POLYNOMIAL APPROXIMATION OF A ONE PARAMETER FAMILY OF (UN)STABLE MANIFOLDS WITH RIGOROUS COMPUTER ASSISTED ERROR BOUNDS

J. D. MIRELES-JAMES*

Abstract. This work describes a method for computing polynomial expansions of a one parameter branch of stable or unstable manifolds associated with hyperbolic fixed points or equilibria of a family of analytic dynamical systems. We develop a-posteriori theorems which provide mathematically rigorous bounds on the truncation errors associated the polynomial expansions. The hypotheses of these theorems are formulated in terms of certain inequalities which can be checked via a finite number of calculations on a digital computer. Exploiting the analytic properties of the dynamical systems we are able to obtain mathematically rigorous bounds on the jets of the manifolds, as well as on the derivatives of the manifolds with respect to the parameter. A number of example computations are given.

1. Introduction.

The existence and geometry of invariant manifolds, especially stable and unstable manifolds of fixed points and equilibria, plays a central role in the qualitative theory of dynamical systems. The intersection of these manifolds gives rise to not only connecting orbits but also to periodic orbits and chaotic motions. When a dynamical system depend on a parameter, then we are interested in how these manifolds and their intersections vary as the parameter is changed. This understanding illuminates the transition from regular to chaotic dynamics, bifurcations of connecting orbits, as well as the location of separatrices.

The present work is concerned with high order approximation of stable (and unstable) manifolds for a parameter dependent family of analytic dynamical systems. We present a constructive method for obtaining polynomial approximations to arbitrary order in both the dynamical variables and the parameter. We also develop analytical tools which facilitate the computation of rigorous computer assisted error bounds on the truncation error. In order to formalize the discussion we establish some notation.

We endow $\mathbb{C}$ with the usual Euclidean norm $|z| = |x+iy| = \sqrt{x^2+y^2}$, and $\mathbb{C}^n$ with the norm

$$|z| = |(z_1, \ldots, z_n)| = \max_{1 \leq i \leq n} |z_i|.$$

These norms induce the balls $B_r(z) = \{w \in \mathbb{C} : |w-z| < r\}$ in the complex plane, and the *poly-disks*

$$B_r(z) = \{w \in \mathbb{C}^n \ : \ |w_i - z_i| < r \quad \text{for} \quad 1 \leq i \leq n\},$$

in the complex vector space $\mathbb{C}^n$. (So $B_r(z)$ could denote a ball in the complex plane of a poly-disk depending on context). We write $B_r = B_r(0)$ to denote balls and poly-disks centered at the origin.

Let $p_0 \in \mathbb{C}^n$, $\rho, \tau > 0$. We consider a one parameter family of analytic vector fields $f\colon B_\rho(p_0) \times B_\tau \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, which is analytic with respect to parameter. We assume that f is continuous and bounded on $\overline{B_\nu} \times \overline{B_\tau}$. Suppose that p_0 is a hyperbolic equilibria of $f(z, 0)$, and that $\partial_\omega f(p_0, \omega)$ is not zero at $\omega = 0$. While the implicit function theorem then guarantees the existence of an analytic branch of

*Rutgers University, Department of Mathematics, 110 Frelinghuysen Rd, Piscataway, NJ 08854, USA.

hyperbolic equilibria, our goal is to obtain more quantitative information about the dynamics. We make the following assumptions.

A1-flows: There is a $\tau > 0$ and an analytic function $p\colon B_\tau \to \mathbb{C}^n$ so that

$$f[p(\omega), \omega] = 0, \qquad \text{for all} \quad \omega \in B_\tau.$$

So p parameterizes an arc of equilibria for the vector field.

A2-flows: For each $\omega \in B_\tau$, $Df[p(\omega), \omega]$ is diagonalizable and hyperbolic in the sense of differential equations. Then there are $k \leq n$ stable eigenvalues, and $n-k$ unstable eigenvalues for each $\omega \in B_\tau$. Each of these eigenvalues is parameterized by a one parameter family of analytic functions $\lambda_i\colon B_\tau \to \mathbb{C}$, $1 \leq i \leq n$ with

$$\det\left(Df[p(\omega), \omega] - \lambda_i(\omega)\mathrm{Id}_n\right) = 0 \qquad \text{for all} \quad \omega \in B_\tau.$$

Moreover for all $\omega \in B_\tau$ we have that $\mathrm{real}(\lambda_i(\omega)) < 0$ for $1 \leq i \leq k$, and $0 < \mathrm{real}(\lambda_i(\omega))$ for $k + i \leq i \leq n$. The eigenvalues are distinct, and undergo no bifurcations on B_τ. Let $\Lambda\colon B_\tau \to \mathrm{Mat}_{k\times k}(\mathbb{C})$ be the diagonal matrix of stable eigenvalues defined by

$$\Lambda(\omega) = \left(\begin{array}{ccc} \lambda_1(\omega) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k(\omega) \end{array}\right).$$

A3-flows: There are analytic functions $\xi_i\colon B_\tau \to \mathbb{C}^n$ parameterizing the eigenvectors associated with each λ_i. Then

$$\left(Df[p(\omega), \omega] - \lambda_i(\omega)\mathrm{Id}_n\right)\xi_i(\omega) = 0, \qquad \text{for all} \quad \omega \in B_\tau.$$

and each $1 \leq i \leq n$. We call $\xi_1(\omega), \ldots, \xi_k(\omega)$ the stable eigenvectors and $\xi_{k+1}(\omega), \ldots, \xi_n(\omega)$ the unstable eigenvectors. Let $A\colon B_\tau \to \mathrm{Mat}_{n\times k}(\mathbb{C})$ denote the matrix of stable eigenvectors given by

$$A(\omega) = [\xi_1(\omega)|\ldots|\xi_k(\omega)].$$

For fixed $\omega \in B_\tau$ let ϕ_ω denote the flow generated by $f(\cdot, \omega)$. Under assumptions **A1-A3-flows** the Stable Manifold Theorem implies that the set

$$W^s[p(\omega)] = \left\{ z \in \mathbb{C}^n \mid \lim_{t\to\infty} \phi_\omega(z, t) = p(\omega) \right\}$$

is an analytic invariant manifold tangent to the span of $A(\omega)$ at $p(\omega)$ (and similarly for the unstable manifold). See for example [16]. Again, the implicit function theorem can be used to show that these manifolds depend analytically on ω in some neighborhood of $\omega = 0$. However, because we are interested again in a more constructive approach we examine the situation in a little more detail.

With $\omega \in B_\tau$ still fixed, let $p_\omega = p(\omega)$, $f_\omega = f(\cdot, \omega)$, $\Lambda_\omega = \Lambda(\omega)$, and $A_\omega = A(\omega)$. We utilize the so called *Parameterization Method* of [8, 10], and obtain that (under some mild non-resonance conditions which we treat in detail later) there exists a $\nu > 0$ and an analytic chart map $P_\omega\colon B_\nu \subset \mathbb{C}^k \to \mathbb{C}^n$ for a neighborhood of the stable manifold at $p(\omega)$ which satisfies the system of first order partial differential equations

$$f_\omega[P_\omega(\theta)] = DP_\omega(\theta)\Lambda_\omega\theta.$$

subject to the linear constraints

$$P_\omega(0) = p_\omega, \qquad DP_\omega(0) = A_\omega.$$

Moreover, [8, 10] develop a method by which a polynomial approximations of the chart map P_ω can be obtained following a constructive procedure. In [5, 17] a-posteriori numerical schemes are developed which allow computer assisted validation of these polynomial approximations, again at a single parameter value.

In [10] the implicit function theorem is invoked in order to show that P_ω is analytic in ω. Then there exists an $\tau' > 0$ and a one parameter of analytic chart maps $P\colon B_\nu \times B_{\tau'} \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ so that

$$P(0,\omega) = p(\omega), \qquad DP(0,\omega) = A(\omega), \tag{1.1}$$

and

$$f[P(\theta,\omega),\omega] = [D_1 P(\theta,\omega)]\Lambda(\omega)\theta \qquad \text{for all} \qquad \theta \in B_\nu,\ \ \omega \in B_{\tau'}. \tag{1.2}$$

Since P is analytic it has a convergent power series expansion in θ and ω on $B_\nu \times B_{\tau'}$. However when we apply the implicit function theorem we obtain no bounds on the size of the parameter neighborhood $B_{\tau'}$ on which the branch is analytic, hence no bounds on the radius of convergence of the Taylor expansion of P with respect to ω.

In the present work we show that Equation (1.2) can be exploited in order to construct polynomial approximations P_{MN} of P which are order N in θ and order M in ω for $M, N \in \mathbb{N}$ large enough. The construction is guided by the earlier work of [8, 9]. Given a polynomial expansion P_{MN}, and a parameterization domain $B_\nu \times B_\tau \subset \mathbb{C}^k \to \mathbb{C}^n$, our next priority is to evaluate the accuracy of the approximation domain in questoin. The main purpose of the present work is to develop an a-posteriori theory which can be implemented on the digital computer and which leads to mathematically rigorous bounds on the magnitude of the truncation errors. In this sense the present work generalizes the numerical methods developed in [5, 17] to parameter dependent families of flows. The following "meta-theorem" summarizes our main result for families of vector fields.

Theorem 1.1 (Theorem (5.8) Paraphrased). *Assume **A1-A3-flows** and suppose that P_{MN} is a "properly constructed" polynomial approximation of a solution of Equatoin (1.2), subject to the linear constraints given by Equation (1.1). Suppose also that P_{MN} is a "good enough" approximation on the domain $B_\nu \times B_{\tau'}$ with $0 < \tau' < \tau$ and τ as in **A1-A3-flows**. Then there is an analytic function $P : B_\nu \times B_{\tau'} \to \mathbb{C}^n$ and a $\delta > 0$ so that P is the unique true solution of Equatoin (1.2) satisfying the linear constraints given by Equation (1.1) and*

$$\sup_{\theta \in B_\nu} \sup_{\omega \in B_{\tau'}} |P(\theta,\omega) - P_{MN}(\theta,\omega)| \le \delta.$$

Much of the present work is devoted to making precise exactly what is meant by "properly constructed" and "good enough" in the previous meta-theorem. Moreover we develop explicit formulas for τ' and δ in terms of know quantities. At the moment we remark that "good enough" will be defined by measuring the defect obtained after plugging the polynomial P_{MN} back into the functional Equation (1.2). We will see that this defect can be bound rigorously using a computer and that in the full version of Theorem (1.1) the constant δ is given explicitly in terms of this defect.

Similar considerations apply to discrete time dynamical systems. Let $f\colon B_\rho(p_0)\times B_\tau \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ be a one parameter family of analytic diffeomorphisms, and $p_0 \in \mathbb{C}^n$ be a hyperbolic fixed point of $f(z,0)$ with $\partial_\omega f(0,\omega)$ not zero at $\omega = 0$. We make the following assumptions.

A1-maps: There is a $\tau > 0$ and an analytic function $p\colon B_\tau \to \mathbb{C}^n$ having

$$f[p(\omega),\omega] - p(\omega) = 0, \qquad \text{for all} \quad \omega \in B_\tau.$$

So p parameterizes a one parameter family of fixed points for f.

A2-maps: For each $\omega \in B_\tau$, $Df[p(\omega),\omega]$ is diagonalizable and hyperbolic in the sense of diffeomorphisms. Then there are $k \le n$ stable eigenvalues, and $n-k$ unstable eigenvalues for each $\omega \in B_\tau$. Each of these eigenvalues is parameterized by a one parameter family of analytic functions $\lambda_i\colon B_\tau \to \mathbb{C}$, $1 \le i \le n$ with

$$\det\left(Df[p(\omega),\omega] - \lambda_i(\omega)\mathrm{Id}_n\right) = 0 \qquad \text{for all} \quad \omega \in B_\tau.$$

Moreover for all $\omega \in B_\tau$ we have that $0 < |\lambda_i(\omega)| < 1$ for $1 \le i \le k$, and $1 < |\lambda_i(\omega)|$ for $k + i \le i \le n$. The eigenvalues are distinct, and undergo no bifurcations on B_τ. Let $\Lambda\colon B_\tau \to \mathrm{Mat}_{k\times k}(\mathbb{C})$ be the diagonal matrix of stable eigenvalues defined by

$$\Lambda(\omega) = \begin{pmatrix} \lambda_1(\omega) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k(\omega) \end{pmatrix}.$$

A3-maps: (Same as **A3-flows**).

In this case the stable manifolds theorem implies that for each $\omega \in B_\tau$ the point set

$$W^s[p(\omega)] = \left\{ z \in \mathbb{C}^n \mid \lim_{n\to\infty} f^n(z,\omega) = p(\omega) \right\},$$

(where f^n denotes the composition of f with itself n times) is an analytic invariant manifold tangent to $A(\omega)$ at $p(\omega)$. [8] shows how to locally parameterize such manifolds, and the implicit function theorem guarantees that there is an analytic branch of parameterizations as ω varies.

In this case the work of [8, 10] (and analogy with the previous discussion above about continuous time dynamical systems) lead us to look for $\nu, \tau' > 0$ and $P\colon B_\nu \times B_{\tau'} \to \mathbb{C}^n$ which solves the functional equation

$$f[P(\theta,\omega),\omega] = P[\Lambda(\omega)\theta,\omega], \qquad \text{for all} \qquad \theta \in B_\nu,\ \omega \in B_{\tau'}. \tag{1.3}$$

subject to the first order constraints

$$P(0,\omega) = p(\omega), \qquad DP(0,\omega) = A(\omega). \tag{1.4}$$

Again, our approach is to develop a formal polynomial approximation of P, and then to validate via a computer assisted argument some bounds on the truncation errors. The main result for maps is Theorem (5.9), which is a precise version of Meta-Theorem (1.1), but formulated for diffeomorphisms. This generalizes the work on [23] to one parameter families of discrete time dynamical systems.

Remark 1.2 (Linear Data for the Branch of Invariant Manifolds). In order to proceed we must first show that ***A1-A3*** (in the case of both maps and flows) are reasonable in practice. The assumptions require that we know exact analytic parameterizations of the fixed point/equilibria as well as the associated eigenvalues and eigenvectors. In Section (4) we explain the method which we use for computing polynomial expansions with rigorous error bounds for the the linear data hypothesized in **A1-A3**. As per the philosophy of our study of invariant manifolds we obtain the formal expansions via the tools of automatic differentiation, and the error bounds via analytic (rather than topological/degree theoretic) arguments.

