius τ about the origin in the complex plane $\mathbb{C}$. We are often interested in a one parameter family of analytic mappings $f\colon D_r(p_0) \times B_\tau \subset \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^n$. Here we employ the norm

$$\|f\|_{r,\tau} \equiv \sup_{|z-p_0|_\infty < r} \sup_{|\omega| < \tau} |f(z, \omega)|_\infty.$$

Again the collection of all such functions is a Banach Space under this norm.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach Spaces. Let $\|\cdot\|_\mathcal{X}$ and $\|\cdot\|_\mathcal{Y}$ denote the norms on these spaces. Suppose that $\mathfrak{L}\colon \mathcal{X} \to \mathcal{Y}$ is a linear operator between them. The norm of the linear operator $\mathfrak{L}$ is defined to be

$$\|\mathfrak{L}\|_{B(\mathcal{X},\mathcal{Y})} \equiv \sup_{\|w\|_\mathcal{X} = 1} \|\mathfrak{L}w\|_\mathcal{Y}.$$

If $\|\mathfrak{L}\|_{B(\mathcal{X},\mathcal{Y})} < \infty$ then we say that the linear operator is bounded. If $\mathfrak{L}$ is invertible and $\|\mathfrak{L}^{-1}\|_{B(\mathcal{Y},\mathcal{X})} < \infty$ then we say that the operator $\mathfrak{L}$ is boundedly invertible. If $\mathcal{X} = \mathcal{Y}$ then we simplify the notation by writing

$$\|\mathfrak{L}\|_{B(\mathcal{X},\mathcal{X})} = \|\mathfrak{L}\|_{B(\mathcal{X})}.$$

Now let A be a $k \times \ell$ matrix of fixed complex numbers. We denote the (i, j) entry of A by either $[A]_{i,j}$ or a_{ij}, depending on context. We take the norm of A to be the maximum of the sum of the absolute values of row entries, where the maximum is taken over all rows; i.e.

$$|A|_M \equiv \max_{1 \le i \le k} \sum_{j=1}^{\ell} |a_{ij}|.$$

When thinking of A as a linear operator from the (finite dimensional) Banach Space $\mathbb{C}^\ell$ to the (finite dimensional) Banach Space $\mathbb{C}^k$ (both endowed with the maximum norm on components) then $\|A\|_{B(\mathbb{C}^\ell,\mathbb{C}^k)} \le |A|_M$. This inequality gives an easy to compute bound for finite dimensional linear operators. When it is clear from context that A is a matrix we use the notation $|A|, |A|_M$ and even $\|A\|$ interchangabally.

Suppose that $A(\omega)$ is a $k \times \ell$ matrix whose entries $a_{ij}\colon B_\tau \subset \mathbb{C} \to \mathbb{C}$ are analytic functions of one complex variable, and that $v\colon B_\tau \subset \mathbb{C} \to \mathbb{C}^\ell$ is a 'column vector' of analytic functions of a single complex variable. Then $A(\omega)$ defines a linear operator $\mathfrak{L}\colon C^\omega(B_\tau, \mathbb{C}^\ell) \to C^\omega(B_\tau, \mathbb{C}^k)$ by the formula

$$\mathfrak{L}[v](\omega) = A(\omega)v(\omega).$$

The discussion of the preceding paragraphs makes it clear that we have

$$\|\mathfrak{L}\|_{B(C^\omega(B_\tau,\mathbb{C}^\ell), C^\omega(B_\tau,\mathbb{C}^k))} \le \sup_{|\omega| \le \tau} |A(\omega)|_M.$$

We write $\|A\|_\tau \equiv \|\mathfrak{L}\|_{B(C^\omega(B_\tau,\mathbb{C}^\ell),C^\omega(B_\tau,\mathbb{C}^k))}$, to simplify when it results in no confusion.

Consider again $p_0 \in \mathbb{C}^m$, $\tau, r > 0$, and take $f\colon D_r(p_0) \times B_\tau \subset \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^n$ an analytic function. Later in the paper $z \in B_r(p_0)$ is thought of as a "dynamical variable" and $\omega \in B_\tau$ is thought of as a parameter. When we consider the derivative of f with respect to z (and ω is fixed) we will denote this derivative (the $n \times n$ matrix of first order partial derivatives of f with respect to z) as $D_1 f(z, \omega)$. Let $\beta \in \mathbb{N}^n$ be a multi-index. Higher order partial derivatives (order $|\beta|$) of f_i, $1 \le i \le n$ with respect to z are denoted by

$$\partial_1^\beta f_i(z,\omega) = \frac{\partial^{|\beta|}}{\partial z_1^{\beta_1} \dots \partial z_n^{\beta_n}} f_i(z,\omega).$$

For n-th order partials with respect to ω we have simply $\partial_2^n f_i(z,\omega) = \partial^n/\partial\omega^n f_i(z,\omega)$.

Let $\alpha \in \mathbb{N}^k$ denote a *multi-index*, $m \in \mathbb{N}$ denote an integer index, $z \in \mathbb{C}^k$, and $a_{(\alpha,m)} \in \mathbb{C}^n$ be a complex number indexed by α and m. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_k)$ we let $|\alpha| = \alpha_1 + \dots + \alpha_k$ and $z^\alpha = z_1^{\alpha_1} \cdot \dots \cdot z_k^{\alpha_k}$. If f is analytic on $B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C}$ then we can write the power series for f as

$$f(z,\omega) = \sum_{|\alpha|=0}^{\infty} \sum_{m=0}^{\infty} a_{(\alpha,m)} \omega^m z^\alpha = \sum_{|\alpha|=0}^{\infty} a_\alpha(\omega) z^\alpha,$$

i.e. as a power series in z whose coefficients are power series in the parameter ω, and have that the series converges to the value of the function for any $|z|_\infty < r$ and $|\omega| < \tau$. Similarly, we denote by

$$f_{MN}(z,\omega) = \sum_{|\alpha|=0}^{N} \sum_{m=0}^{M} a_{(\alpha,m)} \omega^m z^\alpha,$$

a polynomial of degree N in several complex variables $z = (z_1, \dots, z_k)$ whose coefficients are polynomials of degree M in the single complex variable ω.

The next estimate follows directly from the Cauchy Theorem of Complex Analysis [1], and provides a bound on the sup norm of the derivative of an analytic function in terms of the sup norm of the function itself, albeit on a smaller disk. A proof which yields the constants given here is found in [54].

Lemma 2.1 (Cauchy Bounds). *Suppose that $f : D_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}^n$ is bounded and analytic. Then for any $0 < \sigma \le 1$ we have that*

$$\|\partial_i f\|_{\nu e^{-\sigma}} \le \frac{2\pi}{\nu\sigma} \|f\|_\nu \quad \textit{so that} \quad \|Df\|_{\nu e^{-\sigma}} \le \frac{2\pi m}{\nu\sigma} \|f\|_\nu. \tag{2.1}$$

Similarly we have the second order bounds

$$\|\partial_i \partial_j f\|_{\nu e^{-\sigma}} \le \frac{4\pi^2}{\nu^2\sigma^2} \|f\|_\nu \quad \textit{so that} \quad \|D^2 f\|_{\nu e^{-\sigma}} \le \frac{4\pi^2 m^2}{\nu^2\sigma^2} \|f\|_\nu. \tag{2.2}$$

2.2. Notation and Formal Problem Statement. Let $p_0 \in \mathbb{C}^n$, $\rho, \tau > 0$, and consider a one parameter family of analytic vector fields $f\colon D_\rho(p_0) \times B_\tau \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, also analytic with respect to parameter, which is uniformly bounded on

$D_\rho(p_0) \times B_\tau$. Suppose that p_0 is a hyperbolic equilibria of $f(z, 0)$, and that $\partial_\omega f(p_0, \omega)$ is not zero at $\omega = 0$. The implicit functions theorem actually tells us that is a branch of hyperbolic equilibria. However we are interested in constructive results and assume that the following additional data is given.

A1-flows: Assume that there is a $\tau > 0$ and an analytic function $p \colon B_\tau \to \mathbb{C}^n$ so that

$$f[p(\omega), \omega] = 0, \qquad \text{for all} \quad \omega \in B_\tau.$$

A2-flows: Assume that for each $\omega \in B_\tau$, $Df[p(\omega), \omega]$ is diagonalizable and hyperbolic in the sense of differential equations, so that there are $k \leq n$ stable eigenvalues, and $n - k$ unstable eigenvalues for each $\omega \in B_\tau$. We assume that $\lambda_i \colon B_\tau \to \mathbb{C}$, $1 \leq i \leq n$ are analytic functions with

$$\det\left(Df[p(\omega), \omega] - \lambda_i(\omega)\mathrm{Id}_n\right) = 0 \qquad \text{for all} \quad \omega \in B_\tau,$$

and $\mathrm{real}(\lambda_i(\omega)) < 0$ for $1 \leq i \leq k$, and $0 < \mathrm{real}(\lambda_i(\omega))$ for $k + 1 \leq i \leq n$. We assume that the eigenvalues are distinct, and undergo no bifurcations on B_τ. We let $\Lambda \colon B_\tau \to \mathrm{Mat}_{k\times k}(\mathbb{C})$ be the diagonal matrix of stable eigenvalues defined by

$$\Lambda(\omega) = \begin{pmatrix} \lambda_1(\omega) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k(\omega) \end{pmatrix}.$$

A3-flows: Assume that for $1 \leq i \leq n$ there are constants K_i, and analytic functions $\xi_i \colon B_\tau \to \mathbb{C}^n$ having

$$\left(Df[p(\omega), \omega] - \lambda_i(\omega)\mathrm{Id}_n\right)\xi_i(\omega) = 0, \quad \text{and} \quad \|\xi_i(\omega)\| = K_i \quad \text{for all} \;\; \omega \in B_\tau,$$

and each $1 \leq i \leq n$. (Here we use the Euclidean norm for the lengths of the eigenvectors). We call $\xi_1(\omega), \dots, \xi_k(\omega)$ the stable eigenvectors and $\xi_{k+1}(\omega), \dots, \xi_n(\omega)$ the unstable eigenvectors. Let $A \colon B_\tau \to \mathrm{Mat}_{n\times k}(\mathbb{C})$ denote the matrix of stable eigenvectors given by

$$A(\omega) = [\xi_1(\omega)| \dots |\xi_k(\omega)].$$

Under these hypotheses our goal is to solve Equation (1.1) under the constraints given by Equation (1.2). We now outline the main steps in our procedure.

- **Step 1:** Solve the Equation

$$f[P(\theta, \omega)] = D_1 P(\theta, \omega)\, \Lambda(\omega)\theta,$$

term by term in the sense of power series. This formal computation is carried out in Section 4.2 for the Hénon family of diffeomorphisms and in Section 4.3 for the Lorenz family of vector fields. This step leads to recurrence relations (also called homological equations) for the coefficients of the solution P and allows us to compute the desired polynomial approximation P_{MN} to any desired finite order.

- **Step 2:** Next we seek an analytic function $\mathrm{Error}(\theta, \omega)$ having that

$$P(\theta, \omega) = P_{MN}(\theta, \omega) + \mathrm{Error}(\theta, \omega),$$

is an exact solution of Equation (1.1) on some domain $D_\nu \times B_{\tau'} \subset \mathbb{C}^k \times \mathbb{C}$ with $\tau' \leq \tau$. In order to prove that there exists such a truncation error function we derive a nonlinear fixed point problem,

$$\Phi[\text{Error}](\theta, \omega) = \text{Error}(\theta, \omega),$$

related to Equation (1.1), whose solution is the desired function Error. This fixed point problem is derived in Section 4.6.1.

- **Step 3:** The last step is to show that, the operator Φ is a contraction. The fixed point operator Φ depends on the approximation P_{MN}, as well as on the sizes ν and τ' of the domain on which we hope to validate the error. Φ also depends indirectly on the vector field f, its branches of equilibria, eigenvalues, and eigenvectors, and the dimension of the problem. We find that Φ is a contraction only if certain nonlinear relationships hold between all of these quantities. Moreover the relationship between these quantities determine the the size of the ball in function space on which Φ is a contraction, and hence the bounds on the error function. This argument is carried out in detail in Section 4.6.1.

Similar considerations apply to discrete time dynamical systems. Let $f \colon D_\rho(p_0) \times B_\tau \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ be a one parameter family of analytic diffeomorphisms, and $p_0 \in \mathbb{C}^n$ be a hyperbolic fixed point of $f(z, 0)$ with $\partial_\omega f(0, \omega)$ not zero at $\omega = 0$. We make the following assumptions.

A1-maps: There is a $\tau > 0$ and an analytic function $p \colon B_\tau \to \mathbb{C}^n$ having

$$f[p(\omega), \omega] - p(\omega) = 0, \qquad \text{for all} \quad \omega \in B_\tau.$$

So p parameterizes a one parameter family of fixed points for f.

A2-maps: For each $\omega \in B_\tau$, $Df[p(\omega), \omega]$ is diagonalizable and hyperbolic in the sense of diffeomorphisms. Then there are $k \leq n$ stable eigenvalues, and $n-k$ unstable eigenvalues for each $\omega \in B_\tau$. Each of these eigenvalues is parameterized by a one parameter family of analytic functions $\lambda_i \colon B_\tau \to \mathbb{C}$, $1 \leq i \leq n$ with

$$\det\left(Df[p(\omega), \omega] - \lambda_i(\omega)\text{Id}_n\right) = 0 \qquad \text{for all} \quad \omega \in B_\tau.$$

Moreover for all $\omega \in B_\tau$ we have that $0 < |\lambda_i(\omega)| < 1$ for $1 \leq i \leq k$, and $1 < |\lambda_i(\omega)|$ for $k + i \leq i \leq n$. The eigenvalues are distinct, and undergo no bifurcations on B_τ. Let $\Lambda \colon B_\tau \to \text{Mat}_{k\times k}(\mathbb{C})$ be the diagonal matrix of stable eigenvalues defined by

$$\Lambda(\omega) = \left(\begin{array}{ccc} \lambda_1(\omega) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k(\omega) \end{array}\right).$$

A3-maps: (Same as **A3-flows**).

In this case our goal is to solve Equation (1.3) under the constraints given by Equation (1.4). This is done by a small modification of the argument described in steps $1-3$ above.

The cartoon solution given by steps $1-3$ above provides useful heuristic insight, but obscures a number of critical details. Since it is these details which account for much of the technical developments to follow, we make a few clarifying comments.

Remark 2.1. **Components of the Error Function:** Suppose that we write the true solution of Equation (1.1) as

$$P(\theta,\omega)=\sum_{|\alpha|=0}^{\infty}a_{\alpha}(\omega)\theta^{\alpha},$$

where for each α the coefficient function $a_\alpha(\omega)$ is an analytic function of one complex variable (the complex variable is the system parameter). Now write the approximate solution as

$$P_{MN}(\theta,\omega)=\sum_{|\alpha|=0}^{N}\left(\sum_{m=0}^{M}a_{(\alpha,m)}\omega^{m}\right)\theta^{\alpha}.$$

Then the truncation error $\mathrm{Error}(\theta,\omega)$ decomposes naturally into two separate components. First note that for each α there is an error associated with truncating $a_\alpha(\omega)$ at order M. In other words we have

$$a_{\alpha}(\omega)=\sum_{m=0}^{M}a_{(\alpha,m)}\omega^{m}+h_{\alpha}(\omega),$$

and we must bound the truncation error arising from the fact that we only compute a finite number of terms in ω for each of the indices with $0\le|\alpha|\le N$.

In addition to this there is the error introduced by the fact that we only approximate P to order N in θ. Then we have to take into account an error H having

$$P(\theta,\omega)=\sum_{|\alpha|=0}^{N}a_{\alpha}(\omega)\theta^{\alpha}+H(\theta,\omega).$$

Note that it is only H which is zero to high order in the variable θ. In the end then we have that

$$\mathrm{Error}(\theta,\omega)=\sum_{|\alpha|=0}^{N}h_{\alpha}(\omega)+H(\theta,\omega).$$

In truth it is difficult to formulate a contraction operator for $\mathrm{Error}(\theta,\omega)$. Instead we make separate arguments for the components $h_\alpha(\omega)$ and the term $H(\theta,\omega)$. We develop an a-posteriori arguments for the components h_α terms in Section 4.5, and apply this argument once for each α with $0\le|\alpha|\le N$. Then in Section 4.6 we derive a contraction mapping operator for H. We will see that the contraction mapping argument relies heavily on the fact that H is zero to high order in the dynamical variable θ.

Remark 2.2. **The Co-Homological Equations:** In the process of deriving the fixed point operator for the truncation error H we encounter certain linear operators which must be inverted and bounded in order to show that the fixed point operator is well defined. These linear equations are similar to the equation which arise in normal form and KAM theory, where they are usually referred to as "co-homological equations". In general the term co-homological equation is used to refer to the linear obstructions to solving semi-conjugacy problems in many branches of non-linear analysis.

In the present work we arrive at one co-homological equation for vector fields and one for diffeomorphisms. We present and solve these equations together in Section 3.1.

REMARK 2.3. **Abstract Fixed Point Problems:** While it is true that diffeomorphisms and vector fields lead to different linear co-homological equations, the subsequent nonlinear analysis of the fixed point problems are almost identical in both cases. For this reason we formulate and solve a certain abstract nonlinear operator equation in Section 3.2. This problem is formulated with an unspecified linear part, so that the abstract problem applies in both the vector field and the diffeomorphism case. This unifies the analysis in Sections 4.6.1 and 4.6.2.

2.3. Analytic N-Tails and One Parameter Families of Analytic-N Tails. In this section we define a class of functions which we use to model truncation errors.

DEFINITION 2.2. [Analytic N-Tails] An analytic function $h\colon D_\nu \subset \mathbb{C}^k \to \mathbb{C}^n$ is called an *analytic N-tail* if

$$h(0) = \partial_\alpha h(0) = 0, \qquad \text{for all} \qquad \alpha \in \mathbb{N}^k, \text{ with } |\alpha| <= N. \tag{2.3}$$

If the analytic N-tail h uniformly bounded on D_ν then we say that h is a *bounded analytic N-tail on* D_ν.

Given a disk D_ν the set of all bounded analytic N-tails on D_ν is a Banach Space under the supremum norm. A key fact is that a bounded analytic N-tail h on D_ν has a power series representation

$$h(z) = \sum_{|\alpha|=N+1}^{\infty} a_\alpha z^\alpha,$$

which converges for $|z| < \nu$. Analytic N-tail are zero to N-th order at the origin and we think of them as "small perturbation of the zero function", in the sense of power series. We have the following useful bounds.

LEMMA 2.4. *Let h be a bounded analytic N-tail on $D_\nu \subset \mathbb{C}^k$, and let Λ be a $k \times k$ diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ having $0 < |\lambda_i| < 1$ for $1 \leq i \leq k$. Let μ^* be any positive constant with $sup_{1\leq i\leq k}|\lambda_i| \leq \mu^*$. Then $(h \circ \Lambda)(z) = h(\Lambda z)$ is a bounded analytic N-tail on D_ν and*

$$\|h \circ \Lambda\|_\nu \leq (\mu^*)^{N+1} \|h\|_\nu. \tag{2.4}$$

See [5] (Lemma 3.2) for an elementary proof. Since the present work deals largely with one parameter families of analytic functions we introduce the following generalization.

DEFINITION 2.3. [One Parameter Family of Analytic N-Tails] We call an analytic function $H\colon D_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ a *one parameter family of bounded analytic N-tails* if $H(z, \omega)$ is an analytic N-tail on D_ν for each fixed $|\omega| \leq \tau$ and H is uniformly bounded on $D_\nu \times B_\tau$.

A family of analytic N-tails has that

$$H(0, \omega) = D_1^\alpha H(0, \omega) = 0 \quad \text{ for each } \quad \alpha \in \mathbb{N}^k,\ 1 \leq |\alpha| \leq N, \quad \text{and for all} \quad |\omega| \leq \tau,$$

and has power series expansion

$$H(z, \omega) = \sum_{|\alpha|=N+1}^{\infty} \sum_{m=0}^{\infty} a_{(\alpha,m)} \omega^m z^\alpha,$$

which converges uniformly to H for all $|z| < \nu, |omega| < \tau$. Let $\Lambda \colon B_\tau \subset \mathbb{C} \to \mathbb{C}^k$ be a diagonal matrix of analytic functions on B_τ and suppose that there is a positive $\mu^* \in \mathbb{R}$ so that

$$\sup_{|\omega|\leq\tau} |\Lambda(\omega)| \leq \mu^* < 1.$$

Lemma (2.4) applies uniformly to $\omega \in B_\tau$. Defining $(H \circ \Lambda)(z, \omega) \equiv H(\Lambda(\omega)z, \omega)$, we have that $H \circ \Lambda$ is an analytic N-tail in z for each fixed ω and

$$\|H \circ \Lambda\|_{\nu,\tau} \leq (\mu^*)^{N+1} \|H\|_{\nu,\tau}. \tag{2.5}$$

Let

$$\mathcal{H}^{k,n}_{\nu,\tau} = \left\{ H \colon D_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n \mid H \text{ is a one parameter family of bounded analytic } N \text{ tails} \right\}.$$

We note that this is a Banach Space under the supremum norm.

2.4. Analytic Taylor Models. The following defines the fundamental data structure needed in the sequel.

Definition 2.4. Let $r > 0$ and $f \colon B_r \subset \mathbb{C} \to \mathbb{C}$ be a bounded analytic function. An M-th order *Analytic Taylor Model of* f consists of:

1. A polynomial *with interval coefficients*

   $$f_M(\omega) = \sum_{m=0}^{M} a_m \omega^m,$$

   having that

   $$\frac{1}{m!} \frac{d^m}{d\omega^m} f(0) \in a_m,$$

   for each $0 \leq m \leq M$.
2. A number $\delta_f > 0$ so that

   $$\|f - f_M\|_r \leq \delta_f.$$

We denote an analytic Taylor model by the triple (f_M, r, δ_f). Let f^T_M denote the true M-th order Taylor polynomial for f and note that $f^T_M \subset f_M$ (meaning that each coefficient of the Taylor polynomial is enclosed by the corresponding interval coefficient of the polynomial f_M). Then the "tail" of f is an analytic function $h_f \colon B_r \to \mathbb{C}$, defined by $h_f \equiv f - f^T_M$, and having $\|h_f\|_r \leq \delta_f$. Indeed, h_f is an analytic M-tail and we have

$$f(\omega) = f^T_M(\omega) + h_f(\omega) \subset f_M(\omega) + h_f(\omega).$$

One could of course define multi-variable analytic Taylor models in an analogous fashion, however Definition (2.4) is sufficient for the present work.

We now state without proof some elementary properties of analytic Taylor models. More sophisticated operations are discussed in the user guide [55]. It is clear that analytic Taylor models form a vector space, and it is clear that we can easily consider vectors of analytic Taylor models.

Lemma 2.5 (Properties of Analytic Taylor Models). *Let (f_M, r, δ) be an analytic Taylor model. Then for any f which is analytic on B_r and which is enclosed by this analytic Taylor model we have that* *(a)* ***(Sup Bounds)*** *$\|f\|_r \le \sum_{m=0}^{M} |a_m| r^m + \delta$.*

(b) ***(Model of a Sum)*** *If (f_M, r, δ_f) and (g_M, r, δ_g) are analytic Taylor models on B_r then $(f_M + g_M, r, \delta_f + \delta_g)$ is an analytic Taylor model for $f + g$.*

(c) ***(Bound Away From Zero)*** *Suppose that $\tau > 0$ has that*

$$\tau \sum_{m=1}^{M} |a_m| \tau^{m-1} + \delta \le |a_0|.$$

Let $\tilde{C}$ be defined by

$$|a_0| - \tau \sum_{m=1}^{M} |a_m| \tau^{m-1} - \delta \equiv \tilde{C}.$$

Then

$$\left\| \frac{1}{f} \right\|_{\tau} \le \frac{1}{\tilde{C}}.$$

(d) ***(Analytic Taylor Model of the Derivative)*** *For any $0 < \sigma \le 1$ we have that*

$$\|f'\|_{re^{-\sigma}} \le \sum_{m=0}^{M-1} (m+1)|a_{m+1}| r^m + \frac{2\pi}{\sigma r} \delta_f,$$

by applying the Cauchy Bounds of Lemma (2.1). It follows that $(f'_{M-1},\ re^{-\sigma},\ 2\pi\delta_f/\sigma r)$ is an analytic Taylor model for f'. (Note that the domain of the new analytic Taylor model is reduced by a factor of $e^{-\sigma}$, the order of the polynomial approximation is reduced by one, and the bound on the truncation error is inverse proportional to the "loss of domain" parameter σ.

We also note that integrals of analytic Taylor Models can be defined in the obvious way, however we have no need for these in the present work.

Remark 2.6 (Taylor versus Analytic Taylor Models). Note that it is Lemma 2.5(c) which exploits the analytic category and justifies the specialized definitions of this section. A brief technical remark is in order. We note that the standard notion of Taylor Models are not usually required to have interval coefficients. Instead the coefficients are floating point numbers and the round off errors associated with operations between Taylor Models are reallocated into the error bound via a process referred to as "shrink wrapping" [41]. Shrink wrapping is a powerful tool for controlling the so called wrapping effect which tends to inflate the truncation term when a Taylor Model is propagated forward in time under some dynamical system. On the other hand shrink wrapping involves a sort of loss of regularity (or loss of information about derivatives) and in this regard it is clear why a standard C^0 Taylor Model should not be differentiated. In the standard formulation the tail term contains continuous functions which are not differentiable.

3. Operator Equations on the Space of One Parameter Families of Analytic N-Tails.

3.1. Two Linear Equations. We now consider certain linear equations on $\mathcal{H}_{\nu,\tau}^{k,n}$ which play a critical role in the a-posteriori truncation error analysis developed in Section (4.6.1). In the following discussion we take A to be an $n \times n$ matrix of analytic functions $a_{ij}\colon D_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$. We assume that $A(z,\omega)$ is invertible for each $|z| \leq \nu$, $|\omega| \leq \tau$ and that $A(0,\omega)$ is diagonalizable for each $|\omega| < \tau$. We let $\lambda_i\colon B_\tau \to \mathbb{C}$ with $1 \leq i \leq n$ denote parameterizations of the eigenvalues of $A(0,\omega)$. We assume that the eigenvalues vary analytically for $\omega \in B_\tau$ and that there are no bifurcations.

Take $Q\colon B_\tau \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ to be a parameterization of the diagonalizing transformation for $A(0,\omega)$. Then if we denote by $\Sigma\colon B_\tau \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ the matrix with $\lambda_i(\omega)$ as diagonal entries and zeros elsewhere then we have

$$A(0,\omega) = Q(\omega)\Sigma(\omega)Q^{-1}(\omega),$$

for each $\omega \in B_\tau$. Note that for each $\omega \in B_\tau$ the columns of $Q(\omega)$ are eigenvectors for $A(0,\omega)$.

Suppose that for each $\omega \in B_\tau$ the matrix $A(0,\omega)$ is hyperbolic in the sense of maps. We have already supposed that there are no eigenvalue bifurcations on B_τ, so the stability of $A(0,\omega)$ does not change on B_τ. Then there are $k \leq n$ stable eigenvalues. We order the eigenvalues so that the stable ones come first, i.e. we require that $0 < |\lambda_i(\omega)| < 1$ for $1 \leq i \leq k$. Let $\Lambda(\omega)$ denote the $k \times k$ matrix having the stable eigenvalues $\lambda_i(\omega)$ for $1 \leq i \leq k$ as diagonal entries and zeros as the off-diagonal entries.