We remark also that while the computation of such "taylor models" is well know and thoroughly documented in the literature (see the opening remarks in Section (4)) since all our work on invariant manifolds depends explicitly on the remainder bounds obtained on the linear data we include a full description of the methods we use to obtain the polynomial approximations, as well as a full description of the theorems we use to bound the remainders. It is essential here that the results are formulated in the analytic category. In this sense our notion of an "analytic Taylor model" is more restrictive, but also more regular than the usual notion.

Remark 1.3 (Multi-Parameter Families of Dynamical Systems). We note that the choice to study one parameter families of dynamical systems largely to simplify the exposition and the implementation of the numerics. A careful reading of Section (4) makes if clear how the definitions, theorems, and computations could be extended to multi-parameter branches of fixed points/equilibria, eigenvalues, and eigenvectors. It is also clear that if one begins with multi-parameter parameterizations of the linear data then the formalism of Section (5.1), as well as the validation techniques of Sections (5.3) and (5.4) apply with little modification.

Remark 1.4 (Rigorous Computation of Jets). An important feature of the computer assisted error bounds for polynomial approximations of stable/unstable manifolds at a single parameter developed in [5, 17] for vector fields and in [23] for diffeomorphism is that in addition to obtaining rigorous C_0 bounds on the truncation error, one actually obtains that the truncation error is an analytic function. Having a representation of the truncation error as a bounded analytic function allows one to bound derivatives of the truncation error using classical estimates of complex analysis in exchange for shrinking the domain of the function. Having some control over the derivatives of the truncation error is essential in certain applications to computer assisted proof of the existence of connecting orbits, chaotic motions, etc. (See also Remark 1.6)

Remark 1.5 (Related Wrok). Part of the present work (the portion pertaining to on parameter families of vector fields) is closely related to the work of *[2]*. There the authors develop polynomial approximations to one parameter families of stable/unstable manifolds for the purpose of proving the existence of a homoclinic tangency in a certain model of cardiac muscle. A difference between the present work and the work of [2] is in the formulation of the fixed point problem which determines the truncation error. In [2] the second order Taylor remainder of the vector field f about the equilibria is exploited in order to obtain a contraction mapping about the approximate solution. In the present work we follow [5, 23] and formulate the contraction mapping problem for the truncation error in terms of the second order Taylor remainder of the vector field f *about the image of the polynomial approximation* P_{MN} *itself.*

In this sense the present work is an attempt to generalize the kinds of techniques developed in [2] for application in a neighborhood of the equilibria where the second order Taylor remainder of f about the origin may not be small. We stress that both methods can be used to produce the same Taylor coefficients for the polynomial approximation. The distinction is in the arguments used to obtain rigorous bounds on the truncation errors on a particular domain.

REMARK 1.6 (Computer Assisted Proofs for Connecting Orbits). Recently a number of authors have developed strategies for obtaining computer assisted proof of the existence of connecting orbits for discrete and continuous time dynamical systems which require that the connecting orbit is formulated as the solution of a certain boundary value problem. The computer is then used to rigorously solve this boundary value problem. See for example [5, 23, 2, 20] and especially the references therein. In the references just mentioned, the boundary conditions are formulated in terms of chart maps for the stable and unstable manifolds. Then these chart maps must be computed rigorously and the Parameterizatoin Method is a powerful tool in this setting.

A natural extension of the methods just mentioned would be to combine them with the something like the methods of [4, 14] for computing rigorous branches of solutions of infinite dimensional equations such as boundary value problems. (See these papers for a much more thorough discussion of the literature). The combination of the methods of for example [5, 23, 2, 20] with for example the methods of [4, 14] could be used in order to computer rigorous one parameter branches of connecting orbits for families of vector fields. However it would be necessary to to control the boundary conditions, and derivatives on the boundary conditions, with respect to parameter. Since the boundary conditions would be formulated in terms of one parameter families of stable/unstable manifolds, this is the problem solved by the present work.

We also remark that some local control over the invariant manifolds is essential for studying degenerate connecting orbits such as tangencies (see again [2] and also Remark (1.5)) , and that this same local control will be essential in future computer assisted studies of bifurcations of connecting orbits for vector fields. Using the high order methods developed here in order to study connecting orbit bifurcatoins will be the topic of a future work.

The remainder of the paper is organized as follows. In Section (2) we establish the notation used throughout the paper, and recall certain theorems and estimates analysis. In Section (2.2) we define and a certain family of analytic function which we call 'one parameter families of analytic N-tails.' These comprise the main technical tools of our error analysis.

Section (3) belongs to the study of certain operator equations on the Banach Space of one parameter families of analytic N-tails. In Section (3.1) solve a pair of linear equations on the space of one parameter families of analytic N-tails. These linear equations play a central in our analysis of invariant manifolds in the sequel. Section (3.1.1) is devoted to an abstract non-linear equation on the space of one parameter families of analytic N-tails, and we prove an existence theorem. We also examine a concrete instantiation of this non-linear equation which allows us to prove our a-posteriori error theorems later in the paper.

In Section (4) we define the data structure which we use throughout the paper in order to model analytic functions on the computer. The data structure consists of a polynomial with interval coefficients, a real number representing the radius of convergence of the model, and a real number representing a bound on the 'tail' of the analytic function. We call this data structure an 'analytic Taylor' model, to distinguish from the usual notion of a Taylor model where the third real number usually represents a bound in the Banach Space of continuous functions. In Section (4.1) we show how we obtain an analytic Taylor model for the inverse of a matrix whose coefficients are analytic Taylor models, while in Section (4.2) we recall that the composition of a Taylor model with an elementary function can usually be computed at the cost of a Cauchy Product. We also record the estimates which we use in order to bound the tail of the analytic Taylor model representing the resulting composition. In Section (4.3) we discuss using analytic Taylor models to parameterize one parameter solutions of finite dimensional nonlinear equations. Finally in Section (4.4) we present several example computations which illustrate that we can in practice, for a given fixed one parameter family of dynamical systems, obtain the data hypothesized in **A1-A3**. We examine also the relationship between the order of the Taylor Model, the size of the radius of convergence, and the tail error bound.

In Section (5) we finally turn to the main topic of the paper, rigorous computation of one parameter families of stable/unstable manifolds. We begin by illustrating the formal computation of the coefficients for the polynomial approximation of the family of invariant manifolds. We discuss conditions which guarantee that the coefficients are formally well defined to all orders, and illustrate the computation for a specific family of diffeomorphisms and another family of differential equations. We focus on the examples of the classical Hénon map and the Lorenz differential equation. In Section (5.2) we provide a method which allows us to compute explicitly a parameter interval on which the formal solution converges. We think of the parameterization of the invariant manifold as a power series in the dynamical variables, whose coefficients are power series in the parameter. In Section (5.3) we show how to bound the truncation errors of a finite number of these coefficient power series. The remaining truncation error is now a one parameter family of analytic N-tails, and in Section (5.4) we apply the theory of Section (2) in order obtain the desired bound. The cases of maps and flows are studied seperately.

Section (6) presents example computations with rigorous error bounds for the Hénon map and the Lorenz system. Specifically we compute one parameter branches of all four stable and unstable manifolds of the two fixed points of the Hénon map. Since the phase space of the map is two dimensional and all the (un)stable manifolds are one dimensional we can represent the resulting one parameter families of invariant manifolds graphically. We also discuss computations of the one parameter family of two dimensional stable manifolds at the origin of the Lorenz system. Since this results in polynomials of three variables we present only tabular results.

Two of the quantities required in the hypotheses of Theorems (5.8) and (5.9) require information about infinitely many terms of some power series in several variables. In Appendix (A) we discuss how these series can be bound in practice using only the finite data available from the Taylor models. We illustrate the computations and derive explicit estimates for the Hénon and Lorenz systems.

2. Background.

2.1. Spaces, Norms, and Theorems of Analysis. Let $z \in \mathbb{C}^m$ and $f\colon B_r(z) \subset \mathbb{C}^m \to \mathbb{C}^n$ be an analytic function. Then we let

$$\|f\|_r \equiv \sup_{|z| \leq r} |f(z)|,$$

denote the (compontenwise) C^0 norm of f on $B_r(z)$. We note that this norm induces a Banach Space structure on the collection of all such functions. We denote this

Banach Space by $C^\omega(B_r(z), \mathbb{C}^n)$.

Let $\tau > 0$ and B_τ denote the ball of radius τ about the origin in the complex plane $\mathbb{C}$. We are often interested in a one parameter family of analytic mappings $f\colon B_r(z) \times B_\tau \subset \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^n$. Here we employ the norm

$$\|f\|_{r,\tau} \equiv \sup_{|w|\leq r} \sup_{|\omega|\leq\tau} |f(w,\omega)|.$$

Again the collection of all such functions is a Banach Space under this norm.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach Spaces. Let $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$ denote the norms on these spaces. Suppose that $\mathfrak{L}\colon \mathcal{X} \to \mathcal{Y}$ is a linear operator between them. The norm of the linear operator $\mathfrak{L}$ is defined to be

$$\|\mathfrak{L}\|_{B(\mathcal{X},\mathcal{Y})} \equiv \sup_{\|w\|_{\mathcal{X}}=1} \|\mathfrak{L}w\|_{\mathcal{Y}}.$$

If $\|\mathfrak{L}\|_{B(\mathcal{X},\mathcal{Y})} < \infty$ then we say that the linear operator is bounded. If $\mathfrak{L}$ is invertible and $\|\mathfrak{L}^{-1}\|_{B(\mathcal{Y},\mathcal{X})} < \infty$ then we say that the operator $\mathfrak{L}$ is boundedly invertible. If $\mathcal{X} = \mathcal{Y}$ then we simplify the notation by writing

$$\|\mathfrak{L}\|_{B(\mathcal{X},\mathcal{X})} = \|\mathfrak{L}\|_{B(\mathcal{X})}.$$

Now let A be a $k \times \ell$ matrix of fixed complex numbers. We denote the (i,j) entry of A by either $[A]_{i,j}$ or a_{ij}, depending on context. We take the norm of A to be the maximum of the sum of the absolute values of row entries, where the maximum is taken over all rows; i.e.

$$|A|_M \equiv \max_{1\leq i\leq k} \sum_{j=1}^{\ell} |a_{ij}|.$$

If we consider A to be a linear operator from the (finite dimensional) Banach Space $\mathbb{C}^\ell$ to the (finite dimensional) Banach Space $\mathbb{C}^k$ (both endowed with the maximum norm on components) then $\|A\|_{B(\mathbb{C}^\ell,\mathbb{C}^k)} \leq |A|_M$. We will always use this inequality when dealing with finite dimensional linear maps as the quantity $|A|_M$ is easy to compute numerically. When it is clear from context that A is a matrix we will sometimes suppress the M subscript and simply write $|A|$.

Suppose that $A(\omega)$ is a $k \times \ell$ matrix whose entries $a_{ij}\colon B_\tau \subset \mathbb{C} \to \mathbb{C}$ are analytic functions, and $w\colon B_\tau \subset \mathbb{C} \to \mathbb{C}^\ell$ be a 'column vector' of analytic functoins of one variable. Then $A(\omega)$ defines a linear operator $\mathfrak{L}\colon C^\omega(B_\tau, \mathbb{C}^\ell) \to C^\omega(B_\tau, \mathbb{C}^k)$ by the formula

$$\mathfrak{L}[f](\omega) = A(\omega)w(\omega).$$

The discussion of the preceding paragraphs makes it clear that we have

$$\|\mathfrak{L}\|_{B(C^\omega(B_\tau,\mathbb{C}^\ell), C^\omega(B_\tau,\mathbb{C}^k))} \leq \sup_{|\omega|\leq\tau} |A(\omega)|_M.$$

Then we simply write $\|A\|_\tau \equiv \|\mathfrak{L}\|_{B(C^\omega(B_\tau,\mathbb{C}^\ell), C^\omega(B_\tau,\mathbb{C}k))}$, and trust that no confusion will result.

Consider again $z \in \mathbb{C}^m$, $\tau, r > 0$, and take $f\colon B_r(z) \times B_\tau \subset \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^n$ an analytic function. Later in the paper $w \in B_r(z)$ will often be thought of as a "dynamical variable" and $\omega \in B_\tau$ will be thought of as a parameter. When we consider the Fréchte derivative of f with respect to w (and ω is fixed) we will denote this derivative as $D_1 f(w,\omega)$. Should we take the derivative with respect to parameter we will write $\partial/\partial\omega f(w,\omega)$ to stress that our parameter is one dimensional.

Let $\alpha \in \mathbb{N}^k$ denote a *multi-index*, $m \in \mathbb{N}$ denote an integer index, $w \in \mathbb{C}^k$, and $a_{(\alpha,m)} \in \mathbb{C}^n$ be a complex number indexed by α and m. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$ we let $|\alpha| = \alpha_1 + \ldots + \alpha_k$ and $w^\alpha = w_1^{\alpha_1} \cdot \ldots \cdot w_k^{\alpha_k}$. If f is analytic on $B_r(z) \times B_\tau \subset \mathbb{C}^k \times \mathbb{C}$ then we can write the power series for f as

$$f(w,\omega) = \sum_{|\alpha|=0}^{\infty} \sum_{m=0}^{\infty} a_{(\alpha,m)} \omega^m w^\alpha . = \sum_{|\alpha|=0}^{\infty} a_\alpha(\omega) w^\alpha,$$

i.e. as a power series in w whose coefficients are power series in the parameter ω, and have that the series converges to the value of the function for any $|w| < r$ and $|\omega| < \tau$. Similarly, we denote by

$$f_{MN}(w,\omega) = \sum_{|\alpha|=0}^{N} \sum_{m=0}^{M} a_{(\alpha,m)} \omega^m w^\alpha,$$

a polynomial of degree N in w whose coefficients are polynomials of degree M in ω.

The following estimate follows directly from the Cauchy Theorem of Complex Analysis [1], and is a standard part of "KAM folklore". An explicit proof (which yields the constants given here) can be found for example in [23].

Lemma 2.1 (Cauchy Bounds). *Suppose that $f : B_\nu(0) \subset \mathbb{C}^m \to \mathbb{C}^n$ is bounded and analytic. Then for any $0 < \sigma \leq 1$ we have that*

$$\|\partial_i f\|_{\nu e^{-\sigma}} \leq \frac{2\pi}{\nu\sigma} \|f\|_\nu \quad \textit{so that} \quad \|Df\|_{\nu e^{-\sigma}} \leq \frac{2\pi m}{\nu\sigma} \|f\|_\nu, \tag{2.1}$$

as well as

$$\|\partial_i \partial_j f\|_{\nu e^{-\sigma}} \leq \frac{4\pi^2}{\nu^2\sigma^2} \|f\|_\nu \quad \textit{and} \quad \|D^2 f\|_{\nu e^{-\sigma}} \leq \frac{4\pi^2 m^2}{\nu^2\sigma^2} \|f\|_\nu. \tag{2.2}$$

We make repeated use of the following standard theorem of non-linear analysis.