Theorem 3.1 (Parameterization Co-Homological Equation for Maps). *Consider the linear operator on $\mathcal{H}_{\nu,\tau}^{k,n}$ given by*

$$\mathfrak{L}_{maps}[H](z,\omega) = A(z,\omega)H(z,\omega) - H[\Lambda(\omega)z,\omega]. \tag{3.1}$$

Assume that there are $0 < \mu^* < 1$ and $M > 0$ so that

$$\max_{1\leq i\leq k}\sup_{|\omega|\leq\tau}|\lambda_i(\omega)| \leq \mu^*,$$

and

$$\sup_{|\omega|\leq\tau}\sup_{|z|\leq\nu}|A^{-1}(z,\omega)| \leq M.$$

Assume in addition that N is any integer large enough that

$$M(\mu^*)^{N+1} < 1. \tag{3.2}$$

Then $\mathfrak{L}_{maps}$ is boundedly invertible. Moreover we have that

$$\|\mathfrak{L}_{maps}^{-1}\|_{\mathcal{H}_{\nu,\tau}^{k,n}} \leq \frac{M}{1 - M(\mu^{*N+1})}. \tag{3.3}$$

Proof: Let $E \in \mathcal{H}_{\nu,\tau}^{k,n}$ and consider the equation

$$A(z,\omega)H(z,\omega) - H[\Lambda(\omega)z,\omega] = E(z,\omega).$$

Inverting the linear operator (3.1) is equivalent to solving the equation above for any such E. We rewrite as

$$\begin{aligned} H(z,\omega) - L[H](z,\omega) &= [(I-L)H]\,(z,\omega) \\ &= A^{-1}(z,\omega)E(z,\omega), \end{aligned} \tag{3.4}$$

where $L\colon \mathcal{H}^{k,n}_{\nu,\tau} \to \mathcal{H}^{k,n}_{\nu,\tau}$ is the linear operator defined by

$$L[H](z,\omega) = A^{-1}(z,\omega)\,H[\Lambda(\omega)z,\omega].$$

Using the estimate given by Equation (2.5) we have that

$$\begin{aligned} \sup_{\|H\|=1} \|L[H]\|_{(\nu,\tau)} &\leq \sup_{\|H\|=1} \|A^{-1}[H\circ\Lambda]\|_{(\nu,\tau)} \\ &\leq \sup_{\|H\|=1} \|A^{-1}\|_{(\nu,\tau)}\|H\circ\Lambda\|_{(\nu,\tau)} \\ &\leq \sup_{\|H\|=1} M(\mu^*)^{N+1}\|H\|_{(\nu,\tau)} \\ &< 1. \end{aligned}$$

Then the Neumann Theorem gives that the operator defined by the left hand side of Equation (3.4) is boundedly invertible so that

$$H(z,\omega) = \left[(I-L)^{-1}\,A^{-1}E\right](z,\omega),$$

is the desired solution. In addition the Neumann Theorem gives

$$\|H\|_{(\nu,\tau)} \leq \frac{M}{1-(\mu^*)^{N+1}M}\|E\|_{(\mu,\tau)}.$$

Let $\mathfrak{L}^{-1}(E) \equiv H$ and take the sup over all E with norm one in order to obtain

$$\|\mathfrak{L}^{-1}_{\text{maps}}\|_{\mathcal{H}^{k,n}_{\nu,\tau}} \leq \frac{M}{1-(\mu^*)^{N+1}M},$$

as desired.

□

For the case of differential equations suppose that for each $\omega \in B_\tau$ the matrix $A(0,\omega)$ is hyperbolic in the sense of differential equations. Since we assume that there are no eigenvalue bifurcations on B_τ, we have that the stability of $A(0,\omega)$ does not change on B_τ. Then there are $k \leq n$ stable eigenvalues. We order the eigenvalues so that the stable ones come first, i.e. we require that $\text{real}\,[\lambda_i(\omega)] < 0$ for $1 \leq i \leq k$. Let $\Lambda(\omega)$ denote the $k \times k$ matrix having the stable eigenvalues $\lambda_i(\omega)$ for $1 \leq i \leq k$ as diagonal entries and zeros as the off-diagonal entries.

Theorem 3.2 (Parameterization Co-Homological Equation for Vector Fields). *Consider the linear operator on $\mathcal{H}^{k,n}_{\nu,\tau}$ defined by*

$$\mathfrak{L}_{\mathit{flows}}[H](z,\omega) = D_1\,H(z,\omega)\,\Lambda(\omega)z - A(z,\omega)\,H(z,\omega). \tag{3.5}$$

Assume that M_1, M_2, μ_*, and μ^* are positive real constants having that

$$0 < \mu_* \leq \min_{1\leq i\leq k}\inf_{|\omega|\leq\tau} |\mathit{real}[\lambda_i(\omega)]| \leq \max_{1\leq i\leq k}\sup_{|\omega|\leq\tau} |\mathit{real}[\lambda_i(\omega)]| \leq \mu^* < \infty, \tag{3.6}$$

and that

$$\|Q\|_\tau \|Q^{-1}\|_\tau \leq M_1, \tag{3.7}$$

$$\sum_{|\alpha|=1}^{\infty} \sum_{m=0}^{\infty} \frac{|A_{(\alpha,m)}|}{\mu_* |\alpha|} \tau^m \nu^{|\alpha|} \leq M_2. \tag{3.8}$$

Assume in addition that N is any positive integer large enough that

$$(N+1)\mu_* \geq \mu^*. \tag{3.9}$$

Then $\mathfrak{L}_{\mathit{flows}}$ is a boundedly invertible linear operator with

$$\left\| \mathfrak{L}_{\mathit{flows}}^{-1} \right\|_{\mathcal{H}_{\nu,\tau}^{k,n}} \leq \frac{M_1 e^{M_2}}{(N+1)\mu_* - \mu^*}. \tag{3.10}$$

Proof: Let $E \in \mathcal{H}_{\nu,\tau}^{k;n}$ and note that inverting the operator given by Equation (3.5) is equivalent to solving the equation

$$D_1 \, H(z,\omega) \, \Lambda(\omega) z - A(z,\omega) \, H(z,\omega) = E(z,\omega), \tag{3.11}$$

for arbitrary E. We make a change of variables $z \to e^{\Lambda(\omega)t} z$ and define the analytic N-tails

$$x(t) = H\left(e^{\Lambda(\omega)t} z, \omega\right), \qquad \text{and} \qquad E(t) = E\left(e^{\Lambda(\omega)t} z, \omega\right),$$

and the matrix of analytic functions

$$A(t) = A\left(e^{\Lambda(\omega)t} z, \omega\right).$$

Consider the ordinary differential equation

$$\frac{d}{dt} x(t) - A(t) x(t) = E(t), \tag{3.12}$$

and note that if $x(t)$ solves Equation (3.12) the $x(0)$ solves Equation (3.11). We define the integrating factor

$$C(t) = \exp\left(-\int_0^t A(s)\, ds\right),$$

and have that

$$x(t) = -C^{-1}(t) \int_t^\infty C(s)\, E(s)\, ds,$$

solves Equation (3.12). Taking the limit as $t \to 0$ we define

$$\mathfrak{L}^{-1}[E](z,\omega) = H(z,\omega) = x(0) = -\int_0^\infty C(s) E(s)\, ds,$$

as the solution of Equation (3.11). The fact that H is an one parameter of analytic N-tails follows from the fact that E is.

In order to obtain bounds on $\mathfrak{L}^{-1}$ we first note by the definition of μ_* given in Equations (3.6) we have that

$$\left| e^{\Lambda(\omega)t} z \right| \leq e^{-\mu_* t} |z|,$$

for all $t > 0$, $\omega \in B_\tau$, and $z \in D_\nu$. Then, since E is a one parameter family of analytic N-tails, the estimates of Equations (2.5) give that

$$|E(t)| \leq \left\| E\left[e^{\Lambda(\omega)t} z, \omega \right] \right\|_{\nu,\tau} \leq e^{-(N+1)\mu_* t} \|E\|_{\nu,\tau}. \tag{3.13}$$

In order to bound the integrating factor we observe that

$$\begin{aligned}
-\int_0^t A(s)\, ds &= -\int_0^t \sum_{|\alpha|=0}^{\infty} \sum_{m=0}^{\infty} A_{(\alpha,m)} \omega^m \left[e^{\Lambda(\omega)s} z \right]^\alpha \, ds \\
&= -\sum_{|\alpha|=0}^{\infty} \sum_{m=0}^{\infty} A_{(\alpha,m)} \omega^m \left(\int_0^t e^{\langle \Lambda(\omega), \alpha \rangle s}\, ds \right) z^\alpha \\
&= Q(\omega)[-\Sigma(\omega)t] Q^{-1}(\omega) - \sum_{|\alpha|=1}^{\infty} \sum_{m=0}^{\infty} A_{(\alpha,m)} \frac{1 - e^{\langle \Lambda(\omega), \alpha \rangle t}}{|\langle \Lambda(\omega), \alpha \rangle|} \omega^m \, z^\alpha \, ds,
\end{aligned}$$

as the sums are uniformly bounded. Then

$$\|C(t)\| \leq \|Q\|_\tau \|Q^{-1}\|_\tau \exp(\mu^* t) \exp\left(\sum_{|\alpha|=1}^{\infty} \sum_{m=0}^{\infty} \frac{|A_{(\alpha,m)}|_M}{\mu_* |\alpha|} \right) \leq M_1 e^{M_2} e^{\mu^* t}.$$

We note that $\langle \Lambda(\omega), \alpha \rangle$ is never zero for $|\alpha| \geq 1$ by the assumption that the eigenvalues are non-zero for $\omega \in B_\tau$. Combining this with the estimate of Equation (3.13) as well as the assumption given by Equation (3.9) we obtain

$$\begin{aligned}
\left\| \mathfrak{L}^{-1}_{\text{flows}}[E] \right\|_{\mathcal{H}^{k,n}_{\nu,\tau}} &\leq \left\| -\int_0^\infty C(t) E(t) dt \right\| \\
&\leq M_1 e^{M_2} \int_0^\infty e^{-[(N+1)\mu_* - \mu^*]t} \|E\|_{\nu,\tau} \, dt \\
&\leq \frac{M_1 e^{M_2}}{(N+1)\mu_* - \mu^*} \|E\|_{\nu,\tau}.
\end{aligned}$$

Taking the sup over all E with norm one gives the estimate claimed in Equation (3.10).

□

3.2. A Nonlinear Operator Equation on $\mathcal{H}^{k,n}_{\nu,\tau}$. Let $\mathfrak{L}$ be a linear operator defined on $\mathcal{H}^{k;n}_{\nu,\tau}$. Assume in addition that $\mathfrak{L}$ is boundedly invertible and let $E \in \mathcal{H}^{k;n}_{\nu,\tau}$ be a fixed one parameter family of bounded analytic N-tails. Suppose that $T\colon D_s \times D_\nu \times B_\tau \subset \mathbb{C}^n \times \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$, with $k < n$ is analytic in all variables. Further we impose the condition that T is zero to second order at the origin in it's first variable and uniformly bounded in the remaining variables. More precisely suppose that there are $M_1, M_2 > 0$ so that for any $0 < \delta < s$ we have

R1:

$$\sup_{|\omega|\le\tau}\sup_{|\theta|\le\nu}\sup_{|z|\le\delta}|T(z,\theta,\omega)|\le M_1\delta^2,$$

R2:

$$\sup_{|\omega|\le\tau}\sup_{|\theta|\le\nu}\sup_{|z|\le\delta}|DT(z,\theta,\omega)|_M\le M_2\delta.$$

Given such T, E, and $\mathfrak{L}$ we are interested in the equation

$$\mathfrak{L}[H](\theta,\omega)=E(\theta,\omega)+T[H(\theta,\omega),\theta,\omega]=0, \tag{3.14}$$

on $\mathcal{H}^{k;n}_{\nu,\tau}$. The next theorem provides conditions under which we can uniquely solve the equation, as well as bounds on the resulting solution.

Theorem 3.3. *Let $T\colon D_s\times D_\nu\times B_\tau\subset\mathbb{C}^n\times\mathbb{C}^k\times\mathbb{C}\to\mathbb{C}^n$ be an analytic function which together with $M_1,M_2>0$ satisfying the estimates of **R1-R2**. Let $E\in\mathcal{H}^{k;n}_{\nu,\tau}$ be a fixed one parameter family of bounded analytic N-tails with $\|E\|_{\nu,\tau}\le\epsilon$. Let $\mathfrak{L}$ be a boundedly invertible linear operator with $\|\mathfrak{L}^{-1}\|_{\mathcal{H}^{k;n}_{\nu,\tau}}\le C$. Suppose that $0<\delta<s$ has that*

$$CM_1\delta^2-\delta+C\epsilon\le 0, \tag{3.15}$$

and

$$CM_2\delta<1. \tag{3.16}$$

Then equation (3.14) has a unique solution $H\in\mathcal{H}^{k;n}_{\nu,\tau}$ with $\|H\|_{\nu,\tau}\le\delta$.

If we think of ϵ as a "small parameter" then the theorem is saying that when ϵ is small enough we can solve Equation (3.14).

Proof: Since $\mathfrak{L}$ is invertible we define the nonlinear operator $\Phi\colon\mathcal{H}^{k;n}_{\nu,\tau}\to\mathcal{H}^{k;n}_{\nu,\tau}$ by

$$\Phi[H](\theta,\omega)=\mathfrak{L}^{-1}[E(\theta,\omega)+T(H(\theta,\omega),\theta,\omega)],$$

and note that H is a solution of Equation (3.14) if and only if H is a fixed point of Φ. Let

$$U_\delta=\left\{H\in\mathcal{H}^{k;n}_{\nu,\tau}\,|\,\|H\|_{\nu,\tau}\le\delta\right\},$$

and note that U_δ is a complete space. Then the theorem is established as soon as we show that Φ is a contraction mapping on U_δ. First we take $H\in U_\delta$ and consider

$$\begin{aligned}\|\Phi[H]\|_{\nu,\tau}&\le\|\mathfrak{L}^{-1}\|_{\mathcal{H}^{k;n}_{\nu,\tau}}\left(\|E\|_{\nu,\tau}+\|T[H]\|_{\nu,\tau}\right)\\&\le C(\epsilon+M_1\delta^2)\\&\le\delta,\end{aligned}$$

by **R1** and (3.15). Then Φ maps U_δ into itself.

Now let $H_1,H_2\in U_\delta$. By hypothesis we have that $\delta<s$ and we apply the mean value theorem to obtain that

$$\begin{aligned}\|\Phi(H_1)-\Phi(H_2)\|_{\nu,\tau}&\le\|\mathfrak{L}^{-1}\|_{\mathcal{H}^{k;n}_{\nu,\tau}}\|T(H_1)-T(H_2)\|_{\nu,\tau}\\&\le\|\mathfrak{L}^{-1}\|_{\mathcal{H}^{k;n}_{\nu,\tau}}\sup_{H\in U_\delta}\|DT(H)\|_{\mathcal{H}^{k;n}_{\nu,\tau}}\|H_1-H_2\|_{\nu,\tau}\\&\le C\sup_{|\omega|\le\tau}\sup_{|\theta|\le\nu}\sup_{|z|\le\delta}\|DT(z,\theta,\omega)\|_M\|H_1-H_2\|_{\nu,\tau}\\&\le CM_2\delta\|H_1-H_2\|_{\nu,\tau},\end{aligned}$$

by **R2**. That Φ is a contraction follows from Equation (3.16).

$\square$

The previous theorem allows us to solve certain non-linear operator equations which arise when we study the second order Taylor expansion of an analytic function about an analytic sub-manifold. To this end let $f\colon D_\rho(p_0) \times B_\tau \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, denote a one parameter family of analytic dynamical systems, and $P\colon D_\nu \times B_\tau \subset \mathbb{C}^k \times B_\tau \to \mathbb{C}^n$ with $k < n$ be an analytic function with image$(P) \subset D_{\rho'}(p_0)$ for some $\rho' < \rho$. Then for each $\omega \in B_\tau$, $P(\cdot, \omega)$ parameterizes an analytic sub-manifold which is contained in the interior of $D_{\rho'}(p_0)$, the domain of $f(\cdot, \omega)$. In fact the image of P is bounded away from the boundary of the domain of f by a know amount $\rho - \rho'$, uniformly in ω.

Choose $\omega \in B_\tau$, $z \in D_{\rho'}(p_0)$, and let $s = \rho - \rho'$. Then for any $\eta \in D_s \subset \mathbb{C}^n$ we have

$$f(z + \eta, \omega) = f(z, \omega) + Df(z, \omega)\eta + \tilde{R}(\eta, z, \omega) \tag{3.17}$$

where $\tilde{R}\colon D_s \to \mathbb{C}$ is analytic and $|R(\eta, z, \omega)|/|\eta| \to 0$ as $|\eta| \to 0$. In fact, since $\rho - |z| < s$ for any $z \in D_{\rho'}(p_0)$ (uniformly in ω) we have that $\tilde{R}\colon D_s \times D_{\rho'}(z) \times B_\tau \subset \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is an analytic function in all variables, and since image$(P) \subset D_{\rho'}(p_0)$ the function $\tilde{R}_P\colon D_s \times D_\nu \times B_\tau \subset \mathbb{C}^n \times \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ defined by

$$R[\eta, \theta, \omega] = \tilde{R}[\eta, P(\theta, \omega), \omega], \tag{3.18}$$

is analytic in all variables. R is the second order Taylor remainder of f expanded about the image of P, uniformly in the parameter ω. If we stipulate some uniform bounds on the second derivatives of f in $D_\rho(p_0)$ then R satisfies assumptions **R1-R2** of Theorem (3.3). This discussion is formalized in the following theorem.

Note that in order to satisfy **R1-R2** we need bounds on both R and it's first derivative. While a standard Lagrange remainder argument gives bounds on the supremum of R in terms of derivatives of f, we use the Cauchy Estimates of Lemma (2.1) in order to obtain bounds on the derivatives. This requires giving up some portion of the domain $D_s \subset \mathbb{C}^n$, and accounts for the appearance of a factor of $e^{-1}s$ in the theorem. Also note that in the following theorem we exploit the fact that some of the second partial derivatives of f may be zero with respect to both z and ω. This is a problem dependent consideration which improves our bounds in practice.

Corollary 3.4. *Suppose that the linear operator $\mathfrak{L}$ defined on $\mathcal{H}^{k,n}_{\nu,\tau}$ is boundedly invertible linear operator with $\|\mathfrak{L}^{-1}\|_{\mathcal{H}^{k,n}_{\nu,\tau}} \le C$. Suppose that $f\colon D_\rho(p_0) \times B_\tau \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is a bounded family of analytic functions, that $\rho' < \rho$ and that $P\colon D_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ with $k < n$ is analytic with $|P(\theta, \omega) - p_0| \le \rho'$ for all $\omega \in B_\tau$, $\theta \in D_\nu$. Let $s = \rho - \rho'$.*

Define M_1 to be the number of second partial derivatives of $f(z, \omega)$ which are not identically zero (with respect to both z and ω), so more explicitly $M_1 \le n^2$ is given by

$$M_1 = \max_{1 \le j \le n} \mathit{card}\left(\left\{\beta \in \mathbb{N}^n \,\big|\, |\beta| = 2 \text{ and } \partial_1^\beta f_j(z, \omega) \not\equiv 0\right\}\right). \tag{3.19}$$

Suppose in addition that M_2 is any bound of the form

$$\sup_{|\omega| \le \tau} \sup_{|p_0 - z| \le \rho} \max_{1 \le i \le n} \max_{|\beta| = 2} |\partial_1^\beta f_i(z, \omega)| \le M_2.$$

Assume that $E \in \mathcal{H}^{k,n}_{\nu,\tau}$ with $\|E\|_{\nu,\tau} < \epsilon$ and that $0 < \delta < e^{-1}s$ is a positive number with

$$CM_1M_2\delta^2 - \delta + C\epsilon \leq 0, \tag{3.20}$$

and

$$2\pi enCM_1M_2\delta < 1. \tag{3.21}$$

Let $R\colon D_{se^{-1}} \times D_\nu \times B_\tau \subset \mathbb{C}^n \subset \times\mathbb{C} \to \mathbb{C}^n$ be defined by

$$R(\eta, \theta, \omega) = \tilde{R}(\eta, P(\theta, \omega), \omega) \quad \text{for all} \quad \eta \in D_{e^{-1}s}, \;\; \theta \in D_\nu, \;\; \omega \in B_\tau,$$

where $\tilde{R}$ second order Taylor remainder of f in $D_{\rho'}(p_0)$ as defined in defined Equations (3.17), and R is its restriction to the image of P. Then the equation

$$\mathfrak{L}[H](\theta, \omega) = E(\theta, \omega) + R[H(\theta, \omega), \theta, \omega],$$

has a unique solution $H \in U_\delta$.

Proof: Fix $(\theta, \omega) \in D_\nu \times B_\tau$, $z = P(\theta, \omega) \in D_{\rho'}(p_0)$, and take $\tilde{R}(\eta, z, \omega)$ to be as defined in Equation (3.17). For any $\eta \in D_s$ the Lagrange form of the Taylor Remainder gives the bound

$$\begin{aligned}
|R(\eta, z, \omega)| &\leq \max_{1 \leq i \leq n} \left| \sum_{|\beta|=2} \frac{2}{\beta!} \eta^\beta \int_0^1 (1-t) \partial_1^\beta f_i(z + t\eta, \omega)\, dt \right| \\
&\leq \max_{1 \leq i \leq n} \sum_{|\beta|=2} \frac{2}{\beta!} |\eta|^{|\beta|} \int_0^1 (1-t) |\partial_1^\beta f_i(z + t\eta, \omega)|\, dt \\
&\leq \max_{1 \leq i \leq n} \left(\sup_{|\omega| \leq \tau} \sup_{|w - p_0| \leq \rho} \partial_1^\beta |f_i(w, \omega)| \right) M_1 s^{|\beta|} \\
&\leq M_1 M_2 s^2.
\end{aligned}$$

Now suppose that $h \in D_\delta$ and define Define $\eta_h \in D_s$ by

$$h = \frac{\delta}{s}\eta_h.$$

Since $R(\cdot, z, \omega)$ and its first partial derivatives are zero at the origin, $R(\cdot, z, \omega)$ is analytic in all variables and is an analytic 2-tail in it's first variable. (These claims are established via Morera's theorem and the Leibniz rule). Lemma (2.4) yields

$$\begin{aligned}
|R(h, z, \omega)| &= \left| R\left(\frac{\delta}{s}\eta_h, z, \omega \right) \right| \\
&\leq \frac{\delta^2}{s^2} \left| R\left(\eta_h, z, \omega\right) \right| \\
&\leq \frac{\delta^2}{s^2} M_1 M_2 s^2 \\
&= M_1 M_2 \delta^2.
\end{aligned} \tag{3.22}$$

Let $H\colon D_\nu \times B_\tau \to \mathbb{C}^n$ be a one parameter family of bounded analytic N-tails with $\|H\|_{\nu,\tau} \leq \delta$. Since the bound of Equation (3.22) is uniform with respect to z, and since $P(\theta,\omega) \in D_{\rho'}(p_0)$ for all $\theta \in D_\nu$ and $\omega \in B_\tau$ we have that

$$\sup_{|\omega|\leq\tau} \sup_{|\theta|\leq\nu} |R[H(\theta,\omega),\theta,\omega]| = \|\tilde{R}[H(\theta,\omega),P(\theta,\omega),\omega]\|_{\nu,\tau} \leq M_1 M_2 \delta^2,$$

so R satisfies **R1**.

In order to bound the derivative R we again take $z = P(\theta,\omega)$ with $\theta \in D_\nu$ and $\omega \in B_\tau$, and now consider only $0 < \delta < e^{-1}s$. For any $0 < \sigma \leq 1$ define $t = \delta/(se^{-\sigma})$. Let $h \in D_\delta(z)$. Then define $h = t\hat{\eta}_h$ with $\hat{\eta}_h \in D_{e^{-\sigma}s}$. Since $D\tilde{R}(\cdot,z,\omega)$ is a matrix whose entries have constant term equal to zero (with respect to the first variable) we have that

$$\begin{aligned}
\left| D\tilde{R}\left(t\hat{\eta}_h,z,\omega\right)\right|_M &\leq t\left| D\tilde{R}\left(\hat{\eta}_h,z,\omega\right)\right|_M \\
&\leq \frac{\delta}{se^{-\sigma}} \sup_{|\eta|=e^{-\sigma}s} |D\tilde{R}(\hat{\eta}_h,z,\omega)|_M \\
&\leq \frac{\delta e^\sigma}{s}\left(\frac{2\pi n}{s\sigma}\sup_{|\eta|=s}|R(\hat{\eta}_h,z,\omega)|\right) \\
&\leq \delta\frac{2\pi n e^\sigma}{\sigma s^2} M_1 M_2 s^2, \qquad (3.23)
\end{aligned}$$

where we pass from line two to line three using the Cauchy Bound Lemma (2.1).

Let $H \in \mathcal{H}^{k,n}_{\nu,\tau}$ with $\|H\|_{\nu,\tau} \leq \delta$. We observe that the estimate of Equation (3.23) is uniform in z, that $P(\theta,\omega) \in D_{\rho'}(p_0)$, and that $\delta < e^{-1}s$. We also note that the estimate given by Equation (3.23) holds for all $0 < \sigma \leq 1$ and that e^σ/σ is minimized at $\sigma = 1$. Then we have the desired estimate

$$\sup_{|\omega|<\tau} \sup_{|\theta|<\nu} \|DR[H(\theta,\omega),\theta,\omega]\|_M \leq 2\pi e n M_1 M_2 \delta,$$

and R satisfies **R2** as well. Then the hypothesis of Equations (3.20) and (3.21) give that the bounds required by Equations (3.15) and (3.16) in the hypotheses of Theorem (3.3) apply. The corollary follows.

□

4. Parameterized Families of Invariant Manifolds.

4.1. Review of Formal Computation of Taylor Coefficients at a Single Fixed Parameter. Before we move on to the formalism for polynomial approximations of one parameter families of invariant manifolds it is highly instructive to recall the basic steps for the formal the computations at a single fixed parameter value. As discussed in the Introduction of the present work, the problem of finding a parameterization of the stable/unstable manifold of a fixed vector field f is equivalent to the problem of solving the partial differential equation

$$f[P(\theta)] = DP(\theta)\Lambda\theta, \qquad (4.1)$$

under the constraints that $P(0)$ an equilibria and that $DP(0)$ the matrix of stable/unstable eigenvectors. Here Λ is a numerical matrix of fixed complex numbers. (Namely the stable or unstable eigenvalues of the differential at the equilibria).

Moreover it is shown in [13, 15] that (under mild non-degeneracy conditions which are recalled momentarily) the coefficients a_α for $|\alpha| \geq 2$ of the power series solution $P(\theta) = \sum_{|\alpha|=0}^{\infty} a_\alpha \theta^\alpha$ themselves solve the *homological equation*

$$[Df(p_0) - (\alpha_1 \lambda_1 + \ldots + \alpha_k \lambda_k)\mathrm{Id}_n] a_\alpha = s_\alpha. \tag{4.2}$$

The equation is derived by a power matching scheme, and s_α is a nonlinear function of the the coefficients $a_{\alpha'}$ with $|\alpha'| < |\alpha|$. The from of the nonlinearity depends on the nonlinearity of f. Then Equation (4.2) is a matrix equation whose only unknown is a_α. For the specific example of the Lorenz system, an explicit formula the s_α associated with the two dimensional invariant manifolds of an equilibria is given by

$$s_{(n_1,n_2)} = \sum_{0<k+j<n_1+n_2} \left(\begin{array}{c} 0 \\ a^1_{(n_1-j,n_2-k)}\, a^3_{(j,k)} \\ -a^1_{(n_1-j,n_2-k)}\, a^2_{(j,k)} \end{array} \right), \tag{4.3}$$

for all two dimensional multi-indices (n_1, n_2) with $n_1+n_2 \geq 2$. See [29] for the details.