Theorem 2.1 (Newton-Kantorovich Method). *Let X, Y be Bancah spaces and $F : X \to Y$ be a differentiable mapping. Assume that there as an $\hat{x} \in X$ and an $r > 0$ such that*

(i) $DF(\hat{x})$ is boundedly invertible and

(ii) $\|DF(x) - DF(y)\|_{B(X,Y)} \leq \kappa \|x - y\|_X$ for all $x, y \in B_r(\hat{x})$.

If

(I)

$$\epsilon_{NK} \geq \|DF(\hat{x})^{-1} F(\hat{x})\|_Y,$$

(II)

$$\epsilon_{NK} \leq \frac{r}{2},$$

and

(III)

$$4\epsilon_{NK}\,\kappa\,\|DF(\hat{x})^{-1}\|_{B(X,Y)}\,\leq 1,$$

then the equation

$$F(x) = 0$$

has a unique solution in $B(r, \hat{x})$*.*

(See [24] for an exposition of the proof in the language of English).

2.2. Analytic N-Tails and One Parameter Families of Analytic-N Tails. We now define a class of functions which are essential in the sequel, as they are the functions which we use in order to model truncation errors.

Definition 2.2. [Analytic N-Tails] An analytic function $h\colon B_r \subset \mathbb{C}^k \to \mathbb{C}^n$ is called an *analytic N-tail* if

$$h(0) = \partial_\alpha h(0) = 0, \qquad \text{for all} \qquad |\alpha| <= N. \tag{2.3}$$

If $h\colon B_\nu \subset \mathbb{C}^k \to \mathbb{C}^n$ satisfies the condition given by Equation (2.3) and in addition is bounded on B_ν then we say that h is a *bounded analytic N-tail on* B_ν. Given a disk B_ν the set of all bounded analytic N-tails on B_ν is a Banach Space under the supremum norm.

Let $z \in \mathbb{C}^k$, $\alpha \in \mathbb{N}^k$, and $a_\alpha \in \mathbb{C}^n$ for each α. A key fact is that a bouhnded analytic N-tail h on B_ν has a power series representation

$$h(z) = \sum_{|\alpha|=N+1}^{\infty} a_\alpha z^\alpha$$

which converges for $|z| \leq \nu$. We think of an analytic N-tail as a function which is zero to N-th order at the origin. In this sense an analytic N-tails is a 'small perturbations' of the zero function, and as such enjoys some useful 'perturbative' properties.

Lemma 2.2. *Let h be a bounded analytic N-tail on $B_\nu \subset \mathbb{C}^k$, and let Λ be a $k \times k$ diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ having $0 < |\lambda_i| < 1$ for $1 \leq i \leq k$. Denote by $\mu^* \equiv sup_{1\leq i \leq k}|\lambda_i|$. Then $(h \circ \Lambda)(z) = h(\Lambda z)$ is a bounded analytic N-tail on B_ν and*

$$\|h \circ \Lambda\|_\nu \leq (\mu^*)^{N+1}\|h\|_\nu. \tag{2.4}$$

See [5] (Lemma 3.2) for an elementary proof. The following estimate of N-tail solutions of a certain ordinary differential equation is useful in the sequel.

Lemma 2.3. *Suppose that $E\colon B_R \to \mathbb{C}$ is an analytic M-tail, $K \in \mathbb{N}$ with $K > 1$, and that $f\colon B_R \to \mathbb{C}$ is an analytic function. Suppose in addition that $\|f\|_R \leq C$ and that*

$$|f(\omega)| \geq M \qquad \text{for all} \qquad \omega \in B_R.$$

Then the differential equation

$$f(\omega)h'(\omega) - Kf'(\omega)h(\omega) = E(\omega)$$

has a unique solution $h\colon B_R \to \mathbb{C}$. *Moreover* h *is an analytic* M*-tail with*

$$\|h\|_R \le R\left(\frac{C}{M}\right)^K \|E\|_R. \tag{2.5}$$

Proof: We note that since f^K is analytic and nonzero on $B_R(0)$, f^{-K} is analytic and bounded on B_R. Multiplying both sides of the equation by $f^{-K}(z)$ we obtain the equivalent equation

$$f^{-K}h' - Kf^{K-1}f'h = f^{-K}E,$$

or

$$\frac{d}{dz}\left(f^{-K}h\right) = f^{-K}E, \tag{2.6}$$

for any $z \in B_R(0)$. Since $B_R(0)$ is a convex neighborhood about the origin we have that the line segment between the origin and z is contained in z. We parameterize this line by $\gamma\colon [0,1] \to B_R(0)$ by the formula

$$\gamma(t) = tz.$$

Taking the line integral over γ of both sides of Equation (2.6) we have

$$\int_0^1 \frac{d}{dz}\left(f^{-K}[\gamma(t)]h[\gamma(t)]\right)\gamma'(t)\ dt = \int_0^1 f^{-K}[\gamma(t)]E[\gamma(t)]\gamma'(t)\ dt.$$

Since $B_R(0)$ is simply connected we have that the left hand side is

$$\begin{aligned}\int_0^1 \frac{d}{dz}\left(f^{-K}[\gamma(t)]h[\gamma(t)]\right)\gamma'(t)\ dt &= f^{-K}[\gamma(1)]h[\gamma(1)] - f^{-K}[\gamma(0)]h[\gamma(0)]\\ &= f^{-K}(z)h(z),\end{aligned}$$

as $\gamma(0) = 0$ and h is an analytic N-tail. Then

$$h(z) = f^K(z)\int_0^1 f^{-K}[\gamma(t)E[\gamma(t)]\gamma'(t)\ dt,$$

and we note that h is an analytic N-tail due to the fact that E is. Now we bound

$$\begin{aligned}\sup_{|z|\le R}|h(z)| &\le \sup_{|z|\le R}\left|f^K(z)\right|\left|\int_0^1 f^{-K}[\gamma(t)]E[\gamma(t)]z\ dt,\right|\\ &\le C^K\frac{1}{M^K}\ \|E\|_R\ R\end{aligned}$$

as desired.

$\square$

Since the present work deals largely with parameterized families of analytic functions we will have use for the concept of a parameterized family N-tails.

Definition 2.3. [One Parameter Family of Analytic N-Tails] We call an analytic function $H\colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ a *one parameter family of bounded analytic N-tails* if $H(z,\omega)$ is a bounded analytic N-tail on B_ν for each fixed $|\omega| \le \tau$.

A family of analytic N-tails has that

$$H(0,\omega) = D_1^\alpha H(0,\omega) = 0 \quad \text{for each} \quad 1 \le |\alpha| \le N, \quad \text{and for all} \quad |\omega| \le \tau.$$

So a family of analytic N-tails has convergent power series expansion

$$H(z,\omega) = \sum_{|\alpha|=N+1}^{\infty} \sum_{m=0}^{\infty} a_{(\alpha,m)} \omega^m z^\alpha.$$

Let $\Lambda\colon B_\tau \subset \mathbb{C} \to \mathbb{C}^k$ be a diagonal matrix of analytic functions on B_τ and suppose that there is a positive $\mu^* \in \mathbb{R}$ so that

$$\sup_{|\omega| \le \tau} |\Lambda(\omega)| \le \mu^* < 1.$$

Lemma (2.2) applies uniformly to $\omega \in B_\tau$ and we have that $(H \circ \Lambda)(z,\omega) = H(\Lambda(\omega)z,\omega)$ is an analytic N-tails for each fixed ω. Then

$$\|H \circ \Lambda\|_{\nu,\tau} \le (\mu^*)^{N+1} \|H\|_{\nu,\tau}. \tag{2.7}$$

Because Λ is diagonal an equivalent statement is $\|H \circ \Lambda\|_{\nu,\tau} \le \|\Lambda\|_\tau^{N+1} \|H\|_{\nu,\tau}$.

3. Operator Equations on the Space of One Parameter Families of Analytic N-Tails.

3.1. Two Linear Equations.

We now consider the solvability of certain linear equations which are defined on spaces of N-tails and families of N-tails. These linear operators play a critical role in the a-posteriori truncation error analysis developed in Section (5.4.1).

Let $\mathcal{X}_N(B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C}, \mathbb{C}^n)$ denote the Banach Space of all one parameter families of bounded analytic N-tails, endowed with the supremum norm. When N, ν, τ, k, and n are clear from context we simply write $\mathcal{X}$. Let

$$\mathcal{X}_\delta = \{H \mid H \in \mathcal{X} \text{ and } \|H\|_{\nu,\tau} \le \delta\}.$$

In the following discussion we take A be an $n \times n$ matrix of analytic functions $a_{ij}\colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$. We assume that $A(z,\omega)$ is invertible for each $|z| \le \nu$, $|\omega| \le \tau$ and that $A(0,\omega)$ is diagonalizable for each $|\omega| < \tau$. We let $\lambda_i\colon B_\tau \to \mathbb{C}$ with $1 \le i \le n$ denote parameterizations of the eigenvalues of $A(0,\omega)$. We assume that the eigenvalues vary analytically for $\omega \in B_\tau$ and that there are no bifurcations. So we assume that the eigenvalues are distinct and non-zero on B_τ.

Take $Q\colon B_\tau \to \text{Mat}_{n\times n}(\mathbb{C})$ to be a parameterization of the diagonalizing transformation for $A(0,\omega)$. Then if we denote by $\Sigma\colon B_\tau \to \text{Mat}_{n\times n}(\mathbb{C})$ the matrix with $\lambda_i(\omega)$ as diagonal entries and zeros elsewhere then we have

$$A(0,\omega) = Q(\omega)\Sigma(\omega)Q^{-1}(\omega)$$

for each $\omega \in B_\tau$. Note that for each $\omega \in B_\tau$ the columns of $Q(\omega)$ are eigenvectors for $A(0,\omega)$.

Theorem 3.1 (Parameterization Co-Homological Equation for Maps). *Suppose that for each $\omega \in B_\tau$ the matrix $A(0,\omega)$ is hyperbolic in the sense of maps. We have already supposed that there are no eigenvalue bifurcations on B_τ. Assume in addition that the stability of $A(0,\omega)$ does not change on B_τ. Then there are $k \leq n$ stable eigenvalues. We order the eigenvalues so that the stable ones come first. More explicitly we require that $0 < |\lambda_i(\omega)| < 1$ for $1 \leq i \leq k$. Let $\Lambda(\omega)$ denote the $k \times k$ matrix of analytic functions having the stable eigenvectors $\lambda_i(\omega)$ for $1 \leq i \leq k$ as diagonal entries and zeros as the off-diagonal entries.*

Define the linear operator $\mathfrak{L}_{maps} \colon \mathcal{X} \to \mathcal{X}$ by

$$\mathfrak{L}_{maps}[H](z,\omega) = A(z,\omega)H(z,\omega) - H[\Lambda(\omega)z,\omega] \tag{3.1}$$

Assume that there are $0 < \mu^* < 1$ and $M > 0$ so that

$$\max_{1\leq i\leq k} \sup_{|\omega|\leq\tau} |\lambda_i(\omega)| \leq \mu^*,$$

and

$$\sup_{|\omega|\leq\tau} \sup_{|z|\leq\nu} |A^{-1}(z,\omega)| \leq M.$$

Assume in addition that N is so large that

$$M(\mu^*)^{N+1} < 1. \tag{3.2}$$

Then $\mathfrak{L}_{maps}$ is boundedly invertible. Moreover we have that

$$\|\mathfrak{L}_{maps}^{-1}\|_{B(\mathcal{X})} \leq \frac{M}{1 - M(\mu^{N+1})}. \tag{3.3}$$

Proof: Let $E \colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ be a one parameter family of bounded analytic N-tails and consider the equation

$$A(z,\omega)H(z,\omega) - H(\Lambda(\omega)z,\omega) = E(z,\omega).$$

The solvability of this equation is equivalent to the invertibility of the operator defined in Equation (3.1). We rewrite as

$$\begin{aligned} H(z,\omega) - L[H](z,\omega) &= \left[(I-L)H\right](z,\omega) \\ &= A^{-1}(z,\omega)E(z,\omega), \end{aligned} \tag{3.4}$$

where $L \colon \mathcal{X} \to \mathcal{X}$ is the linear operator defined by

$$L[H](z,\omega) = A^{-1}(z,\omega)\, H[\Lambda(\omega)z,\omega].$$

Using the estimate given by Equation (2.7) we have that

$$\begin{aligned} \sup_{\|H\|=1} \|L[H]\|_{(\nu,\tau)} &\leq \sup_{\|H\|=1} \|A^{-1}[H\circ\Lambda]\|_{(\nu,\tau)} \\ &\leq \sup_{\|H\|=1} \|A^{-1}\|_{(\nu,\tau)} \|H\circ\Lambda\|_{(\nu,\tau)} \\ &\leq \sup_{\|H\|=1} M(\mu^*)^{N+1}\|H\|_{(\nu,\tau)} \\ &< 1. \end{aligned}$$

Then the Neumann Theorem gives that the operator defined by the left hand side of Equation (3.4) is boundedly invertible so that

$$H(z,\omega) = \left[(I - L)^{-1} A^{-1} E \right] (z,\omega)$$

is the desired solution. In addition the Neumann Theorem gives

$$\|H\|_{(\nu,\tau)} \le \frac{M}{1 - (\mu^*)^{N+1} M} \|E\|_{(\mu,\tau)}.$$

Then $\mathfrak{L}^{-1}(E) \equiv H$ and taking the sup over all E with norm one gives

$$\|\mathfrak{L}^{-1}\| \le \frac{M}{1 - (\mu^*)^{N+1} M}$$

as desired.

□

Theorem 3.2 (Parameterization Co-Homological Equation for Vector Fields). *Suppose that for each $\omega \in B_\tau$ the matrix $A(0,\omega)$ is hyperbolic in the sense of differential equations. We have already supposed that there are no eigenvalue bifurcations on B_τ. Assume in addition that the stability of $A(0,\omega)$ does not change on B_τ. Then there are $k \le n$ stable eigenvalues. We order the eigenvalues so that the stable ones come first. More explicitly we require that $0 < real[\lambda_i(\omega)] < 0$ for $1 \le i \le k$. Let $\Lambda(\omega)$ denote the $k \times k$ matrix of analytic functions having the stable eigenvectors $\lambda_i(\omega)$ for $1 \le i \le k$ as diagonal entries and zeros as the off-diagonal entries.*

Define linear operator $\mathfrak{L}_{flows} \colon \mathcal{X} \to \mathcal{X}$

$$\mathfrak{L}_{flows}[H](z,\omega) = [D_1 H(z,\omega)] \Lambda(\omega) z - A(z,\omega) H(z,\omega). \tag{3.5}$$

Assume that M_1, M_2, μ_*, and μ^* are positive real constants having that

$$0 < \mu_* \le \min_{1 \le i \le k} \inf_{|\omega| \le \tau} |real[\lambda_i(\omega)]| \le \max_{1 \le i \le k} \sup_{|\omega| \le \tau} |real[\lambda_i(\omega)]| \le \mu^* < \infty, \tag{3.6}$$

and that

$$\|Q\|_\tau \|Q^{-1}\|_\tau \le M_1, \tag{3.7}$$

$$\sum_{|\alpha|=1}^{\infty} \sum_{m=0}^{\infty} \frac{|A_{(\alpha,m)}|}{\mu_* |\alpha|} \tau^m \nu^{|\alpha|} \le M_2. \tag{3.8}$$

Assume in addition that N is so large that

$$(N+1)\mu_* \ge \mu^*, \tag{3.9}$$

Then $\mathfrak{L}_{flows}$ is a boundedly invertible with

$$\left\| \mathfrak{L}_{flows}^{-1} \right\|_{B(\mathcal{X})} \le \frac{M_1 e^{M_2}}{(N+1)\mu_* - \mu^*}. \tag{3.10}$$

Proof: Let $E\colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ be a one parameter family of bounded analytic N-tails and note that inverting the operator given by Equation (3.5) is equivalent to solving the equation

$$[D_1\, H(z,\omega)]\, \Lambda(\omega)z - A(z,\omega)\, H(z,\omega) = E(z,\omega). \tag{3.11}$$

We make a change of variables $z \to e^{\Lambda(\omega)t}z$ and define the analytic N-tails

$$x(t) = H\left(e^{\Lambda(\omega)t}z, \omega\right), \qquad \text{and} \qquad E(t) = E\left(e^{\Lambda(\omega)t}z, \omega\right),$$

and the matrix of analytic functions

$$A(t) = A\left(e^{\Lambda(\omega)t}z, \omega\right).$$

We consider the ordinary differential equation

$$\frac{d}{dt}x(t) - A(t)x(t) = E(t), \tag{3.12}$$

and note that if $x(t)$ solves Equation (3.12) the $x(0)$ solves Equation (3.11). We define the integrating factor

$$C(t) = \exp\left(-\int_0^t A(s)\, ds\right)$$

and have that

$$x(t) = -C^{-1}(t)\int_t^\infty C(s)\, E(s)\, ds$$

solves Equation (3.12). Taking the limit as $t \to 0$ we define

$$\mathfrak{L}^{-1}[E](z,\omega) = H(z,\omega) = x(0) = -\int_0^\infty C(s)E(s)\, ds,$$

as the solution of Equation (3.11). The fact that H is an one parameter of analytic N-tails follows from the fact that E is.

In order to obtain bounds on $\mathfrak{L}^{-1}$ we first note by the definition of μ_* given in Equations (3.6) we have that

$$\left|e^{\Lambda(\omega)t}z\right| \leq e^{-\mu_* t}|z|$$

for all $t > 0$, $\omega \in B_\tau$, and $z \in B_\nu$. Then, since E is a one parameter family of analytic N-tails, the estimates of Equations (2.7) give that

$$|E(t)| \leq \left\| E\left[e^{\Lambda(\omega)t}z, \omega\right]\right\|_{\nu,\tau} \leq e^{-(N+1)\mu_* t}\|E\|_{\nu,\tau}. \tag{3.13}$$

In order to bound the integrating factor we observe that

$$\begin{aligned}
-\int_0^t A(s)\, ds &= -\int_0^t \sum_{|\alpha|=0}^\infty \sum_{m=0}^\infty A_{(\alpha,m)}\omega^m \left[e^{\Lambda(\omega)s}z\right]^\alpha\, ds \\
&= -\sum_{|\alpha|=0}^\infty \sum_{m=0}^\infty A_{(\alpha,m)}\omega^m \left(\int_0^t e^{\langle \Lambda(\omega),\alpha\rangle s}\, ds\right) z^\alpha \\
&= Q(\omega)[-\Sigma(\omega)t]Q^{-1}(\omega) - \sum_{|\alpha|=1}^\infty \sum_{m=0}^\infty A_{(\alpha,m)} \frac{1 - e^{\langle \Lambda(\omega),\alpha\rangle t}}{|\langle \Lambda(\omega),\alpha\rangle|}\omega^m\, z^\alpha\, ds.
\end{aligned}$$

Then

$$\|C(t)\| \leq \|Q\|_\tau \|Q^{-1}\|_\tau \exp(\mu^* t) \exp\left(\sum_{|\alpha|=1}^{\infty} \sum_{m=0}^{\infty} \frac{|A_{(\alpha,m)}|}{\mu_* |\alpha|} \right) \leq M_1 e^{M_2} e^{\mu^* t}.$$

We note that $\langle \Lambda(\omega), \alpha \rangle$ is never zero for $|\alpha| \geq 1$ by the assumption that the eigenvalues are non-zero for $\omega \in B_\tau$. Combining this with the estimate of Equation (3.13) as well as the assumption given by Equation (3.9) we obtain

$$\begin{aligned} \|\mathfrak{L}^{-1}[E]\| &\leq \left\| -\int_0^\infty C(t)E(t)dt \right\| \\ &\leq \int_0^\infty e^{-[(N+1)\mu_* - \mu^*]t} \|E\|_{\nu,\tau}\, dt \\ &\leq \frac{1}{(N+1)\mu_* - \mu^*} \|E\|_{\nu,\tau} \end{aligned}$$

Taking the sup over all E with norm one gives the estimate claimed in Equation (3.10).

$\square$

3.1.1. A Non-Linear Operator Equation on $\mathcal{X}$. In this section we take $\mathfrak{L}\colon \mathcal{X} \to \mathcal{X}$ to be a bounded linear operator and E be a fixed one parameter family of bounded analytic N-tails. Suppose that $R\colon B_s \times B_\tau \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is a one parameter family of analytic functions. Assume that there are $C_1, C_2 > 0$ having so that for any $0 < \delta < s$ we have that

R1:

$$\sup_{|\omega| \leq \tau} \sup_{|z| \leq \delta} |R(z,\omega)| \leq C_1 \delta^2,$$

R2:

$$\sup_{|\omega| \leq \tau} \sup_{|z| \leq \delta} \|DR(z,\omega)\|_M \leq C_2 \delta.$$

We are interested in equations of the form

$$\mathfrak{L}[H](\theta,\omega) = E(\theta,\omega) + R[H(\theta,\omega),\omega] = 0, \tag{3.14}$$

with $H\colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ (and $k < n$) a one parameter family of bounded analytic N-tails. The next theorem provides conditions under which we can uniquely solve such equations.

Theorem 3.3. *Suppose that $R\colon B_s \times B_\tau \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is a one parameter family of bounded analytic functions, that $0 < \delta < s$, and that there are $C_1, C_2 > 0$ so that the estimates of **R1-R2** are satisfied. Let $E\colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ be a fixed one parameter family of bounded analytic N-tails with $\|E\|_{\nu,\tau} \leq \epsilon$. Let $\mathfrak{L}$ be a boundedly invertible linear operator with $\|\mathfrak{L}^{-1}\|_{B(\mathcal{X})} \leq C$. If there is a $\delta > 0$ having*

$$2C\epsilon < \delta < \min\left(\frac{1}{2CC_1}, \frac{1}{CC_2}, s \right), \tag{3.15}$$

then Equation (3.14) has a unique solution $H \in \mathcal{X}_\delta$.

We remark that if we think of ϵ as a "small parameter" then the theorem is saying that; if ϵ is small enough then we can solve Equation (3.14).

Proof: Since $\mathfrak{L}$ is invertible we define the nonlinear operator $\Phi\colon \mathcal{X} \to \mathcal{X}$ by

$$\Phi[H](z,\omega) = \mathfrak{L}^{-1}[E(z,\omega) + R[H](z,\omega)],$$

and note that H is a solution of Equation (3.14) if and only if H is a fixed point of Φ. Then the theorem is proved as soon as we show that Φ is a contraction mapping on $\mathcal{X}_\delta$. First we take $H \in \mathcal{X}_\delta$ and consider

$$\begin{aligned} \|\Phi[H]\|_{\nu,\tau} &\leq \|\mathfrak{L}^{-1}\|_{B(\mathcal{X})}\left(\|E\|_{\nu,\tau} + \|R[H]\|_{\nu,\tau}\right) \\ &\leq C(\epsilon + C_1\delta^2) \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} \\ &\leq \delta \end{aligned}$$

as the left hand side of (3.15) gives

$$C\epsilon < \frac{\delta}{2}$$

and the right hand side gives

$$CC_1\delta^2 < \frac{\delta}{2}.$$

Here we have used **R1**. Then Φ maps $\mathcal{X}_\delta$ into itself.

Now let $H_1, H_2 \in \mathcal{X}_\delta$. Noting that the left hand side of Equation (3.15) gives that $\delta < s$ we have that

$$\begin{aligned} \|\Phi(H_1) - \Phi(H_2)\|_{\nu,\tau} &\leq \|\mathfrak{L}^{-1}\|_{B(\mathcal{X})}\|R(H_1) - R(H_2)\|_{\nu,\tau} \\ &\leq \|\mathfrak{L}^{-1}\|_{B(\mathcal{X})} \sup_{H \in \mathcal{X}_\delta} \|DR[H]\|_{B(\mathcal{X})}\|H_1 - H_2\|_{\nu,\tau} \\ &\leq C \sup_{|\omega|\leq\tau} \sup_{|z|\leq\delta} \|DR(z,\omega)\|_M \|H_1 - H_2\|_{\nu,\tau} \\ &\leq CC_2\delta\|H_1 - H_2\|_{\nu,\tau}. \end{aligned}$$

Here we pass from line three to line four by **R2**. Now it is again by the right hand side of Equation (3.15) that $CC_2\delta < 1$, so that Φ is a contraction. Then Φ has a unique fixed point on $\mathcal{X}_\delta$ by the Banach Fixed Point Theorem.

$\square$

We now discuss a situation where the previous theorem applies. The operator we define is the second order Taylor remainder of a one parameter family analytic vector fields on $\mathbb{C}^n$, where the for each fixed parameter the vector field is expanded not about a single point, but about the image of some analytic function P from $\mathbb{C}^k$ to $\mathbb{C}^n$ with $k < n$ (so that the image of P is then a sub-manifold of $\mathbb{C}^n$). If we have a certain amount of control over the vector field in a neighborhood of the sub-manifold then we can show that the second order Taylor term, restricted to the sub-manifold satisfies assumptions **R1-R2** above.

Corollary 3.4. *Suppose that $\mathfrak{L}\colon \mathcal{X} \to \mathcal{X}$ is boundedly invertible linear operator with $\|\mathfrak{L}\|_{B(\mathcal{X})} \leq C$, that $0 < \rho' < \rho$, that $f\colon B_\rho(p_0) \times B_\tau \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is a bounded family of analytic functoins, and that $P\colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ with $k < n$ is analytic with $P[B_\nu, \omega] \subset B_{\rho'}(p_0)$ for all $\omega \in B_\tau$.*

Assume that M, K, have that

$$\sup_{|\omega| \leq \tau} \sup_{|z - p_0| \leq \rho} \max_{|\beta| = 2} |\partial_\beta f(z, \omega)| \leq K,$$

and

$$\sum_{|\beta|=2} \frac{2}{\beta!} = M.$$

Assume that $E\colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ is a fixed one parameter family of analytic N-tails with $\|E\|_{\nu,\tau} < \epsilon$ and that there is a $\delta > 0$ with $\delta < (\rho - \rho')e^{-1}$ so that

$$2C\epsilon < \delta < \min\left(\frac{1}{2CMK}, \frac{1}{2\pi enCMK}, (\rho - \rho')e^{-1}\right)$$

For every $(\theta, \omega) \in B_\nu \times B_\tau$ and all $\eta \in B_{se^{-1}}$, let $R_P\colon B_{se^{-1}} \times B_\tau \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ be the analytic function defined by

$$f[P(\theta, \omega) + \eta, \omega] = f[P(\theta, \omega), \omega] + Df[P(\theta, \omega), \omega]\eta + R_{P(\theta,\omega)}[H(\theta, \omega), \omega].$$

Then the equation

$$\mathfrak{L}[H](\theta, \omega) = E(\theta, \omega) + R_{P(\theta,\omega)}[H(\theta, \omega), \omega]$$

has a unique solution $H \in \mathcal{X}_\delta$.

Proof: Define $s = \rho - \rho'$, and note that $s > 0$. We want to bound the second order error term associated with the Taylor expansion of f at $P(\theta, \omega)$ on the ball $B_s[P(\theta, \omega)]$. Then note that for any $\theta \in B_\nu$, $\omega \in B_\tau$ we have that the ball $B_s[(P(\theta, \omega)] \subset B_\rho(p_0)$, and that $B_\rho(p_0)$ is the ball on which we have control of the second order partial derivatives of f.

The argument reduces to a local computation. To this end we fix $(\theta, \omega) \in B_\nu \times B_\tau$ and define $z = P(\theta, \omega)$. For any $\eta \in B_s(z)$ we have that

$$f(z + \eta, \omega) = f(z, \omega) + Df(z, \omega)\eta + R_z(\eta, \omega),$$

with $R_z(\eta, \omega)/\eta \to 0$ as $|\eta| \to 0$, as f is analytic on $B_\rho(p_0)$. $R_z(\eta, \omega)$ can be bound using the Lagrange form of the Taylor Remainder. We see that

$$\begin{aligned}
|R_z(\eta, \omega)| &\leq \max_{1 \leq i \leq n} \left| \sum_{|\beta|=2} \frac{2}{\beta!} \eta^\beta \int_0^1 (1 - t)\partial_\beta f_i(z + t\eta, \omega)\, dt \right| \\
&\leq \max_{1 \leq i \leq n} \sum_{|\beta|=2} \frac{2}{\beta!} |\eta|^{|\beta|} \int_0^1 (1 - t)|\partial_\beta f_i(z + t\eta, \omega)|\, dt \\
&\leq \max_{1 \leq i \leq n} \left(\sup_{|\omega| \leq \tau} \sup_{|w - p_0| \leq \rho} \partial_\beta |f_i(w, \omega)| \right) \sum_{|\beta|=2} \frac{2}{\beta!} s^{|\beta|} \\
&\leq MKs^2.
\end{aligned}$$

Recall that $\delta < se^{-1} < s$ and fix any $h \in B_\delta$. Then there is an $\eta_h \in B_s$ so that

$$h = \frac{\delta}{s}\eta_h.$$

Since $R_z(\cdot,\omega)$ and its first partial derivatives are zero at the origin, $R_z(\cdot,\omega)$ is an analytic 2-tail. Exploiting the bound given by Lemma (2.2) we have that

$$\begin{aligned}
|R_z(h,\omega)| &= \left| R_z\left(\frac{\delta}{s}\eta_h,\omega\right)\right| \\
&\leq \frac{\delta^2}{s^2}\left|R_z\left(\eta_h,\omega\right)\right| \\
&\leq \frac{\delta^2}{s^2}MKs^2 \\
&= MK\delta^2
\end{aligned} \tag{3.16}$$

Now if H is a one parameter family of bounded analytic N-tails with $\|H\|_{\nu,\tau} \leq \delta$, then since the Estimates of Equation (3.16) are uniform in z, h, and ω we have that

$$\sup_{|\omega|\leq\tau}\sup_{|\theta|\leq\nu}\left|R_{P(\theta,\omega)}[H(\theta,\omega),\omega]\right| = \|R_P[H]\|_{\nu,\tau} \leq MK\delta^2,$$

which of the same form as **R1**.