Similarly a parameterization of the stable/unstable manifold of a fixed point of a diffeomorphism f solves the problem

$$f[P(\theta)] = P(\Lambda\theta), \tag{4.4}$$

with $P(0)$ an equilibria and $DP(0)$ the matrix of stable/unstable eigenvalues. Again under mild non-degeneracy conditions, the coefficients a_α for $|\alpha| \geq 2$ of the power series solution $P(\theta) = \sum_{|\alpha|=0}^{\infty} a_\alpha \theta^\alpha$ solve the *homological equation*

$$[Df(p_0) - (\lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_k^{\alpha_k})\mathrm{Id}_n] a_\alpha = s_\alpha. \tag{4.5}$$

Again s_α is a non-linear function of the the coefficients $a_{\alpha'}$ with $|\alpha'| < |\alpha|$. For the Hénon map one can work out that the homological equation for a stable/unstable manifold is

$$\left(\begin{array}{cc} -2aa_0^1 - \lambda^n & 1 \\ b & -\lambda^n \end{array} \right) \left[\begin{array}{c} a_n^1 \\ a_n^2 \end{array} \right] = \left[\begin{array}{c} a \sum_{k=1}^{n-1} a^1_{n-k} a^1_k \\ 0 \end{array} \right]. \tag{4.6}$$

The explicit derivation of this equation is found in [27].

The following lemmas provide conditions under which we can define (at least formally) the chart maps parameterizing the stable/unstable manifolds discussed above. In other words we desire conditions under which we can solve the homological equations for every multi-index of order $|\alpha| \geq 2$. The proofs of the lemmas follow immediately from the discussion above (see also [13, 5, 54, 42]). The idea is that the left hand sides of the homological equations are characteristic equations for the differential at the fixed point/equilibria. Then the coefficient a_α fails to be defined if and only if the sum $(\alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k)$ (for flows) or the product $\lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_k^{\alpha_k}$ (for maps) is itself equal to an eigenvalue. Should equality occur for some multi-index $\alpha \in \mathbb{N}^k$, $|\alpha| \geq 2$ we say that there is a *resonance* of order α. Since $\lambda_1, \ldots, \lambda_k$ are eigenvalues of like stability, there are only a finite number of possible resonances, and no "small divisors". (This is in contrast to the situation in KAM/normal form theory where one encounters homological equations where λ_1, $1 \leq i \leq k$ may be a set eigenvalues of mixed or elliptic stability).

Lemma 4.1 (Existence of a Formal Solution of Equation (4.1)). *Assume that p_0 is an equilibria of an analytic vector field f and that $Df(p_0)$ is hyperbolic. Let $\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_n$ be the eigenvalues of $Df(p_0)$ and suppose that the first k eigenvalues are stable and the remaining $n-k$ eigenvalues are unstable (in the sense of differential equations). Define*

$$\mu_* = \min_{1 \leq i \leq k} |real(\lambda_i)|, \qquad \textit{and} \qquad \mu^* = \max_{1 \leq i \leq k} |real(\lambda_i)|.$$

Assume that for each $\alpha \in \mathbb{N}^k$ with $2 \leq |\alpha| \leq \lceil \mu^*/\mu_* \rceil$ we have that the non-resonance condition

$$\alpha_1 \lambda_1 + \ldots + \alpha_k \lambda_k \neq \lambda_i, \qquad 1 \leq i \leq k,$$

holds. Then the solution of Equation (4.1) is formally well defined to all orders.

Lemma 4.2 (Existence of a Formal Solution of Equation (4.4)). *Assume that p_0 is a fixed point of an analytic diffeomorphism f and that $Df(p_0)$ is hyperbolic. Let $\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_n$ be the k stable and $n-k$ unstable (in the sense of maps) eigenvalues of $Df(p_0)$. Define*

$$\mu_* = \min_{1 \leq i \leq k} |\lambda_i| \qquad \textit{and} \qquad \mu^* = \max_{1 \leq i \leq k} |\lambda_i|.$$

Assume that for each $\alpha \in \mathbb{N}^k$ with $2 \leq |\alpha| \leq \lceil \ln(\mu_*)/\ln(\mu^*) \rceil$ we have that the non-resonance condition

$$\lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_k^{\alpha_k} \neq \lambda_i, \qquad 1 \leq i \leq k,$$

holds. Then the solution of Equation (4.4) is formally well defined to all orders.

Remarks 4.3.

(A) **(Unstable Manifold Parameterization)** When considering the parameterization of an unstable manifold for differential equations, we apply Lemma (4.1) to $-f$. Since $-f$ has differential $-Df(p_0)$ the unstable eigenvalues of f become the stable eigenvalues of $-f$ at the same equilibria. Similarly when considering the parameterization of an unstable manifold for diffeomorphisms, we apply Lemma (4.2) to f^{-1}. We have that $Df^{-1}(p_0) = [Df(p_0)]^{-1}$, and since the differential is diagonalizable it follows that the reciprocals of the unstable eigenvalues of $Df(p_0)$ become the stable eigenvalues of $Df^{-1}(p_0)$.

(B) **(Systems With a Single Stable/Unstable Direction)** Suppose that $k = 1$ so that the system has only one stable eigendirection, and hence the stable manifold is one dimensional. Then the multi-indices are one dimensional (i.e. $\alpha = n \in \mathbb{N}$) and Equations (4.2) and (4.5) reduce to

$$[Df(p_0) - n\lambda \mathrm{Id}]a_n = s_n, \quad \text{and} \quad [Df(p_0) - \lambda^n \mathrm{Id}]a_n = s_n,$$

respectively. Then since $n \geq 2$ *and* λ is the only stable eigenvalue it is impossible to have either $n\lambda = \lambda$ (in the case of differential equations) or $\lambda^n = \lambda$ (in the case of maps). We conclude that in the case of one stable direction there are never resonances, and the parameterizations are formally defined to all orders. A similar remark holds for the case of a singe unstable direction.

(C) **(Real Systems With a Single Complex Stable/Unstable Direction)** Similarly if f is real, $k = 2$, and λ_1 is complex, then it follows that $\lambda_2 = \bar{\lambda}_1$. Considering a two dimensional multi-index $\alpha = (n_1, n_2) \in \mathbb{N}^2$ we see that both $n_1\lambda_1 + n_2\bar{\lambda_1} = \lambda_{1,2}$ (for differential equations) and $\lambda_1^{n_1} \cdot \bar{\lambda_1}^{n_2} = \lambda_{1,2}$ (for maps) are impossible. So here again there are no possible resonances.

4.2. Formal Solution of Equation (1.3) for the Hénon Map. In this section we develop the homological equation which defines the power series coefficients of the solution of Equation (1.3). These homological equations allow us to compute interval enclosures of the Taylor expansion P_{MN} to any desired order N in the dynamical variables, and M in the parameter.

Consider again the Hénon Family at the classical parameter values. At $\omega = 0$ choose p_0 one of the maps two fixed points and let λ_0 and ξ_0 be the stable eigenvalue and associated eigenvector of $Df(p_0, 0)$. Computation of analytic Taylor Models for

$$p(\omega) = \sum_{m=0}^{\infty} p_m \omega^m, \qquad \lambda(\omega) = \sum_{m=0}^{\infty} \lambda_m \omega^m \qquad \text{and} \qquad \xi(\omega) = \sum_{m=0}^{\infty} \xi_m \omega^m,$$

is discussed in detail in [55]. There we also find methods for computing analytic Taylor models for the inverse of the diagonalizing transformation

$$Q^{-1}(\omega) = \sum_{m=0}^{\infty} Q_m \omega^m,$$

(where $Q(\omega)$ is just the matrix whose columns are the analytic Taylor models of the eigenvectors), and also the powers

$$[\lambda(\omega)]^n = \sum_{m=0}^{\infty} \lambda_{(m,n)} \omega^m,$$

for each $1 \leq n \leq N$.

As mentioned in the Introduction (and proved in [15]) there exists an analytic branch of parameterizations $P(\theta, \omega)$ for the invariant stable/unstable manifold at p_0. We denote its unknown power series by

$$P(\theta, \omega) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_{(m,n)} \theta^n \omega^m.$$

For the Hénon family, the invariance equation reduces to

$$f[P(\theta, \omega), \omega] = P[\lambda(\omega)\theta, \omega]. \tag{4.7}$$

By imposing the linear constraints given by Equation (1.4) we have that $p_{(m,0)} = p_m$, $p_{(m,1)} = \xi_m$.

The coefficients $p_{(0,n)}$ are the coefficients of the parameterization when $\omega = 0$. These are computed by solving the homological equation for the Hénon map given by Equation (4.6). We obtain the equations for the coefficients $p_{(mn)}$ when $n \geq 2, m \geq 1$ by plugging the unknown power series representation for P into Equation (4.7) and matching like powers of ω and θ.

We expand the right hand side of Equation (4.7) and obtain

$$
\begin{aligned}
P[\lambda(\omega)\theta,\omega] &= \sum_{n=0}^{\infty} p_n(\omega)[\lambda(\omega)]^n\theta^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} p_{(m,n)}\omega^m\right)\left(\sum_{m=0}^{\infty}\lambda_{(m,n)}\omega^m\right)\theta^n \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\lambda_{(m-k,n)}\begin{bmatrix} p^1_{(k,n)} \\ p^2_{(k,n)} \end{bmatrix}\omega^m\theta^n. \qquad (4.8)
\end{aligned}
$$

Expanding the left hand side of Equation (4.7) as a power series gives

$$
f[P(\theta,\omega),\omega] = \begin{bmatrix} 1 + P_2(\theta,\omega) - a[P_1(\theta,\omega)]^2 \\ (b+\omega)P_1(\theta,\omega) \end{bmatrix},
$$

which we expand component-wise to obtain

$$
f[P(\theta,\omega),\omega]_1 = 1 + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p^2_{mn}\theta^n\omega^m
$$

$$
-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{m} a\, p^1_{(m-j,n-k)}p^1_{(j,k)}\theta^n\omega^m, \qquad (4.9)
$$

and

$$
f[P(\theta,\omega),\omega]_2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} bp^1_{mn}\omega^m\theta^n + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} p^1_{(m-1,n)}\omega^m\theta^n. \qquad (4.10)
$$

Now we equate the power series expressions for the left and right hand sides, match like powers, and isolate the highest order terms to obtain the homological equation

$$
\begin{bmatrix} -2ap^1_{(00)} - \lambda_0^n & 1 \\ b & \lambda_0^n \end{bmatrix}\begin{bmatrix} p^1_{(m,n)} \\ p^2_{(m,n)} \end{bmatrix} = \begin{bmatrix} s^1_{(m,n)} \\ s^2_{(m,n)} \end{bmatrix}, \qquad (4.11)
$$

where

$$
s^1_{(m,n)} = \sum_{j=0}^{m-1}\lambda_{(m-j,n)}p^1_{(j,n)} + \sum_{k=0}^{n}\sum_{j=0}^{m} a\ \delta_{(m-j,n-k)}\delta_{(j,k)}\ p^1_{(m-j,n-k)}p^1_{(j,k)},
$$

with

$$
\delta_{(i,\ell)} = \begin{cases} 0 & \text{if} \quad i = \ell = 0, \\ 1 & \text{otherwise} \end{cases}
$$

and

$$
s^2_{(m,n)} = -p^1_{(m-1,n)} + \sum_{j=0}^{m-1}\lambda_{(m-j,n)}p^2_{(j,n)},
$$

for $n \geq 2, m \geq 1$. Note that the $\delta_{(i,j)}$ is not the usual Kronecker delta, but that ours could be written as $\delta_{(i,j)} = \delta^{\mathrm{kr}}_{(i,j,0)}$ where δ^{kr} is the usual Kronecker delta. The purpose of this factor is to 'zero out' the terms which have been 'moved to the other side of the equation', namely the zeroth order terms.

4.3. Formal Solution of Equation (1.1) for the Lorenz System. We illustrate the formal computation for the one parameter branch of two dimensional stable manifolds based at the origin of the Lorenz System. Let

$$P(\theta, \omega) = P(\theta_1, \theta_2, \omega) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m=0}^{\infty} p_{(m,n_1,n_2)} \omega^m \theta_1^{n_1} \theta_2^{n_2},$$

denote the parameterization of the one parameter branch of two dimensional stable manifolds through the origin. Then P satisfies the functional equation

$$f[P(\theta_1, \theta_2, \omega), \omega] = [D_1 P(\theta_1, \theta_2, \omega)] \Lambda(\omega) \left[\begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right],$$

where

$$\Lambda(\omega) = \left[\begin{array}{cc} \lambda^1(\omega) & 0 \\ 0 & \lambda^2(\omega) \end{array} \right] = \sum_{m=0}^{\infty} \left[\begin{array}{cc} \lambda_m^1 & 0 \\ 0 & \lambda_m^2 \end{array} \right] \omega^m.$$

Since the origin is a fixed point for all ω the series expansion of $p(\omega)$ is trivial to all orders. Moreover since we take $\beta > 0$, we have that $\lambda^1(\omega) = -\beta$ and $\xi^1(\omega) = (0,0,1)$ are a stable eigenvalue/eigenvector pair for all ω. The remaining unstable eigenvalue/eigenvector pair $\lambda^2(\omega)$ and $\xi_2(\omega)$ do depend on ω. Computations of analytic Taylor models for $\lambda^2(\omega)$ and $\xi_2(\omega)$ are discussed in the user guide [55]. For now note that $p_{(m,0,0)} = 0$ for all $m \geq 0$, $p_{(m,1,0)} = \xi_m$ for all $m \geq 0$, $p_{(0,0,1)} = \xi_3 = (0,0,1)$, and $p_{(m,0,2)} = 0$ for all $m \geq 1$.