Similarly, in order to bound the derivative R_P we take again $z = P(\theta,\omega)$, and now consider any $0 < \delta < e^{-1}s$. Take $0 < \sigma \leq 1$, and define $\tau = \delta/(se^{-\sigma})$. Let $h \in B_\delta(z)$. Then $h = \tau\hat{\eta}_h$ for some $\hat{\eta}_h \in B_{e^{-\sigma}s}(z)$. Since $DR_z(\cdot,\omega)$ is a matrix whose entries are analytic 1-Tails we have that

$$\begin{aligned}
\left|DR_z\left(\tau\hat{\eta}_h,\omega\right)\right|_M &\leq \tau\left|DR_z\left(\hat{\eta}_h,\omega\right)\right|_M \\
&\leq \frac{\delta}{se^{-\sigma}}\sup_{|\eta|=e^{-\sigma}s}|DR_z(\eta,\omega)|_M \\
&\leq \frac{\delta e^\sigma}{s}\left(\frac{2\pi n}{s\sigma}\sup_{|\eta|=s}|R_z(\eta,\omega)|\right) \\
&\leq \delta\frac{2\pi n e^\sigma}{\sigma s^2}MKs^2,
\end{aligned} \tag{3.17}$$

where we pass from line two to line three using the Cauchy Bound Lemma (2.1).

Taking H a one parameter family of bounded analytic N-tails with $\|H\|_{\nu,\tau} \leq \delta$, we observe again that the estimate of Equation (3.17) is uniform in θ, ω, and σ. Also note that e^σ/σ is minimized at $\sigma = 1$, so that

$$\sup_{|\omega|<\tau}\sup_{|\theta|<\nu}\left\|DR_{P(\theta,\omega)}[H(\theta,\omega),\omega]\right\|_M \leq 2\pi enMK\delta.$$

This is of the form of **R2**. Then Theorem (3.3) applies to $R_P(\cdot,\omega)$ and we have the corollary.

□

We remark that while R_P is defined, analytic, and bounded on B_s, because we have used the Cauchy Bound we only have the needed estimate on the derivative of R_P on the strictly smaller ball $B_{e^{-1}s}$. The is the reason for the restricted domain of definition for R_P.

4. Analytic Taylor Models: Formalism and Validation. In principle (if not in name) the use of so called "Taylor models" for the purposes of computer assisted proofs in analysis appears in the literature as early as the works of [11, 12, 13, 18, 19] on universality, renormalization, and the Feigenbaum conjectures. (We remark that these works appear inaugurate the birth of the field of computer assisted proof in dynamical systems). Taylor models seem to have been developed independently (and so named) beginning with the works of [21, 22, 6], and leading to the development of the COSY Infinity software for computing and manipulating Taylor models [7].

The fundamental idea behind Taylor models is to discretize the space of continuous functions using polynomials of a fixed order, restricted to a fixed domain, and concatenated with a floating point number defining a neighborhood of the polynomial in function space. While the polynomial part is often interpreted as the Taylor coefficients of an analytic function, techniques such as "shrink wrapping" sometime lessen this requirement [15]. We also note that the coefficients of the polynomial of a Taylor may or may not be intervals. In any event round-off errors associated with Taylor model manipulations can be periodically reorganized into the remainder term. Typical operations on such Taylor models include the usual operations of arithmetic, function composition, and integration. However it is generally not possible to differentiate a Taylor model, because of optimizations which redistribute the truncation/round-off errors.

In the present work we require a stricter notion of a Taylor models, in order to recover some regularity. The following defines the fundamental data structure needed in the sequel.

Definition 4.1. labeldef:analyticTaylor An *Analytic Taylor Model* (centered at the origin) for an analytic function $f\colon B_r \subset \mathbb{C} \to \mathbb{C}$ is a polynomial *with interval coefficients*

$$f_M(\omega) = \sum_{m=0}^{M} a_m \omega^m$$

and a number $\delta_f > 0$ so that

$$\frac{1}{n!}\frac{d^n}{d\omega^n} f(0) \in a_m$$

and

$$\|f - f_M\|_r \leq \delta_f.$$

Noting that $h_f\colon B_r \to \mathbb{C}$ defined by $h_f \equiv f - f_M$ is an analytic M-tail we have that f can be represented by

$$f(\omega) = f_M(\omega) + h_f(\omega)$$

where the coefficients of f_M enclose the first M Taylor coefficients of f and the truncation error h_f is uniformly bounded on B_r by δ_f. We represent an analytic Taylor model by the triple (f_M, r, δ_f), i.e an interval polynomial f_M, a positive number $r > 0$ describing radius of the domain of the model, and the bound $\delta > 0$ on the truncation error.

Remark 4.1 (Interval Arithmetic and MatLab). For the implementation of all the numerical computations discussed in the present work we utilize the MatLab software known as IntLab [28]. See [27] for a thorough introduction to interval arithmetic and the algorithms used in the IntLab software.

It is clear that analytic Taylor models form a vector space, and it is clear that we can easily consider vectors of analytic Taylor models. The following Lemma enumerates several basic useful properties of analytic Taylor models.

Lemma 4.2 (Properties of Analytic Taylor Models). *Let (f_M, r, δ) be an analytic Taylor model. Then for any f which is analytic on B_r and which is enclosed by this analytic Taylor model we have that*

(a) $\|f\|_r \leq \sum_{m=0}^{M} |a_m| r^m + \delta \equiv \|f_M\|_{(\Sigma, r)} + \delta$.

(b) Suppose that $\tau > 0$ has that

$$\tau \sum_{m=1}^{M} |a_m| \tau^{m-1} + \delta \leq |a_0|.$$

Let $\tilde{C}$ be defined by

$$|a_0| - \tau \sum_{m=1}^{M} |a_m| \tau^{m-1} - \delta \equiv \tilde{C}.$$

Then

$$\left\| \frac{1}{f} \right\|_{\tau} \leq \frac{1}{\tilde{C}}.$$

(c) For any $0 < \sigma \leq 1$ we have that

$$\|f'\|_{re^{-\sigma}} \leq \sum_{m=0}^{M-1} (m+1)|a_{m+1}| r^m + \frac{2\pi}{\sigma r} \delta_f,$$

by applying the Cauchy Bounds of Lemma (2.1). Then it is moreover the case that $(f'_{M-1}, re^{-\sigma}, 2\pi\delta_f/\sigma r)$ is an analytic Taylor model for f'. (Note that the domain of the new analytic Taylor model is reduced by a factor of $e^{-\sigma}$, the order of the polynomial approximation is reduced by one, and the bound on the truncation error is inverse proportional to the "loss of domain" parameter σ. However the amount of domain given up is exponential in σ.

(d) If (f_M, r, δ_f) and (g_M, r, δ_g) are analytic Taylor models on B_r then $(f_M + g_M, r, \delta_f + \delta_g)$ is an analytic Taylor model for $f + g$.

(e) Let (f_M, r, δ_f) be an M-th order analytic Taylor model and $K < M$ Then there is an analytic M-tail h so that

$$f(\omega) = \sum_{m=0}^{M} a_m \omega^m + h(\omega)$$

with $\|h\|_r \leq \delta_f$. This can be re-written as

$$f(\omega) = \sum_{m=0}^{K} a_m \omega^m + \sum_{m=K+1}^{M} a_m \omega^m + h(\omega),$$

and we note that

$$\hat{h}(\omega) = \sum_{m=K+1}^{M} a_m \omega^m + h(\omega),$$

is an analytic K-tail with

$$\|\hat{h}\|_r \leq \sum_{m=K+1}^{M} |a_M| r^m + \delta_f \equiv \hat{\delta}_f.$$

Then $(f_K, r, \hat{\delta}_f)$ is a K-th order analytic Taylor model for f. This allows us to "step down" the order of a model. In this way we can for exampe add analytic Taylor models of different orders and obtain a model whose order is the minimum of the orders of the summands.

Remark 4.3 (Taylor versus Analytic Taylor Models). Note that it is Lemma (4.2(c)) which exploits the analytic category and justifies the specialized definitions of this section, and distinguishes our analytic Taylor models from the usual ones.

We now state and prove a simple Lemma providing an analytic Taylor model for the product of two analytic Taylor models. The Lemma itself is almost trivial, but it will be instructive later to compare the cost and accuracy of other operations to the cost and accuracy of a product. The Lemma also illustrates the *a-posteriori* philosophy in an simple setting.

Lemma 4.4 (Product of Analytic Taylor Models). *Let (f_M, r, δ_f) and (g_M, r, δ_g) be two analytic Taylor models. Then an analytic Taylor model (p_M, r, δ_p) for the product $(f \cdot g)(\omega)$ is given by the M-th order polynomial p_M whose coefficients given by the Cauchy Product formula*

$$p_m = \sum_{k=0}^{M} a_{m-k} b_k \tag{4.1}$$

(where a_k and b_k are the coefficient of f_M and g_M respectively). Moreover, defining the a-posteriori error

$$E_M(\omega) = f_M(\omega) g_M(\omega) - p_M(\omega),$$

we have the explicit bound

$$\delta_p \leq \|E_M\|_r + \|f_M\|_r \delta_g + \|g_M\|_r \delta_f + \delta_f \delta_g. \tag{4.2}$$

Proof: That the coefficients of p_M are given by Equation (4.1) is just the standard Cauchy Product. We note that while we could obtain a bound on the product p simply by bounding $f \cdot g$, this does not provide an explicit truncation estimate for p_M. So we let $\hat{h}$,$\bar{h}$, and h denote the analytic M-tails of f, g, and p respectively. We have that

$$(f_M + \hat{h})(g_M + \bar{h}) = p_M + h.$$

From this we obtain the bound

$$\delta_p = \|h\|_r \leq \|f_M g_M - p_M + f_M \bar{h} + g_M \hat{h} + \bar{h}\hat{h}\|_r,$$

from which Equation (4.2) follows.

$\square$

The cost of the computation is the cost of a Cauchy Product, plus the cost of the evaluation of $\|E_M\|_r$, $\|f_M\|_r$ and $\|g_M\|_r$. Note that E_M is a $2M$-the order polynomial as this is the order of the product $f_M \cdot g_M$. However, because E_M is obtained by taking p_M from $f_M g_M$ and because the coefficients p_M are determined by the Cauchy Product, E_M will be almost zero to M-th order. (The low order terms of E_M capture the "round off errors" associated with computing the Cauchy Product coefficients). The cost of computing the norms using the sigma norm is the cost of an inner product. Then computing (p_M, r, δ_p) is the cost of a Cauchy product, the cost of a polynomial multiplication, and the cost of three inner products. The bound on the truncation error of the product is the a-posteriori error plus terms proportional to the individual truncation errors of the products.

4.1. Matrix Inversion and Linear Equations. Consider a $K \times K$ matrix of analytic functions

$$B(\omega) = \begin{pmatrix} b_{11}(\omega) & \ldots & b_{1K}(\omega) \\ \vdots & \ddots & \vdots \\ b_{K1}(\omega) & \ldots & b_{KK}(\omega) \end{pmatrix}.$$

Suppose that each of the $b_{ij}(\omega)$ are analytic on the ball $B_r \subset \mathbb{C}$. Suppose further that associated with each b_{ij} is an analytic Taylor model $(b_{ij}^M, r, \delta_{ij})$. We define the matrix valued polynomial

$$B_M(\omega) = \sum_{m=0}^{M} B_m \omega^m$$

with coefficients

$$B_m = \begin{pmatrix} b_{11}^m & \ldots & b_{1K}^m \\ \vdots & \ddots & \vdots \\ b_{K1}^m & \ldots & b_{KK}^m \end{pmatrix}$$

and truncation error with $\delta_B = K\, max_{ij}(\delta_{ij})$. Then we consider the data (B_M, r, δ_B) an analytic Taylor model for the matrix of functions B. Supposing that B is invertible at the origin, we are interested in developing an analytic Taylor model for the matrix inverse of B.

LEMMA 4.5 (Matrix Inversion). *Assume that $B(0) = B_0$ is invertible, that B_0^{-1} is an interval enclosure of its inverse, and that (B_M, r, δ_B) is an analytic Taylor model of B. Moreover assume that there are $M, \tau > 0$ so that*

$$|B_0^{-1}| \left(\tau \sum_{m=1}^{M} |B_m| \tau^{m-1} + \delta_B \right) \leq M < 1. \tag{4.3}$$

Then there is an M-th order analytic Taylor model (C_M, τ, δ_C) for $C(\omega) \equiv B^{-1}(\omega)$, where the coefficients of C_M are defined recursively by

$$C_0 = B_0^{-1}, \qquad \textit{and} \qquad C_m = -B_0^{-1} \sum_{k=0}^{m-1} B_{m-k} C_k \qquad \textit{for} \quad 1 \leq m \leq M. \tag{4.4}$$

Defining the a-posteriori error

$$E_M(\omega) = Id - B_M(\omega)\, C_M(\omega).$$

we have that the truncation error $\delta_C > 0$ satisfies the explicit bound

$$\delta_C \leq \frac{|B_0^{-1}|}{1 - M} \left(\|E_M\|_\tau + \|C_M\|_\tau \ \delta_B \right). \tag{4.5}$$

Proof: The first part of the proof is formal. We seek

$$C(\omega) = \sum_{m=0}^{\infty} C_m \omega^m$$

so that

$$B(\omega) C(\omega) = \mathrm{Id}$$

Expanding as series we have

$$\begin{aligned} B(\omega)C(\omega) &= \left(\sum_{m=0}^{\infty} B_m \omega^m \right) \left(\sum_{m=0}^{\infty} C_m \omega^m \right) \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{m} B_{m-k} C_k \omega^m \\ &= \mathrm{Id} + 0\omega + 0\omega^2 + \dots \end{aligned}$$

Matching like powers of ω we have for $m = 0$ that

$$B_0 C_0 = \mathrm{Id}$$

so that indeed $C_0 = B_0^{-1}$. Similarly when $m \geq 1$ we have that

$$\sum_{k=0}^{m} B_{m-k} C_k = 0,$$

which we solve for C_m in order to obtain that

$$C_m = -B_0^{-1} \sum_{k=0}^{m-1} B_{m-k} C_k,$$

as desired.

This formula is now used in order to compute the M-th order polynomial C_M. Let $G \colon B_r \to L\left(\mathbb{C}^K\right)$ denote the truncation error associated with B. Then G is a $K \times K$ matrix of analytic M-tails with $\|G\|_r \leq \delta$ so that $B(\omega) = B_M(\omega) + G$. Now we seek a constant $\delta_C > 0$ and a $K \times K$ matrix of analytic M-tails $H \colon B_r \to L\left(\mathbb{C}^K\right)$ so that $\|H\|_\tau \leq \delta_C$ and

$$C(\omega) = C_M(\omega) + H(\omega).$$

Since C is the inverse of B, this last equation is equivalent to the condition

$$(B_M(\omega) + G(\omega))(C_M(\omega) + H(\omega)) = \mathrm{Id},$$

for each $\omega \in B_r$. Then formally we have that

$$H(\omega) = B^{-1}(\omega)\left[E_M(\omega) + G(\omega)\, C_M(\omega)\right]. \tag{4.6}$$

for each ω such that B is invertible. The Neuman series is used in order to obtain in fact that B is invertible on B_τ. Moreover we have the explicit bound

$$\begin{aligned}
\|B^{-1}\|_\tau &= \left\| \left(B_0 + \sum_{m=1}^{M} B_m \omega^m + G(\omega) \right)^{-1} \right\|_\tau \\
&\leq \left\| \left(\mathrm{Id} + B_0^{-1} \omega \sum_{m=1}^{M} B_m \omega^{m-1} + B_0^{-1} G(\omega) \right)^{-1} \right\|_\tau |B_0^{-1}| \\
&\leq \frac{|B_0^{-1}|}{1 - M}
\end{aligned} \tag{4.7}$$

where we use that

$$\left\| B_0^{-1} \omega \sum_{m=1}^{M} B_m \omega^{m-1} + B_0^{-1} G(\omega) \right\|_\tau \leq |B_0^{-1}| \; \left(\tau \sum_{m=1}^{M} |B_m| \tau^{m-1} + \delta_B \right) \leq M < 1$$

by the hypothesis that Equation (4.3) holds. Applying the bound of Equation (4.7) to Equation (4.6) gives the bound on δ_C claimed in Equation (4.5)

$\square$

Now if (B_M, r, δ_B) is an analytic Taylor model for an analytic matrix function $B(\omega)$ and (q_M, r, δ_q) is an analytic Taylor modle for a vector of analytic functions q, then we consider the equation $Bp = q$. The following Lemma shows that we can obtain an analytic Taylor model for p without first computing $B^{-1}(\omega)$ directly. The proof of the is almost identical the proof of the previous Lemma and is omitted.

Lemma 4.6 (Solutions of Linear Equations). *Assume that $B(0) = B_0$ is invertible, that B_0^{-1} is an interval enclosure of its inverse and that (B_M, r, δ_B) is an analytic Taylor model of B. Let (q_M, r, δ_q) be an analytic Taylor model of the analytic function q. Assume in addition that there are $M, \tau > 0$ satisfying the bound given in Equation (4.3). Then there is an M-th order analytic Taylor model (p_M, τ, δ_p) for the analytic function p having*

$$B(\omega)p(\omega) = q(\omega) \qquad \textit{for} \qquad \omega \in B_\tau.$$

The coefficients for the polynomial p_M are defined recursively by

$$p_0 = B_0^{-1} q_0 \qquad \text{and} \qquad p_m = B_0^{-1}\left(q_m - \sum_{k=0}^{m-1} B_{m-k} p_k,\right) \tag{4.8}$$

and that truncation estimate satisfies

$$\delta_p \le \frac{|B_0^{-1}|}{1-M}\left(\|E_M\|_\tau + \|p_M\|_\tau \ \delta_B + \delta_q\right)$$

where E_M is the a-posteriori error defined by

$$E_M(\omega) = q_M(\omega) - B_M(\omega) p_M(\omega).$$

4.2. Elementary Functions of Analytic Taylor Models. LEMMA 4.7. *Suppose that (f_M, r, δ_f) is an analytic Taylor model for the analytic function f and $K \in \mathbb{R}$, $K \neq 0$. Denote the coefficients of the polynomial f_M by a_m for $0 \le m \le M$. Assume also that $f(0) = a_0 \neq 0$ and in fact that there are $M, \tau > 0$ so that*

$$|a_0| - \tau \sum_{m=1}^{M} |a_m| \tau^{m-1} - \delta_f \ge M > 0. \tag{4.9}$$

Then for any $0 < \sigma \le 1$ an analytic Taylor model for $p(\omega) \equiv f^K(\omega)$ is given by (p_M, R, δ_p) where

$$R = \min\left(\tau, r e^{-\sigma}\right),$$

the coefficients of p_M are defined recursively by

$$p_0 = a_0^K \qquad \text{and} \qquad p_m = \frac{1}{m a_0} \sum_{k=0}^{m-1} (mK - k(K+1)) a_{m-k} p_k \quad \text{for } 1 \le m \le M, \tag{4.10}$$

and moreover we have an explicit bound on the truncation error of $f^K = p$ given by

$$\delta_p \le \left(\frac{\|f_M\|_r + \delta_f}{M}\right)^K R \left(\|E_M\|_R + K\|p_M\|_R \frac{2\pi}{r\sigma}\delta_f + \|p_M'\|_R \delta_f\right) \tag{4.11}$$

where the a-posteriori error is defined by

$$E_M(\omega) = K f_M'(\omega) p_M(\omega) - f_M(\omega) p_M'(\omega). \tag{4.12}$$

Proof: Of course we actually know that $p = f^K$ is analytic on the same disk B_r as f regardless of the magnitude of a_0. The additional constraints are imposed in order to obtain explicit bounds on the truncation error associated with the M-th order approximation of p.

The coefficients of p_M are computed formally as follows. Let p' denote the derivative of f^K. Then we have

$$p' = K f^{K-1} f'.$$

Multiplying both sides by f gives

$$fp' = Kpf'. \tag{4.13}$$

Expanding f, f', p, and p' as power series (with the coefficients of p unknown), taking the Cauchy Products, matching like powers, and isolating the m-th coefficient of p leads to the recursion relations given in Equation (4.10).

The functional relation given by Equation (4.13) also leads to an effective a-posteriori analysis scheme for p. Let g denote the analytic M-tail so that $f(\omega) = f_M(\omega) + g(\omega)$ on B_r and $\|g\|_r \leq \delta_f$. We seek an analytic M-tail h defined on B_R so that $f^K(\omega) = p_M(\omega) + h(\omega)$ on B_R and $\|h\|_R \leq \delta_p$. Expanding Equation (4.13) gives the first order linear differential equation for h defined by

$$f(\omega)h'(\omega) - Kf'(\omega)h(\omega) = E(\omega) \tag{4.14}$$

where

$$E(\omega) = E_M(\omega) + K\, p_M(\omega)g'(\omega) - p_M'(\omega)g(\omega),$$

and $E_M(\omega)$ is the a-posteriori error given by Equation (4.12). The right hand side has the bound

$$\|E\|_R \leq \|E_M\|_R + \frac{2K\,\pi\|p_M\|_R}{r\sigma}\delta_f + \|p_M'\|_R\delta_f,$$

and the bound in Equation (4.11) is obtainde once we realize that Equation (4.14) has the form discussed in Lemma (2.3) so that the Estimate given by Equation (2.5) provides the needed bound on h.

□

Remark 4.8. Similar Lemmas can be derived for all the so called 'elementary functions' by utilizing that such functions can be expressed as solutions of linear differential equations. For example if we want to compute sin and cos of an analytic Taylor model then we define p by $p(\omega) \equiv e^{if(\omega)} = \sin(f(\omega)) + i\cos(f(\omega))$ and note that

$$p' = ipf'.$$

Again the coefficients can be computed for the cost of a Cauchy Product. Taking real and imaginary parts gives the sine and cosine series. The a-posteriori analysis of the truncations errors can be done by exploiting the differential equation.

4.3. One Parameter Branches of Zeros for Finite Dimensional Non-Linear Problems. Definition 4.2. [Validation Values for a One Parameter Branch of Solutions] Suppose that $f\colon \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is a one parameter family of analytic maps, that $p_0 \in \mathbb{C}^n$ has $f(p_0, 0) = 0$, and that $D_1 f(p_0, 0)$ is invertible. (Here D_1 applied to $f(x, \omega)$ means the differential with respect to the 'first' variable, namely the variable x. Since $x \in \mathbb{C}^n$ $D_1 f$ is an $n \times n$ matrix of analytic functions. The entries of $D_1 f$ are functions in the variables $x \in \mathbb{C}^n$ and $\omega \in \mathbb{C}$). In addition we assume the existence of the following data.

(1) Assume that B_0^{-1} is an interval inclosure of $D_1 f(p_0, 0)^{-1}$, and suppose that (B_M, r, δ_B) is an analytic Taylor model for

$$B(\omega) = D_1 f(p_M(\omega), \omega).$$

(2) Assume that there exist $M, \tau > 0$ having that $0 < \tau < r$ and

$$|B_0^{-1}| \left(\tau \sum_{m=1}^{M} |B_m| \tau^{m-1} + \delta_B \right) \leq M < 1.$$

Then Lemma (4.5) allows us to construct an analytic Taylor model (C_M, τ, δ_C) so that $C(\omega) = B^{-1}(\omega)$. In particular we have that

$$\sup_{\omega \in B_\tau} \left| [D_1 f(p_M(\omega), \omega)]^{-1} \right| \leq \|C_M\|_\tau + \delta_C.$$

(3) Assume that there is an $\epsilon > 0$ and an M-th order polynomial $p_M : B_\tau \subset \mathbb{C} \to \mathbb{C}^N$ having

$$|f\,(p_M(\omega), \omega)\,| < \epsilon \qquad \text{for all} \qquad \omega \in B_\tau.$$

Lemma 4.9 (A-Posteriori Validation of a Branch of Zeros). *Suppose that f, p_0, B_0^{-1}, B_M, τ, δ_B, ϵ, p_M, M, C_M, and δ_C are as in Definition (4.2). Let $\epsilon_{NK} > 0$ be any constant with*

$$(\|C_M\|_\tau + \delta_C)\, \epsilon \leq \epsilon_{NK}.$$

Define

$$R = 2\epsilon_{NK}.$$

Let $C = \sum_{m=1}^{M} |p_m| \tau^m + R$ and define the set

$$U = \{z \in \mathbb{C}^n \,:\, |z - p_0| \leq C\}.$$

Now let $\kappa > 0$ have that

$$n^2 \sup_{x \in U} \max_{1 \leq i \leq j \leq n} \|\partial_{ij} f(x)\| \leq \kappa.$$

Suppose that

$$4\epsilon_{NK} \kappa (\|C_M\|_\tau + \delta_C) < 1. \tag{4.15}$$

Then there is a unique analytic M-tail $h : B_\tau \to \mathbb{C}^N$ with

$$\|h\|_\tau \leq R$$

so that $p(\omega) = p_M(\omega) + h(\omega)$ is a one parameter analytic branch of zeros of f. In other words (p_M, τ, R) is an analytic Taylor model for the analytic function $p : B_\tau \to \mathbb{C}^n$ having

$$f\,[p(\omega), \omega)] = 0 \quad \textit{for all} \quad \omega \in B_\tau.$$

Proof: Let $\mathcal{X} = C^\omega(\mathbb{C}, \mathbb{C}^n)$ and define the operator $\Phi : \mathcal{X} \to \mathcal{X}$ by

$$\Phi[q](\omega) = f[q(\omega), \omega].$$

We note that the Frechet Differential of Φ is the linear operator $D\Phi \in \mathfrak{L}(\mathcal{X}, \mathcal{X})$ given by

$$D\Phi[q](\omega) = D_1 f[q(\omega), \omega].$$

$B(\omega) = D_1 f[p_M(\omega), \omega] \equiv D\Phi[p_M]$ is invertible for all $\omega \in B_\tau$ by hypothesis. Moreover we have that

$$\|[D\Phi(p_M)]^{-1}\|_\tau \leq \|C_M\|_\tau + \delta_C.$$

Then

$$\|[D\Phi(p_M)]^{-1}\Phi[p_M]\|_\tau \leq (\|C_M\|_\tau + \delta_C)\epsilon \leq \epsilon_{\text{NK}}.$$

Let $V = \{v \in \mathcal{X} \,:\, \|v\|_\tau \leq R\}$, where we recall that $R = 2\epsilon_{\text{NK}}$. Then for any $q \in p_M + U$ and $\omega \in B_\tau$ we have that

$$\|p_0 - (p_M(\omega) + q(\omega))\| \leq \sum_{m=1}^{M} |p_m|\tau^m + R = C.$$

From this we see that image$(p_M + q) \subset p_0 + U \subset \mathbb{C}^n$. It follows that for any $q_1, q_2 \in V$ we have that

$$\begin{aligned}
\|D\Phi[p_M + q_1] - D\Phi[p_M + q_2]\| &\leq \left(\sup_{q \in p_M + V} \|D^2\Phi[q]\|_\tau \right) \|q_1 - q_2\|_\tau \\
&= \left(\sup_{q \in p_M + V,\ \omega \in B_\tau} \|D^2 f[p_M(\omega) + q(\omega), \omega]\right) \|q_1 - q_2\|_\tau \\
&\leq \left(\sup_{x \in U,\ \omega \in B_\tau} \|D^2 f[x, \omega]\| \right) \|q_1 - q_2\|_\tau \\
&\leq \kappa \|q_1 - q_2\|_\tau,
\end{aligned}$$

by the Mean-Value Theorem and the definition of κ. Recalling Equation (4.15), the Newton-Kantorovich Theorem applied to $\Phi[p_M](\omega)$ provides a unique $h \in V$ so that $\Phi[p_M + h](\omega) = f[p_M(\omega) + h(\omega), \omega] = 0$ for all $\omega \in B_\tau$.