The $p_{(0,n_1,n_2)}$ coefficients are the coefficients for the two dimensional manifold in the $\omega = 0$ system. These are computed using the homological equation (4.2) with $\alpha = (n_1, n_2)$ a two dimensional multi-index and with the right hand side given by Equation (4.3). What remains is to compute the coefficients $p_{(n_1,n_2,m)}$ for $n_1 + n_2 \geq 2$ and $m \geq 1$. As in the previous example for the Hénon map we compute a recursive expression for the remaining coefficients by a power matching scheme and arrive at

$$\left[\begin{array}{ccc} -\sigma - (n_1\lambda_0^1 + n_2\lambda_0^2) & \sigma & 0 \\ \rho - a_{(00)}^3 & -1 - (n_1\lambda_0^1 + n_2\lambda_0^2) & -a_{(00)}^1 \\ a_{(00)}^2 & a_{(00)}^1 & -\beta - (n_1\lambda_0^1 + n_2\lambda_0^2) \end{array} \right] \left(\begin{array}{c} p_{(m,n_1,n_2)}^1 \\ p_{(m,n_1,n_2)}^2 \\ p_{(m,n_1,n_2)}^3 \end{array} \right)$$

$$= \left(\begin{array}{c} s_{(m,n_1,n_2)}^1 \\ s_{(m,n_1,n_2)}^2 \\ s_{(m,n_1,n_2)}^3 \end{array} \right), \tag{4.12}$$

where

$$s_{(m,n_1,n_2)}^1 = \sum_{k=0}^{m-1} \left[n_1\lambda_{m-k}^1 + n_2\lambda_{m-k}^2 \right] p_{(k,n_1,n_2)}^1$$

$$s_{(m,n_1,n_2)}^2 = -p_{(m-1,n_1,m_2)} + \sum_{k=0}^{m-1} \left[n_1\lambda_{m-k}^1 + n_2\lambda_{m-k}^2 \right] \; p_{(k,n_1,n_2)}^2$$

$$+\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{m}\bar{p}^1_{(m-k,n_1-i,n_2-j)}\bar{p}^3_{(kij)},$$

and

$$s^3_{(m,n_1,n_2)} = \sum_{k=0}^{m-1}\left[n_1\lambda^1_{m-k} + n_2\lambda^2_{m-k}\right]\ p^3_{(k,n_1,n_2)} - \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{m}\bar{p}^1_{(m-k,n_1-i,n_2-j)}\bar{p}^2_{(k,i,j)}.$$

Here $\bar{p^i}_{(j,k,\ell)} = \delta_{(j,k,\ell)}p^i_{(j,k,\ell)}$ for $i = 1,2,3$ and,

$$\delta_{(j,k,\ell)} = \begin{cases} 0 & \text{if} \quad j = k = l = 0, \\ 1 & \text{otherwise.} \end{cases}$$

(Again this is not precisely the Kronecker delta).

4.4. Analytic One Parameter Family of Non-Resonance Conditions: Necessary Conditions for Convergence of the Formal Solutions. We note that Equation (4.11), which is the homological equation defining the coefficients of the one parameter branch of chart maps for the stable/unstable manifolds of the Hénon map, has the form

$$[Df(p_0) - \lambda_0^n \mathrm{Id}]a_{(m,n)} = \hat{s}_{(m,n)},$$

where the characteristic matrix on the left-hand side is *exactly* the same matrix as in the left hand side of the homological equation (4.6) for the coefficients of the parameterization for the $\omega = 0$ system. So while the right hand sides of Equations (4.11) and (4.6) are different, we see that the coefficients of $P(\theta,\omega)$ are well defined under precisely the same conditions given in Lemma (4.2). We conclude that if the eigenvalues of the $\omega = 0$ system are non-resonant in the sense of Lemma (4.2), then the formal series for the one parameter branch of parameterizations is well defined to all orders. To put it another way: when we decide to compute a one parameter branch of invariant manifolds *we need impose no extra conditions* in order that the formal solution is well defined to all orders.

Similar comments are seen to apply for the Lorenz system by observing that the matrix on the left-hand-side of the homological Equation (4.12) is the same matrix as on the left-hand-side of the homological Equation for the $\omega = 0$ system of differential equations given by Equation (4.2). So again we see that the one parameter branch of parameterizations is formally well defined under precisely the conditions of Lemma (4.1).

These considerations give rise to an *a-priori* necessary condition on the radius of convergence of the formal series defined above. Namely, for a one parameter branch of invariant manifolds for differential equations we must find a $\tau > 0$ so that

$$\alpha_1\lambda_1(\omega) + \ldots + \alpha_k\lambda_k(\omega) \neq \lambda_i(\omega),$$

for all $2 \leq |\alpha| \leq \lceil \mu^*/\mu_* \rceil$, $1 \leq i \leq k$, and all $\omega \in B_\tau$. On the other hand, for a one parameter branch of invariant manifolds for diffeomorphisms we must find a $\tau > 0$ so that

$$[\lambda_1(\omega)]^{\alpha_1} \cdot \ldots \cdot [\lambda_k(\omega)]^{\alpha_k} \neq \lambda_i(\omega),$$

for all $2 \leq |\alpha| \leq \lceil \ln(\mu_*)/\ln(\mu^*) \rceil$, $1 \leq i \leq k$, and all $\omega \in B_\tau$.

We focus for the moment on the case of differential equations. Consider the analytic Taylor models

$$\lambda_i(\omega) = \left(\lambda_M^i(\omega), \tau_i, \delta_i\right), \qquad 1 \le i \le k,$$

for the eigenvalues at an equilibria of a one parameter family of analytic vector fields. Then there is a resonance at the parameter $\hat{\omega} \in B_\tau$ if and only if $\hat{\omega}$ has

$$\alpha_1 \lambda_1(\hat{\omega}) + \ldots + \alpha_k \lambda_k(\hat{\omega}) - \lambda_i(\hat{\omega}) = 0.$$

for some $2 \le |\alpha| \le \lceil \mu^*/\mu_* \rceil$. Under the assumptions of Lemma (4.1) we know that $\hat{\omega} \neq 0$. We now want to find a $\tau > 0$ so that if $\omega \in B_\tau$ then there are no solutions for any multi-index α with $2 \le |\alpha| \le \lceil \mu^*/\mu_* \rceil$. We know that there is such a τ by the implicit function theorem. However we require an explicit bound.

For any $\tau > 0$ we define the quantities

$$b_\alpha(\tau) \equiv \min_{1 \le i \le k} \inf_{|\omega| \le \tau} |\alpha_1 \lambda_1(\omega) + \ldots + \alpha_k \lambda_k(\omega) - \lambda_i(\omega)|.$$

Let $\lambda_M^i(\omega) = \sum_{m=0}^{M} \lambda_m^i \omega^m$, so that λ_m^i are the polynomial coefficients associated with $\lambda_i(\omega)$. Then we have the bound

$$b_\alpha(\tau) \ge \min_{1 \le i \le k} \left|\alpha_1 \lambda_0^1 + \ldots + \alpha_k \lambda_0^k - \lambda_0^i\right| - B_\alpha(\tau), \tag{4.13}$$

where

$$B_\alpha(\tau) \equiv \tau \sum_{m=1}^{M} \left|\alpha_1 \lambda_m^1 + \ldots + \alpha_k \lambda_m^k - \lambda_m^i\right| \tau^{m-1} + |\alpha_1 \delta_1 + \ldots + \alpha_k \delta_k + \delta_i|.$$

If

$$\left|\alpha_1 \lambda_0^1 + \ldots + \alpha_k \lambda_0^k - \lambda_0^i\right| > \alpha_1 \delta_1 + \ldots + \alpha_k \delta_k + \delta_i, \tag{4.14}$$

for each $1 \le i \le k$, then there exists a $\tau > 0$ so that $b_\alpha(\tau) > 0$ for every α. If we can find a $\tau > 0$ for which all of the Equations (4.14) holds for each multi-index α with $2 \le |\alpha| \le \lceil \mu^*/\mu_* \rceil$, then there are no resonances on B_τ with this choice of τ.

Since in the present work we consider only the two dimensional Hénon map with one stable and one unstable direction, there are no possible resonances. Then the only restrictions on the parameter domain come from assumptions **A1-A3(Maps)**; namely we must choose a B_τ so that for all $|\omega| \le \tau$ the differential is invertible and there are no eigenvalue bifurcations. If we were to consider the secondary equilibria (or "eyes") of the Lorenz System near the classical parameters then again there would be no possible resonances, as at the classic parameters the eyes have one stable direction and one complex unstable direction, and again there are no possible resonances.

On the other hand when we consider the stable manifold associated with the equilibria at the origin of the Lorenz System near the classical parameter values then the eigenvalues are real distinct and we must rule out any possible resonances. An numerical example is discussed in the user guide [55].

4.5. Validated Truncation Error for Coefficient Tails. Let

$$P_{MN}(\theta, \phi) = \sum_{|\alpha|=0}^{N} \sum_{m=0}^{M} a_{(\alpha,m)} \omega^m \theta^\alpha,$$

be the approximate solution of Equation (1.1). In this section we are interested in the error which is made when we approximate the full coefficients

$$a_\alpha(\omega) = \sum_{m=0}^{\infty} a_{(\alpha,m)}\omega^m,$$

by its M-th order Taylor expansion

$$a_\alpha^M(\omega) = \sum_{m=0}^{M} a_{(\alpha,m)}\omega^m.$$

The key to our argument is that $a_\alpha(\omega)$ is itself the solution of a certain operator equation, namely the one parameter family of homological equations at the α order given by

$$(Df[p(\omega),\omega] - (\alpha_1\lambda_1(\omega) + \ldots + \alpha_k\lambda_k(\omega))\mathrm{Id}_n)\, a_\alpha(\omega) = s_\alpha(\omega), \tag{4.15}$$

The following theorem provides the desired a-posteriori bounds on the truncation error.

Theorem 4.4 (h_α Bounds for Differential Equations). *Assume that $\lambda_i(\omega)$, $1 \leq i \leq k$ are non-resonant on B_τ in the sense of differential equations. Assume that $(\lambda_i^M, \tau, \delta_i)$, $1 \leq i \leq k$, $(s_\alpha^M, \tau, \delta_s)$, and (p_M, τ, δ_p) are analytic Taylor Models for the functions λ_i, s_α, and the fixed point branch p. Define $\delta_\Lambda = \max_i \delta_i$. Additionally let (A_M, τ, δ_A) be an analytic Taylor model of the differential of f at p having*

$$A(\omega) = Df[p(\omega),\omega] = Df[p_M(\omega),\omega] + H_A(\omega),$$

with $A_M = Df[p_M(\omega),\omega]$ an M-th order polynomial in ω with matrix coefficients, and $\|H_A\|_\tau \leq \delta_A$. Let $Q(\omega)$ be the matrix of eigenvectors for $Df[p(\omega),\omega]$.

Suppose that M_α is any positive constant with

$$\max_{1\leq i\leq k} \sup_{|\omega|\leq\tau} |\lambda_i(\omega) - \alpha_1\lambda_1(\omega) - \ldots - \alpha_k\lambda_k(\omega)|^{-1} \leq M_\alpha.$$

Let $a_\alpha^M(\omega)$ be the M-th order solution of Equation (4.15) obtained by solving the homological Equation (4.12). Define the a-posteriori error polynomial

$$E_M^\alpha(\omega) = s_\alpha^M(\omega) - \left(Df[p_M(\omega),\omega] - (\alpha_1\lambda_1^M(\omega) + \ldots + \alpha_k\lambda_k^M(\omega)Id_n\right) a_\alpha^M(\omega),$$

and the total a-posteriori error bound

$$\epsilon_\alpha = \|E_M^\alpha\|_\tau + \delta_s + (\delta_A + |\alpha|\delta_\Lambda)\|a_\alpha^M\|_\tau.$$

Then there is a unique analytic M-tail $h_\alpha \colon B_\tau \to \mathbb{C}^n$ so that $a_\alpha^M + h_\alpha = a_\alpha$ is the exact solution of Equation (4.15). Moreover we have the bound

$$\|h_\alpha\|_\tau \leq \|Q\|_\tau\, \|Q^{-1}\|_\tau M_\alpha \epsilon_\alpha.$$

Proof: Let $a_\alpha(\omega) = a_\alpha^M(\omega) + h_\alpha(\omega)$ where the function h_α is to be determined. We re-write Equation (4.15) as

$$(Df[p_M(\omega) + h_p(\omega),\omega] - \langle \Lambda_M(\omega) + H_\Lambda(\omega), \alpha\rangle\, \mathrm{Id})\left[a_\alpha^M(\omega) + h_\alpha(\omega)\right]$$

$$= s_\alpha^M(\omega) + h_s(\omega).$$

or

$$[Df[p(\omega),\omega] - \langle \Lambda(\omega), \alpha \rangle \, \mathrm{Id}] \, h_\alpha(\omega) =$$

$$E_M^\alpha + h_s(\omega) - [H_A(\omega) - \langle H_\Lambda(\omega), \alpha \rangle \, \mathrm{Id}] \, a_\alpha^M(\omega). \tag{4.16}$$

Let $\hat{E}(\omega)$ denote the right hand side of Equation (4.16) and note that $\hat{E}$ is an analytic M-tail with $\|\hat{E}\|_\tau \leq \epsilon_\alpha$. Utilizing the diagonalizing transformation $Q(\omega)$ we have

$$\left[Q(\omega)\Sigma(\omega)Q^{-1}(\omega) - \langle \Lambda(\omega), \alpha \rangle \, \mathrm{Id}\right] h_\alpha(\omega) = \hat{E}(\omega).$$

We now make the change of variables

$$Q(\omega) w_\alpha(\omega) = h_\alpha(\omega),$$

and re-write the equation as

$$(\Sigma(\omega) - \langle \Lambda(\omega), \alpha \rangle \, \mathrm{Id}) \, w_\alpha(\omega) = Q^{-1}(\omega)\hat{E}(\omega),$$

which is diagonalized. Under the assumption that the eigenvalues are a non-resonant branch, we obtain the component equations

$$[w_\alpha(\omega)]_j = \frac{1}{\lambda_j(\omega) - \alpha_1\lambda_1(\omega) - \ldots - \alpha_k\lambda_k(\omega)} [Q^{-1}(\omega)\hat{E}(\omega)]_j \qquad \text{for} \quad 1 \leq j \leq n.$$

Then w_α exists and is an analytic M-tail. Now since $h_\alpha = Q w_\alpha$ and we see that h_α is an analytic M-tail as desired. Moreover we have the estimate

$$\|h_\alpha\|_\tau \leq \|Q\|_\tau \max_{1 \leq j \leq n} \sup_{|\omega| \leq \tau} \left| \frac{1}{\lambda_j(\omega) - \alpha_1\lambda_1(\omega) - \ldots - \alpha_k\lambda_k(\omega)} [Q^{-1}(\omega)\hat{E}(\omega)]_j \right|$$

$$\leq \|Q\|_\tau \|Q^{-1}\|_\tau M_\alpha \epsilon_\alpha,$$

as desired.