□

4.4. Examples: Parameterized Arcs of Fixed Points, Equilibria, Eigenvalues, Eigenvectors, and Diagonalizing Matrices. Lemma (4.9) can be applied directly in order to validate analytic Taylor models for one parameter families of equilibria. Since fixed points of diffeomorphisms can be expressed as zeros finding problems, the Lemma can also be used to validate analytic Taylor models for families of fixed points as well. In fact one parameter branches eigenvalues and eigenvectors can also be viewed as zero sets, and so can be fit into the framework of Lemma (4.9) as well. We consider several examples.

Example 1: A One Parameter Branch of Fixed Points for the Hénon Map. Consider the one parameter family of Hénon mappings defined by

$$f(x, y, \omega) = \begin{bmatrix} y + 1 - ax^2 \\ (b + \omega)x \end{bmatrix}, \tag{4.16}$$

where we think of a and b as fixed. We begin by developing a formal expansion for a branch of fixed points for the family. Let

$$x(\omega) = \sum_{n=0}^{\infty} x_n \omega^n$$

parameterize an analytic branch of the first component of a fixed point of Equation (4.16). Then $x(\omega)$ solves

$$a[x(\omega)]^2 + x(\omega)(1 - b - \omega) - 1 = 0. \tag{4.17}$$

From this we see that

$$x_0 = \frac{b - 1 \pm \sqrt{(1-b)^2 + 4a}}{2a}, \qquad \text{and} \qquad x_1 = \frac{d}{d\omega} x(0) = \frac{x_0}{2ax_0 - b + 1}. \tag{4.18}$$

Matching like powers of ω in equation 4.17 gives that

$$x_n = \frac{1}{2ax_0 - b + 1}\left[x_{n-1} - \sum_{k=1}^{n-1} a\, x_{n-k} x_k\right]. \qquad \text{for} \qquad n \geq 2. \tag{4.19}$$

We note that since the second component of the fixed point is given by $y(\omega) = (b + \omega)x(\omega)$ we now have

$$y_0 = bx_0, \qquad y_1 = bx_1 + x_0 \qquad \text{and} \qquad y_n = bx_n + x_{n-1} \qquad n \geq 2. \tag{4.20}$$

These recursion relations can be used to define a polynomial approximation

$$p_M(\omega) = \sum_{m=0}^{M} \begin{bmatrix} x_m \\ y_m \end{bmatrix} \omega^m$$

of a branch of fixed points for this Hénon family to any desired finite order M. Then Lemma (4.9) can be applied in order to validate a branch of zeros of the map

$$F(p_M(\omega), \omega) = f[p_M(\omega), \omega] - p_M(\omega).$$

An analytic Taylor model for a branch of zeros of F is a model of a branch of fixed points of the Hénon family. This calculation is carried out by the program *paperCodeEx1*. Performance results for several program parameters at the classic values of $a = 1.4$, $b = 0.3$ are given in Table (4.1).

Example 2: One Parameter Family of Eigenvalues and Eigenvectors for a Fixed Point of the Hénon Map. We now consider the eigenvalue problem at a fixed point of the Hénon family. If λ_0 is an eigenvalue of $D_{(x,y)}f(x, y, 0)$ then we let

$$\lambda(\omega) = \sum_{n=0}^{\infty} \lambda_n \omega^n$$

parameterize a branch of eigenvalues passing through λ_0. Then $\lambda(\omega)$ satisfies the equation

$$\lambda(\omega)^2 + 2a\, x(\omega)\, \lambda(\omega) - \omega - b = 0, \tag{4.21}$$

M	τ	δ_{p_1}	δ_{p_2}	r_1	r_2	t
2	10^{-4}	1.66×10^{-13}	1.75×10^{-13}	6.91×10^{-14}	6.92×10^{-15}	0.31(sec)
2	10^{-2}	1.67×10^{-7}	1.73×10^{-7}	6.89×10^{-8}	6.90×10^{-8}	0.31(sec)
2	0.23	0.0024	0.0028	8.57×10^{-4}	8.57×10^{-4}	0.3(sec)
5	0.23	3.47×10^{-7}	2.85×10^{-7}	1.13×10^{-7}	1.13×10^{-7}	0.71(sec)
10	0.23	8.51×10^{-13}	5.89×10^{-13}	2.45×10^{-13}	2.45×10^{-13}	1.7(sec)
15	0.23	4.73×10^{-15}	7.33×10^{-15}	7.77×10^{-16}	8.88×10^{-16}	3.1(sec)

TABLE 4.1

Fixed Point Branch Performance Data for the Hénon Family: *M is the parameterization order, τ is the radius of the domain of the analytic Taylor model, i.e. each model is validated for the real interval $\omega \in [-\tau, \tau]$. We compute models of a branch of fixed points for both p_1 and p_2. The associated truncation errors (the δ values) are given for each model. The columns labeled r_1 and r_2 are qualitative assessments of the error. For each branch we evaluate the polynomial at $\omega = \pm\tau$. We include the truncation errors into the interval results. We compare this to values of the fixed points given by the explicit formulas. r_1 is the maximum error over $\pm\tau$ for the first fixed point and similarly for r_2. Then r_1 and r_2 represent the observed error, while the δ's give theoretical bounds on the error. Note that the r's are always smaller than the δ's. The computation time for each fixed point branch is given as well. We note that the proof fails for $\tau = 0.23$ due to loss of control of the bounds on the norm of the inverse of the differential. For $\tau = 0.23$ the accuracy is not noticeably increased by computing to higher order than fifteen.*

with $\lambda(0) = \lambda_0$. We have that

$$\lambda_0 = -ax_0 \pm \sqrt{a^2x_0^2 + b}, \qquad \lambda_1 = \frac{1 - 2ax_1\lambda_0}{2\lambda_0 + 2ax_0} \tag{4.22}$$

and

$$\lambda_n = \frac{-1}{2\lambda_0 + 2ax_0}\left(\sum_{k=1}^{n-1}\lambda_{n-k}\lambda_k + \sum_{k=0}^{n-1} 2ax_{n-k}\lambda_k\right) \qquad \text{with} \qquad n \geq 2. \tag{4.23}$$

Then the λ_n are formally well defined as long as $\lambda_0 \neq -ax_0$, i.e. as long as λ_0 is not a repeated eigenvalue. Also note that the coefficient λ_n depends on the coefficients of x_i of $x(\omega)$ but only for $0 \leq i \leq n$. Then if we want to compute $\lambda(\omega)$ to order M we need only compute $x(\omega)$ up to order M. Now we choose an eigenvector ξ_0 with $\|\xi\|^2 = \hat{K}$ for some $\hat{K} > 0$, associated with the eigenvalue λ_0. Denote by

$$\xi(\omega) = \sum_{n=0}^{\infty} \xi_n \omega^n$$

a parameterizatoin of the branch of eigenvectors through ξ_0, where the entire branch is normalized to have length $\sqrt{\hat{K}}$. Then $\xi(\omega)$ satisfies the system of nonlinear equations

$$\begin{bmatrix} -2ax(\omega) - \lambda(\omega) & 1 \\ b + \omega & -\lambda(\omega) \end{bmatrix} \begin{pmatrix} \xi_1(\omega) \\ \xi_2(\omega) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\xi_1(\omega)^2 + \xi_2(\omega)^2 = \hat{K}$$

but since the rows of the matrix equation are linearly dependent, we throw away the first row of the matrix and have that $\xi(\omega)$ solves

$$\begin{pmatrix} (b+\omega)\xi_1(\omega) - \lambda(\omega)\xi_2(\omega) \\ \xi_1(\omega)^2 + \xi_2(\omega)^2 - \hat{K} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.24}$$

Matching like powers leads to

$$\begin{bmatrix} b & -\lambda_0 \\ 2\xi_0^1 & 2\xi_0^2 \end{bmatrix} \begin{pmatrix} \xi_1^1(\omega) \\ \xi_1^2(\omega) \end{pmatrix} = \begin{pmatrix} \lambda_1 \xi_0^2 - \xi_0^1 \\ 0 \end{pmatrix}$$

for the coefficient ξ_1 and

$$\begin{bmatrix} b & -\lambda_0 \\ 2\xi_0^1 & 2\xi_0^2 \end{bmatrix} \begin{pmatrix} \xi_n^1(\omega) \\ \xi_n^2(\omega) \end{pmatrix} = \begin{pmatrix} -\xi_{n-1}^1 + \sum_{k=0}^{n-1} \lambda_{n-k}\xi_k^2 \\ -\sum_{k=1}^{n-1} \xi_{n-k}^1 \xi_k^1 + \xi_{n-k}^2 \xi_k^2 \end{pmatrix} \tag{4.25}$$

for ξ_n when $n \geq 2$. The coefficient ξ_n depends recursively on the coefficients of $\lambda(\omega)$ to n-th order.

Now suppose that we use the recursion relations above and compute M-th order polynomial approximations $\lambda_M(\omega)$ and $\xi_M(\omega)$ for a branch of eigenvalues and eigenvectors for Hénon. We need to approximate the truncation error associated with these polynomial approximations in order to obtain rigorous analytic Taylor models. To do this we simply define the maps $F_{\text{eigenvalue}} \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $F_{\text{eigenvector}} \colon \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$ by

$$F_{\text{eigenvalue}}(\lambda, \omega) = \lambda^2 + 2ax(\omega)\lambda - \omega - b$$

and

$$F_{\text{eigenvector}} = \begin{pmatrix} (b+\omega)\xi_1 - \lambda(\omega)\xi_2 \\ \xi_1^2 + \xi_2^2 - \hat{K} \end{pmatrix}.$$

Since $F_{\text{eigenvalue}}(\lambda_M(\omega), \omega)$ and $F_{\text{eigenvector}}(\xi_M(\omega), \omega)$ are approximately zero we again use Lemma (4.9) in order to obtain rigorous rigorous bounds of the truncation errors for the eigendata.

Note the first component of the branch of fixed points $x(\omega)$ in the definition of $F_{\text{eigenvalue}}$ and the branch of eigenvalues $\lambda(\omega)$ in the definition of $F_{\text{eigenvectors}}$ are only known up to analytic Taylor approximation. More precisely let (x_M, τ, δ_x) be the analytic Taylor model for the first component of the fixed point branch, and $(\lambda_M, \tau, \delta_\lambda)$ be the analytic Taylor model for the branch of eigenvalues through λ_0. The Newton-Kanrorivich a-posteriori errors have

$$\|F_{\text{eigenvalue}}(\lambda_M(\omega), \omega)\|_{tau} \leq \|\lambda_M(\omega)^2 + 2a\, x_M(\omega)\lambda_M(\omega) - \omega - b\|_\tau + 2a\|\lambda_M\|_\tau \delta_x,$$

and

$$\|F_{\text{eigenvector}}(\xi_M(\omega), \omega)\|_\tau \leq \left\| \begin{pmatrix} b\xi_M^1(\omega) + \omega\xi_M^1(\omega) - \lambda_M(\omega)\xi_M^2(\omega) \\ \left[\xi_M^1(\omega)\right]^2 + \left[\xi_M^2(\omega)\right]^2 - \hat{K} \end{pmatrix} \right\|_\tau + \|\xi_M^2\|_\tau \delta_\lambda$$

Similarly, in both cases we must provide analytic Taylor models for the differentials as these are functions of models themselves. For example if $h(\omega)$ is the truncation error associated with the analytic Taylor model (x_M, τ, δ_x) then we have

$$D_1 F_{\text{eigenvalue}}(\lambda_M(\omega), \omega) = 2\lambda_M(\omega) + 2a\, x(\omega) + 2a\delta_x$$

So that $(2\lambda_M + 2a\xi_M^1, \tau, 2|a|\delta_x)$ is an analytic Taylor model for the differential. Similarly if $h_\lambda(\omega)$ is the truncation error associated with the analytic Taylor model $(\lambda_M, \tau, \delta_\lambda)$ we have that

$$D_1 F_{\text{eigenvector}}(\xi_M(\omega), \omega) = \sum_{m=0}^{M} B_m \omega^m \quad + \quad \begin{pmatrix} 0 & -h_\lambda(\omega) \\ 0 & 0 \end{pmatrix}$$

M	τ_1	τ_2	$\delta_{\lambda_1,\lambda_2}$	$\delta_{\lambda_3,\lambda_4}$	δ_{ξ_1,ξ_2}	δ_{ξ_3,ξ_4}	time
5	0.001	0.001	2.5×10^{-14}	3.7×10^{-14}	1.3×10^{-14}	7.9×10^{-15}	3.4 (sec)
5	0.13	0.14	1.5×10^{-6}	1.4×10^{-7}	1.9×10^{-5}	3.5×10^{-7}	3.4 (sec)
10	0.13	0.14	8.92×10^{-11}	9.7×10^{-14}	1.2×10^{-9}	2.9×10^{-14}	8.8 (sec)
25	0.13	0.14	4.4×10^{-14}	6.4×10^{-14}	5.3×10^{-14}	3.8×10^{-14}	43 (sec)

TABLE 4.2

Branch of Eigenvalues/vectors Performance Data for the Hénon Family: *M is the parameterization order, τ_1 and τ_2 are the parameterization domains for the branches associated with fixed points one and two respectively. The eigenvalues and eigenvectors associated with fixed point one are subscripted one and two, while the eigenvalues and eigenvectors associated with fixed point two are subscripted three and four. $\delta_{\lambda_1,\lambda_2}$ is the maximum truncation error over the two eigenvalues and similarly for the remaining deltas.*

where

$$B_0 = \begin{pmatrix} b & -\lambda_0 \\ 2\xi_0^1 & 2\xi_0^2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & -\lambda_1 \\ 2\xi_1^1 & 2\xi_1^2 \end{pmatrix}, \text{and } B_m = \begin{pmatrix} 0 & -\lambda_m \\ 2\xi_m^1 & 2\xi_m^2 \end{pmatrix},$$

so that $(B_M, \tau, \delta_\lambda)$ is an analytic Taylor model for $D_1 F_{\text{eigenvectors}}(\xi_M(\omega), \omega)$. Then we can compute analytic Taylor models for

$$[D_1 F_{\text{eigenvalues}}(\lambda_M(\omega), \omega)]^{-1} \quad \text{and} \quad [D_1 F_{\text{eigenvectors}}(\xi_M(\omega), \omega)]^{-1}$$

using Lemma (4.5). Once this is done we have all the ingredients needed to apply Lemma(4.9) in order to validate the branches. An implementation of these computations can be found in *paperCodeEx2*, and some performance data is recorded in Table (4.2).