$\square$

Note that the $M_\alpha = 1/b_\alpha(\tau)$, where $b_\alpha(\tau)$ is defined as in Section (4.4) is used to obtain the M_α bounds in practice. Similar considerations apply for diffeomorphisms and lead to the following theorem.

THEOREM 4.5 (h_α Bounds for Diffeomorphisms). *Assume that $\lambda_i(\omega)$, $1 \leq i \leq k$ are non-resonant on B_τ in the sense of diffeomorphisms. Assume that we have analytic Taylor model representations $(\lambda_i^M, \tau, \delta_i)$, $1 \leq i \leq k$, $(s_\alpha^M, \tau, \delta_s)$, and (p_M, τ, δ_p) respectively for the functions λ_i, s_α and the fixed point branch p. Define $\delta_\Lambda = \max_i \delta_i$. Let (A_M, τ, δ_A) be an analytic Taylor model of the differential of f at p having*

$$A(\omega) = Df[p(\omega), \omega] = Df[p_M(\omega), \omega] + H_A(\omega),$$

with $\|H_A\|_\tau \leq \delta_A$. We also assume that $(\Lambda_M^\alpha, \tau, \delta_{\Lambda^\alpha})$ is an analytic Taylor model for the scalar function $\Lambda^\alpha(\omega)$. Let $Q(\omega)$ be the matrix of eigenvectors of $Df[p(\omega), \omega]$.

Define

$$M_\alpha = \max_{1\le i\le k} \sup_{|\omega|\le\tau} \left|\lambda_i(\omega) - \lambda_1^{\alpha_1}(\omega)\cdot\ldots\cdot\lambda_k^{\alpha_k}(\omega)\right|^{-1}.$$

Let $a_\alpha^M(\omega)$ be the M-th order solution of Equation (4.15) obtained by solving the homological equations. Define the a-posteriori error polynomial

$$E_M^\alpha(\omega) = s_\alpha^M(\omega) - \left(Df[p_M(\omega),\omega] - \Lambda_M^\alpha(\omega) Id_n\right) a_\alpha^M(\omega),$$

and the total a-posteriori error bound

$$\epsilon_\alpha = \|E_M^\alpha\|_\tau + \delta_s + (\delta_A + \delta_{\Lambda^\alpha})\|a_\alpha^M\|_\tau.$$

*Then there is a unique analytic M-tail $h_\alpha\colon B_\tau \to \mathbb{C}^n$ so that $a_\alpha^M + h_\alpha = a_\alpha$ is the exact solution of Equation (**??**). Moreover we have the bound*

$$\|h_\alpha\|_\tau \le \|Q\|_\tau \, \|Q^{-1}\|_\tau M_\alpha \epsilon_\alpha.$$

The proof is almost identical to the proof of Theorem (4.4.)

Remark 4.6 (Computational Cost of Computing δ_α). The theorems say that in order to bound h_α we must compute the a-posteriori error polynomial E_M^α, as well as the sigma-norms of E_M^α and a_α^M. Note that the cost of computing E_M^α in both cases is the cost of a Cauchy product of two polynomials of order M. The cost of evaluating the sigma norms are the cost of an inner product.

4.6. A-Posteriori Analysis of The Full Truncation Error. In this section we state and prove the main theorems of the paper; one theorem for flows and one for maps. Throughout the section we take f, ρ, ν, $p(\omega)$, $D_1 f[p(\omega),\omega]$, k, $\lambda_i(\omega)$ and $\xi_i(\omega)$ for $1 \le i \le k$, $\Lambda(\omega)$, and $A(\omega)$ to be as in either **A1-A3-flows** or **A1-A3-maps** from Section (1) depending on wether we are discussing differential equations or diffeomorphisms. In either case we assume that that $P_N\colon D_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ is an N-th order polynomial in $\theta \in D_\nu(0) \subset \mathbb{C}^k$ with coefficients analytic in the variable $\omega \in B_\tau(0) \subset \mathbb{C}$, so that P_N has power series expansion

$$P_N(\theta,\omega) = \sum_{|\alpha|=0}^{N} a_\alpha(\omega)\theta^\alpha = \sum_{|\alpha|=0}^{N}\sum_{m=0}^{\infty} a_{(m,\alpha)}\omega^m\theta^\alpha,$$

convergent on $D_\nu \times B_\tau$. Moreover suppose that P_N satisfies the first order constraints

$$P_N(0,\omega) = p(\omega), \quad \text{and} \quad D_1 P_N(0,\omega) = [\xi_1(\omega)|\ldots|\xi_k(\omega)].$$

Suppose that the power series of the differential

$$Df[P_N(\theta,\omega),\omega] = \sum_{|\alpha|=0}^{\infty}\sum_{m=0}^{\infty} A_{(\alpha,m)}\,\omega^m\,\theta^\alpha,$$

also converges on $D_\nu \times B_\tau$. Take $Q, Q^{-1}\colon B_\tau \subset \mathbb{C} \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ to be the transformations which diagonalize $Df[p(\omega),\omega]$; i.e. Q is the matrix whose columns are all of the stable and unstable eigenvectors, and Q^{-1} it's inverse.

Finally, assume that P_N solves either Equation (1.1) or Equation (1.3) in the case of diffeomorphisms exactly to N-th order in the sense of power series: i.e. suppose that the coefficients of P_N are exact solutions of the homological equations for a one parameter family of stable manifolds given by Equation (4.15) or its analogue for diffeomorphisms. In the next two subsections we consider the bounds on the remainder associated with P_N.

4.6.1. Differential Equations. Define the total a-posteriori error

$$E_N(\theta, \omega) = f[P_N(\theta, \omega), \omega] - D_1 P_N(\theta, \omega)\Lambda(\omega)\theta, \tag{4.17}$$

for the case of vector fields.

DEFINITION 4.1. [Validation Values for an N-th Order Solution of Equation (1.1)] A set of positive real constants, ϵ, ρ', μ_*, μ^*, M_1, M_2, C_1, and C_2 are called *validation values* for P_N if

(i): $\|E\|_{\nu,\tau} \leq \epsilon$,

(ii):

$$\min_{1\leq i\leq k} \inf_{|\omega|\leq\tau} \mathrm{real}\,(\lambda_i(\omega)) \leq \mu^* \quad \text{and} \quad \mu_* \leq \max_{1\leq i\leq k} \sup_{|\omega|\leq\tau} \mathrm{real}\,(\lambda_i(\omega))$$

(iii)

$$\|Q\|_\tau \|Q^{-1}\|_\tau \leq C_1,$$

(iv)

$$\sum_{|\alpha|=1}^{\infty} \sum_{m=0}^{\infty} \frac{|A_{(\alpha,m)}|_M}{\mu_* |\alpha|} \tau^m \nu^{|\alpha|} \leq C_2.$$

(v) M_1 is the number of second partial derivatives of $f_i(z, \omega)$, $1 \leq i \leq n$ with respect to z which are not identically zero for all z, and ω, and M_2 is any uniform bound of the form

$$\max_{1\leq i\leq n} \max_{|\beta|=2} \sup_{\omega\in B_\tau} \sup_{|z-p_0|\leq\rho} \left\| \partial_1^\beta f_i(z, \omega) \right\| \leq M_2.$$

(vi): ρ' has $0 < \rho' < \rho$, with

$$\sup_{|\theta|\leq\nu} \sup_{|\omega|} \|P_N(\theta, \omega) - P_N(0, 0)\| \leq \rho'.$$

so that for each $\omega \in B_\tau$ the image of P_N is contained in the interior of $D_\rho(p_0)$, the ball where we have bounds on the second derivatives of f.

THEOREM 4.7 (A-Posteriori Error for a One Parameter Branch of Stable Manifolds for an Equilibria of a Vector Field). *Suppose that ϵ, ρ', μ_*, μ^*, M_1, M_2, C_1, and C_2 are validation values for a one parameter branch of local stable manifolds at an equilibria of a vector field.*

Assume that $N \in \mathbb{N}$ and $\delta > 0$ are such that

- $$(N+1) > \frac{\mu^*}{\mu_*}, \tag{4.18}$$

- $$\delta < e^{-1} \ \min\left\{ \frac{(N+1)\mu_* - \mu^*}{2n\pi M_1 M_2 C_1 e^{C_2}}, \rho - \rho', \right\} \tag{4.19}$$

- *and*

$$\frac{2C_1 e^{C_2}}{(N+1)\mu_* - \mu^*}\epsilon < \delta. \tag{4.20}$$

Then there is a unique one parameter family of analytic N-tails $H\colon D_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ with

$$\|H\|_{\nu,\tau} \leq \delta,$$

so that

$$P(\theta,\omega) = P_N(\theta,\omega) + H(\theta,\omega),$$

is the exact solution of Equation (1.1) on $D_\nu \times B_\tau$.

Proof. We seek a one parameter family of bounded analytic N-tails so that

$$f[P_N(\theta,\omega) + H(\theta,\omega),\omega] = D_1[P_N(\theta,\omega) + H(\theta,\omega)]\Lambda(\omega)\theta \tag{4.21}$$

for all $(\theta,\omega) \in D_\nu \times B_\tau$. Image$(P_N) \subset B(p_0,\rho')$ and f is analytic on $B(p_0,\rho)$ so we can expand the left hand side of Equation (4.21) to second order and obtain

$$f[P_N(\theta,\omega)+H(\theta,\omega),\omega] = f[P_N(\theta,\omega)]+Df[P_N(\theta,\omega)]H(\theta,\omega)+R[H(\theta,\omega),P_N(\theta,\omega),\omega],$$

where R is the quadratic remainder term. Rearranging Equation (4.21) we have

$$D_1H(\theta,\omega)\Lambda(\omega)\theta - Df[P_N(\theta,\omega),\omega]H(\theta,\omega) = E_N(\theta,\omega) + R[H(\theta,\omega),P_N(\theta,\omega),\omega].$$

Letting

$$A(\theta,\omega) = Df[P_N(\theta,\omega),\omega],$$

and recalling the definition of $\mathfrak{L}_{\text{flow}}$ from Section (3.1) we note that this is

$$\mathfrak{L}_{\text{flow}}[H](\theta,\omega) = E_N(\theta,\omega) + R[H(\theta,\omega),P_N(\theta,\omega),\omega], \tag{4.22}$$

which has the form of the non-linear operator equation considered in Corollary (3.4). An application of Lemma 3.10 shows that $\mathfrak{L}_{\text{flow}}$ is boundedly invertible, and gives the needed bound on the norm of the inverse. The proof follows by a straight forward application of Corollary 3.4. □

4.6.2. Maps. Define the total a-posteriori error

$$E_N(\theta,\omega) = f[P_N(\theta,\omega),\omega] - P_N(\Lambda(\omega)\theta,\omega), \tag{4.23}$$

for the case of diffeomorphisms.

Definition 4.2. [Validation Values for an N-th Order Solution of Equation (1.3)] A set of positive real constants, ϵ, ρ', μ_*, μ^*, M_1, M_2, and C are called *validation values* for P_N if

(i): $\|E\|_{\nu,\tau} \leq \epsilon$,

(ii):

$$0 < \mu_* \leq \min_{1\leq i\leq k} \inf_{|\omega|\leq\tau} |\lambda_i(\omega)|, \quad \text{and} \quad \max_{1\leq i\leq k} \sup_{|\omega|\leq\tau} |\lambda_i(\omega)| \leq \mu^* < 1,$$

(iii)

$$\sum_{|\alpha|=1}^{\infty}\sum_{m=0}^{\infty}\frac{|A_{(\alpha,m)}|_M}{\mu_*|\alpha|}\tau^m\nu^{|\alpha|}\le C.$$

(iv) M_1 is the number of partial derivatives of $f(z,\omega)$ with respect to z which are not identically zero for all ω, and M_2 is any uniform bound of the form

$$\max_{1\le i\le\in n}\max_{|\beta|=2}\sup_{\omega\in B_\tau}\sup_{|z-p_0|\le\rho}\|\partial_\beta f_i(z,\omega)\|\le M_2.$$

(v) and that there is a $0<\rho'<\rho$ with

$$\sup_{|\theta|\le\nu}\sup_{|\omega|}\|P_N(\theta,\omega)-P_N(0,0)\|\le\rho',$$

so that the image of P_N is contained in the interior of $D_\rho(p_0)$.

THEOREM 4.8 (A-Posteriori Error for a Solution of Equation (1.3)). *Suppose that ϵ, ρ', μ_*, μ^*, M_1, M_2, and C are a collection of validation values for P_N.*

Assume that $N\in\mathbb{N}$ and $\delta>0$ are such that

- $$(\mu^*)^{N+1}C<1, \tag{4.24}$$

- $$\delta<e^{-1}\ \min\left\{\frac{1-C\,(\mu^*)^{N+1}}{2n\pi CM_1M_2},\rho-\rho',\right\} \tag{4.25}$$

- *and*
$$\frac{2C}{1-C(\mu^*)^{N+1}}\epsilon<\delta. \tag{4.26}$$

Then there is a unique one parameter family of analytic N-tails $H\colon D_\nu\times B_\tau\subset\mathbb{C}^k\times\mathbb{C}\to\mathbb{C}^n$ with

$$\|H\|_{\nu,\tau}\le\delta,$$

so that

$$P(\theta,\omega)=P_N(\theta,\omega)+H(\theta,\omega),$$

is the exact solution of Equation (1.3) on $D_\nu\times B_\tau$.

The proof is almost identical to the proof of Theorem 4.7. The difference is that now we are now looking for a one parameter family of bounded analytic N-tails so that

$$f[P_N+H](\theta,\omega)=[P_N+H](\Lambda(\omega)\theta,\omega) \tag{4.27}$$

for all $(\theta,\omega)\in D_\nu\times B_\tau$. Image$(P_N)\subset B(p_0,\rho')$ and f is analytic on $B(p_0,\rho)$. As before for the case of differential equations we take the Taylor expansion of the left hand side of Equation (4.27) and rearrange in order to obtain

$$H[\Lambda(\omega)\theta,\omega]-Df[P_N(\theta,\omega),\omega]H(\theta,\omega)=E_N(\theta,\omega)+R[H(\theta,\omega),P_N(\theta,\omega),\omega].$$

M	N	τ	$\bar{\tau}$	ν	δ_H	δ	time
1	1	10^{-6}	0.995×10^{-6}	10^{-8}	4.48×10^{-13}	3.22×10^{-11}	0.3 (sec)
3	5	10^{-2}	0.995×10^{-2}	0.1	2.1×10^{-9}	1.95×10^{-6}	1.7 (sec)
6	10	10^{-2}	0.995×10^{-2}	0.1	5.1×10^{-15}	3.28×10^{-12}	6.1 (sec)
10	10	10^{-1}	0.995×10^{-1}	0.5	2.5×10^{-10}	1.04×10^{-6}	10.7 (sec)
20	10	10^{-1}	0.995×10^{-1}	0.5	1.4×10^{-13}	6.12×10^{-10}	28.6 (sec)
20	10	0.2	0.1991	0.75	2.1×10^{-11}	2.46×10^{-7}	28.5 (sec)
20	10	0.25	0.248	0.75	9.36×10^{-9}	1.3×10^{-4}	28.4 (sec)

TABLE 5.1

Branch of Stable Manifold Performance Data for the Hénon Family.

Recalling the definition of $\mathfrak{L}_{\text{map}}$ from Section (3.1) we note that this is

$$\mathfrak{L}_{\text{map}}[H](\theta, \omega) = E_N(\theta, \omega) + R[H(\theta, \omega), P_N(\theta, \omega), \omega], \tag{4.28}$$

with

$$A(\theta, \omega) = Df[P_N(\theta, \omega), \omega].$$

The remainder of the proof is now an application of Theorem (3.1) and Corollary (3.4),

REMARK 4.9. In practice the methods of Section (2.4) provide us with only analytic Taylor models for the analytic branches of fixed points/equilibria $p(\omega)$, stable eigenvalues $\lambda_i(\omega)$, stable eigenvectors $\xi_i(\omega)$, the inverse transformation $Q^{-1}(\omega)$, and for the case of diffeomorphisms the powers of the stable eigenvalues. Similarly the methods of Section (4.5) provide analytic Taylor models for the coefficients

$$a_\alpha(\omega) = \sum_{m=0}^{\infty} a_{(\alpha,m)} \omega^m.$$

In other words all terms are known up to interval enclosures of the M -th order taylor polynomials, plus a validated error term on the complex parameter disk B_τ, and we don't actually know exactly the polynomial P_N hypothesized in Definitions (4.1) and (4.2). Instead we have an interval inclosure with validated error bounds. However we do know that a polynomial satisfying all the conditions of the theorems is enclosed by our Taylor model. Moreover all the conditions of the theorems are checked using only the information provided by the Taylor model. The reader interested in the details can consult the user guide [55] and the source code [56].

5. Numerical Computation of Families of Invariant Manifolds with Rigorous Error Bounds for Hénon and Lorenz. We now discuss some numerical results for computations based on the techniques developed in the present work. The reader interested in more details for the numerical computations should consult the user guide [55] and also the source code itself [56].

We consider first the Hénon map with the classical parameter values of $a = 1.4$ and $b = 0.3$. We expand in parameter space about b (i.e. our parameter is $b + \omega$ with $b = 0.3$). We compute analytic Taylor Models of the fixed points, eigenvalues, and eigenvectors of the system to order $M = 20$ on a validated domain of $\tau = 0.1$ using the methods discussed in the user guide [55]. Table (5.1) shows the results of a number of validated computations for the one parameter family of stable manifold at one of the fixed points. The results of a set of validated computations for the stable and unstable manifolds at both of the system fixed points is shown in Figure 5.1. The MatLab/IntLab implementations of these computations are found at [56]. The specific files for the stable/unstable computations at both fixed points are in the separate files `paperCode_henonBranchProof.m` versoins $I - IV$. The program `paperCodePushProof.m` uses some hand optimized estimates for the eigenvalues and eigenvectors of the Hénon map, and is used to obtain the results reported in the last two rows of Table (5.1).

For the sake of comparison we also include Figure 5.2, which shows the plots of the same manifolds on a domain of $\tau = 0.15$ and $\nu \approx 4$ for each of the manifolds (see the caption under the figure for more details). We note that while we have not validated the parameterizations for domains of this size, the approximations are still "good" in the sense that the residuals are small for each expansion (numerical residual smaller than 10^{-6} in all cases). Figure 5.2 gives some idea of what the global dynamics are for parameters near $b = 0.3$ and also highlights that the methods of the present work provide heuristic insight into the dynamics of the one parameter family of dynamical systems even in the absence of rigorous proofs.

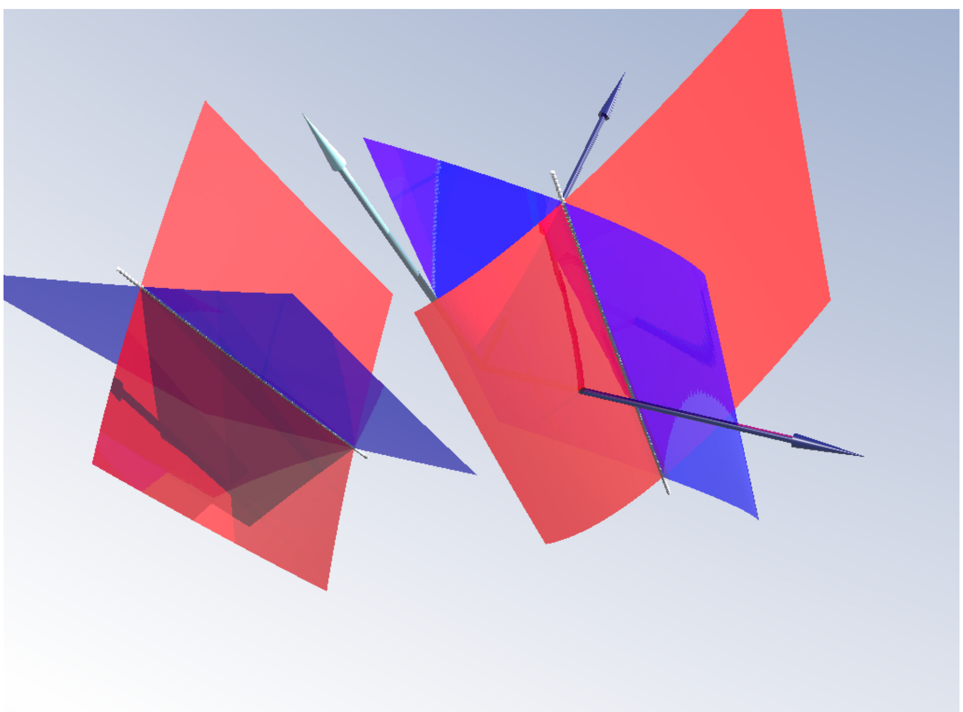

FIG. 5.1. *Stable and Unstable Manifolds of the Fixed Points of the Hénon Family: the stable manifolds are shown in red and the unstable in blue. The parameterized arcs of fixed points are shown as white arcs. The blue axes are the phase space variable and the white axis is the family parameter. The manifolds are plotted for* $-0.12 \le \tau \le 0.12$. *The manifolds shown in the picture have been validated and have truncation errors smaller than* 10^{-5}.

We cary out similar computations at the origin of Lorenz System in order to obtain a three variable polynomial approximation to the one parameter family of two

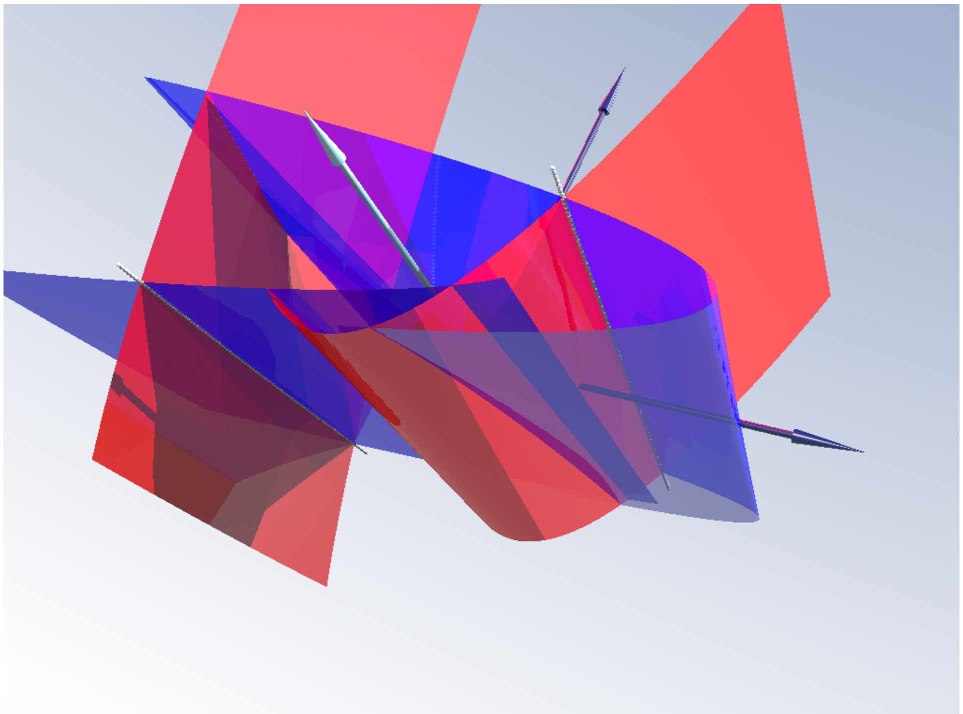

FIG. 5.2. *The figure highlights the fact that the the Parameterization method has value as both a rigorous and non-rigorous numerical method. The manifolds are shown over the same parameter range as in Figure (5.1), but now but the polynomials are evaluated on a larger domain than where the computer aided estimates hold. Nevertheless the figure gives some indication of the global dynamics over the parameter region where the rigorous computations were carried out. Note that numerous transverse intersections of the stable and unstable manifolds are detected. We also point out that in the top left corner of the picture the blue (unstable) manifold folds back sharply on itself, leading to a tangency with the red (stable) manifold. The development of this tangency signals the birth/death of the Hénon attractor, as for parameters before the tangency the system has and attractor, while after the tangency orbits can cross the stable manifold on the left and escape to infinity along the other unstable manifold. This is verified by iterating some test orbits before and after the tangency. The point of this comment is simply to indicate that the parameter range in the validated computation is large enough for interesting global dynamics to occur, even though none are visible in the validated figure. We remark also that the intersections seen in this non-rigorous figure could be validated using "shooting" methods as in [54]*

dimensional stable manifolds. We center the expansion at the parameter set $\sigma = 10$, $\beta = 8/3$, and $\rho = 13.9265$, and expand the family of manifolds in the ρ parameter. The results of this computation for several different program inputs are shown in Table (5.2). Since the resulting manifolds are three dimensional we omit graphical results. The MatLab/IntLab program which perform the computations for the Lorenz system just described is called `validated2DLorenzBranch.m` and is found at [56].

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M	N	τ	ν	δ_H	δ	time
1	6	10^{-8}	0.0001	4.32×10^{-18}	1.85×10^{-15}	8.5 (sec)
1	10	10^{-8}	0.1	4.1×10^{-12}	9.84×10^{-10}	32.7 (sec)
2	10	10^{-4}	0.1	6.5×10^{-14}	1.6×10^{-12}	39.5 (sec)
3	10	10^{-2}	0.5	7.9×10^{-12}	3.2×10^{-10}	46.3 (sec)
3	10	10^{-2}	1.5	3.4×10^{-10}	1.4×10^{-7}	46.3 (sec)
6	10	0.1	1.5	2.1×10^{-11}	9×10^{-8}	69.7 (sec)
8	15	0.25	1.5	1.3×10^{-8}	6.9×10^{-6}	4.7 (min)
8	20	0.25	2.5	1.4×10^{-8}	4.9×10^{-6}	11.7 (min)
3	30	10^{-4}	3.0	2.6×10^{-8}	1.1×10^{-5}	28.9 (min)

TABLE 5.2

Branch of Stable Manifold Performance Data for the Lorenz System.

Waves on a Contractile Substratum. (In Preperation)

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