Example 3: We now consider powers of the analytic Taylor models of the one parameter expansions of the eigenvalues computed in the previous section. Consider for example the stable eigenvalue associated with p_1 for the Hénon map with $a = 1.4$ and $b = 0.3$. Recall that the stable eigenvalue associated with the fixed point p_1 has

$$\lambda_s \in [0.155946322302793, 0.155946322302794]$$

We begin by computing an analytic Taylor model for the one parameter branch of eigenvalues through λ_s. We take a model $(\lambda_s^M(\omega), \tau, \delta)$ for the eigenvalue branch with $M = 10$, $\tau = 0.1$, and $\delta = 4.8 \times 10^{-12}$. Using Lemma (4.7) we compute an analytic Taylor model for the fifth power of $\lambda_s(\omega)$ with $M = 10$, $\bar{\tau} = 0.995$, and validated error $\delta_5 = 1.6 \times 10^{-10}$. Here we have used a loss of domain parameter $\sigma = 0.005$ as required by Lemma (4.7). An analytic Taylor model for the twelfth power of $\lambda_s(\omega)$ with the same loss of domain parameter has $\delta_{12} = 7.4 \times 10^{-13}$ while the analytic Taylor model for the twentieth and thirtith powers have $\delta_{20} = 6.2 \times 10^{-15}$ and $\delta_{30} = \times 1.9 \times 10^{-16}$. This decay in the truncation error is due to the fact that we are working with the expansion of an eigenvalue whose norm is less than one.

Suppose instead we work with the unstable eigenvalue associated with p_1, which we recall has

$$\lambda_u \in [-1.923738858153409, -1.923738858153407],$$

and compute an analytic Taylor model for $\lambda_u(\omega)$ with $M = 10$, $\tau = 0.1$, and $\delta = 4.8 \times 10^{-12}$. Now we compute analytic Taylor models for say the second, fifth, tenth and twentieth powers of $\lambda_u(\omega)$ and obtain $\delta_2 = 5.1 \times 10^{-8}$, $\delta_5 = 1.1 \times 10^{-6}$, $\delta_10 = 7.5 \times 10^{-5}$, and $\delta_{20} = 0.299$. We see that the truncation errors don't decay, but rather grow when the constant term of the original analytic Taylor model is greater than one.

This is not terribly surprising when we observe that the same phenomenon occurs when simply computing powers of intervals. For example if we just compute powers of an interval of radius 10^{-12} about the number 2 then we see that the radii of the intervals enclosing the powers of the initial interval grow in a similar way.

Fortunately when we validate the errors of stable/unstable manifolds for maps using the techniques of Section (5) we never need to compute validated bounds on powers of analytic functions whose constant terms have absolute value greater than one. The reason for this is that we validate a polynomial expansion for the unstable manifold of a diffeomorphism f by treating it as the stable manifold of the inverse map f^{-1}. For example when we validate the stable manifold of the inverse of the Hénon (i.e. the unstable manifold of Hénon) then instead of computing analytic Taylor models for the powers $\lambda_u^n(\omega)$ we compute analytic Taylor models for the function $\lambda_u^{-1}(\omega)$ (which of course has constant term with absolute value less than one) and consider powers of this. We recall that an analytic Taylor model can be computed for $\lambda_u^{-1}(\omega)$ by utilizing Lemma (4.5) and the fact that a number can be thought of as a 1×1 matrix and the reciprocal as the matrix inverse. Using this scheme we obtain results for the powers of the reciprocal of the unstable eigenvalues which are as good as the results above for powers of the stable eigenvalues, i.e. we have the the truncation errors decay as a function of the powers of n.

Note also that for differential equations we can have stable or unstable eigenvalues with absolute value greater than one because for differential equations stability is determined instead by the sign of the real part of the eigenvalue. However for differential equations we are do not have to consider powers of eigenvalues at all (rather we must only contend with linear combinations of eigenvalues) so this particular problem does not arise at all.

5. Parameterized Families of Invariant Manifolds.

5.1. Formal Computation of Coefficients. In this section we develop the formalism for polynomial approximations of one parameter families of invariant manifolds, as promised in the Introduction. First however it is highly instructive to recall several basic facts about formal the computations at a single parameter value. As discussed in the Introduction of the present work, the problem of finding a parameterizatoin of the stable/unstable manifold of a fixed vector field f is equivalent to the problem of solving the partial differential equation

$$f[P(\theta)] = DP(\theta)\Lambda\theta \tag{5.1}$$

under the constraints that $P(0)$ an equilibria and that $DP(0)$ the matrix of stable/unstable eigenvectors. Here Λ is a numerical matrix of fixed complex numbers. (Namely the stable/unstable eigenvalues of the differential at the equilibria).

Moreover it is shown in [8, 10] that (under mild non-degeneracy conditions which will be recalled momentarial) the coefficients a_α for $|\alpha| \geq 2$ of the power series solution $P(\theta) = \sum_{|\alpha|=0}^{\infty} a_\alpha \theta^\alpha$ themselves solve the *homological equation*

$$[Df(p_0) - (\alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k)\mathrm{Id}_n]a_\alpha = s_\alpha. \tag{5.2}$$

The equation is derived by a power matching scheme. Here s_α is a non-linear function of the the coefficients $a_{\alpha'}$ with $|\alpha'| < |\alpha|$. The from of the nonlinearity depends on the nonlinearity of f. Then Equation (5.2) is a matrix equation whose only unknown is a_α. For the specific example of the Lorenz system, an explicit formula the s_α associated with the two dimensional invariant manifolds of any equilibria is given by

$$s_{(n_1,n_2)} = \sum_{0<k+j<n_1+n_2} \begin{pmatrix} 0 \\ a^1_{(n_1-j,n_2-k)}\, a^3_{(j,k)} \\ -a^1_{(n_1-j,n_2-k)}\, a^2_{(j,k)} \end{pmatrix} \tag{5.3}$$

for all two dimensional multi-indices (n_1, n_2) with $n_1 + n_2 \geq 2$.

Similarly a parameterizatoin of the stable/unstable manifold of a fixed point of a diffeomorphism f solves the problem

$$f[P(\theta)] = P(\Lambda\theta) \tag{5.4}$$

with $P(0)$ an equilibria and $DP(0)$ the matrix of stable/unstable eigenvalues. Again under mild non-degeneracy conditions, the coefficients a_α for $|\alpha| \geq 2$ of the power series solution $P(\theta) = \sum_{|\alpha|=0}^{\infty} a_\alpha \theta^\alpha$ solve the *homological equation*

$$[Df(p_0) - (\lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_k^{\alpha_k})\mathrm{Id}_n]a_\alpha = s_\alpha. \tag{5.5}$$

Again s_α is a non-linear function of the the coefficients $a_{\alpha'}$ with $|\alpha'| < |\alpha|$. For the Hénon map one can for example work out that the homological equation for a stable/unstable manifold is

$$\begin{pmatrix} -2aa_0^1 - \lambda^n & 1 \\ b & -\lambda^n \end{pmatrix} \begin{bmatrix} a_n^1 \\ a_n^2 \end{bmatrix} = \begin{bmatrix} a\sum_{k=1}^{n-1} a^1_{n-k} a^1_k \\ 0 \end{bmatrix} \tag{5.6}$$

The explicit derivation of this homological equation can be found for example in (CITE FRECHENCII).

Then the following lemmas provide conditions under which we can define (at least formally) the chart maps parameterizing the stable/unstable manifolds discussed above. The proofs of the lemmas follow immediately from the discussion above, however reader interested in the details can consult [8, 5] (KONST ME MAPS, TuCKER). The idea is that the left and side of the homological equations are characteristic equations for the differential of the fixed point/equilibria. Then the coefficient a_α fails to be defined if and only if the sum $(\alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k)$ (for flows) or the product $\lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_k^{\alpha_k}$ (for maps) is itself equal to an eigenvalue. Should this equality occur we say that there is a *resonance*. Since $\lambda_1, \ldots, \lambda_k$ are eigenvalues of like stability, there are only a finite number of possible resonances, and no "small divisors". (This is in contrast to the situation in KAM/normal form theory where one encounters homological equations with eigenvalues of mixed stability, i.e. some stable and some unstable or even some with elliptic stability).

Lemma 5.1 (Existence of a Stable Formal Solution for Differential Equations). *Assume that p_0 is an equilibria of an analytic vector field f and that $Df(p_0)$ is hyperbolic. Let $\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_n$ be the eigenvalues of $Df(p_0)$ and suppose that the first k eigenvalues are stable and the remaining $n-k$ eigenvalues are unstable (in the sense of differential equatoins). Define*

$$\mu_* = \min_{1\leq i\leq k} |\mathit{real}(\lambda_i)|, \qquad \mathit{and} \qquad \mu^* = \max_{1\leq i\leq k} |\mathit{real}(\lambda_i)|.$$

Assume that for each $\alpha \in \mathbb{N}^k$ with $2 \leq |\alpha| \leq \lceil \mu^*/\mu_* \rceil$ we have that the non-resonance condition

$$\alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k \neq \lambda_i, \qquad 1 \leq i \leq k$$

holds. Then the solution of Equatoin (5.1) is formally well define to all orders.

Lemma 5.2 (Existence of a Stable Formal Solution for Maps). *Assume that p_0 is a fixed point of an analytic diffeomorphism f and that $Df(p_0)$ is hyperbolic. Let $\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_n$ be the k stable and $n-k$ unstable (in the sense of maps) eigenvalues of $Df(p_0)$. Define*

$$\mu_* = \min_{1\leq i\leq k} |\lambda_i| \qquad \text{and} \qquad \mu^* = \max_{1\leq i\leq k} |\lambda_i|.$$

Assume that for each $\alpha \in \mathbb{N}^k$ with $2 \leq |\alpha| \leq \lceil \ln(\mu_*)/\ln(\mu^*) \rceil$ we have that the non-resonance condition

$$\lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_k^{\alpha_k} \neq \lambda_i, \qquad 1 \leq i \leq k$$

holds. The the solution of Equatoin (5.4) is formally well define to all orders.

Remarks 5.3.

(A) **(Unstable Manifold Parameterization)** When considering the parameterization of an unstable manifold for differential equations we apply Lemma (5.1) to $-f$. Since this has differential $-Df(p_0)$ the unstable eigenvalues of f become the stable eigenvalues of $-f$. Similarly when considering the parameterization of an unstable manifold for diffeomorphisms we apply Lemma (5.2) to $Df^{-1}(p_0)$, and again the unstable eigenvalues of $Df(p_0)$ become the stable eigenvalues of $Df^{-1}(p_0)$.

(B) **(Systems With a Single Stable/Unstable Direction)** Suppose that $k = 1$ so that the system has only one stable eigendirection, and hence the stable manifold is one dimensional. Then the multi-indices are one dimensional (i.e. $\alpha = n \in \mathbb{N}$) and Equations (5.2) and (5.5) reduce to

$$[Df(p_0) - n\lambda Id]a_n = s_n, \qquad \text{and} \qquad [Df(p_0) - \lambda^n Id]a_n = s_n,$$

respectively. Then since $n \geq 2$ and λ is the only stable eigenvalue it is impossible to have either $n\lambda = \lambda$ (in the case of differential equations) or $\lambda^n = \lambda$ (in the case of maps). We conclude that in the case of one stable direction there are never resonances, and the parameterizations are formally defined to all orders. A similar remark holds for the case of a singe unstable direction.

(C) **(Real Systems With a Single Complex Stable/Unstable Directoin)** Similarly if f is real, $k = 2$, and λ_1 is complex, then it follows that $\lambda_2 = \bar{\lambda}_1$ (i.e. in real systems complex eigenvalues occur in complex conjugate pairs). Considering a two dimensional multi-index $\alpha = (n_1, n_2) \in \mathbb{N}^2$ we see that both $n_1\lambda_1 + n_2\bar{\lambda_1} = \lambda_{1,2}$ (for differential equations) and $\lambda_1^{n_1} \cdot \bar{\lambda_1}^{n_2} = \lambda_{1,2}$ (for maps) are impossible. So here again there are no possible resonances.

5.1.1. Formal Computation of a One Parameter Branch of Invariant Manifolds for the Hénon Map. Consider again the Hénon Family given by Equation (4.16). At $\omega = 0$ choose p_0 one of the maps two fixed points and let λ_0 and ξ_0 be the stable eigenvalue and associated eigenvector of $Df(p_0, 0)$. As discussed in Section (4.4) we can compute analytic Taylor Models for

$$p(\omega) = \sum_{m=0}^{\infty} p_m \omega^m, \qquad \lambda(\omega) = \sum_{m=0}^{\infty} \lambda_m \omega^m \qquad \text{and} \qquad \xi(\omega) = \sum_{m=0}^{\infty} \xi_m \omega^m,$$

using Equations (4.19), (4.20), (4.23), and (4.25) to compute the first M coefficients and Lemma (4.7) to validate the truncation errors.

As mentioned in the Introduction (and proved in [10]) there exists an analytic branch of parameterizations $P(\theta, \omega)$ for the invariant stable/unstable manifold at p_0. We denote its unknown power series by

$$P(\theta, \omega) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_{(m,n)} \theta^n \omega^m.$$

For the Hénon family the Equation (1.3) reduces to

$$f[P(\theta, \omega), \omega] = P[\lambda(\omega)\theta, \omega]. \tag{5.7}$$

By imposing the linear constraints given by Equation (1.4) we have that $p_{(m,0)} = p_m$, $p_{(m,1)} = \xi_m$.

The coefficients $p_{(0,n)}$ are the coefficients of the parameterization when $\omega = 0$. These are computed by solving the homological equation for the Hénon map given by Equation (5.6). We obtain the equations for the coefficients $p_{(mn)}$ when $n \geq 2, m \geq 1$ by plugging the unknown power series representation for P into Equation (5.7) and matching like powers of ω and θ. First we define the coefficients $\lambda_{(m,n)}$ by the series expansion of $\lambda(\omega)^n$. So

$$[\lambda(\omega)]^n \equiv \sum_{m=0}^{\infty} \lambda_{(m,n)} \omega^m.$$

We expand the right hand side of Equation (5.7) and obtain

$$\begin{aligned} P[\lambda(\omega)\theta, \omega] &= \sum_{n=0}^{\infty} p_n(\omega) [\lambda(\omega)]^n \theta^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} p_{(m,n)} \omega^m \right) \left(\sum_{m=0}^{\infty} \lambda_{(m,n)} \omega^m \right) \theta^n \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \lambda_{(m-k,n)} \begin{bmatrix} p^1_{(k,n)} \\ p^2_{(k,n)} \end{bmatrix} \omega^m \theta^n. \end{aligned} \tag{5.8}$$

Expanding the left hand side of Equation (5.7) as a power series gives

$$f[P(\theta, \omega), \omega] = \begin{bmatrix} 1 + P_2(\theta, \omega) - a[P_1(\theta, \omega)]^2 \\ (b + \omega) P_2(\theta, \omega) \end{bmatrix}$$

which we expand component-wise to obtain

$$f[P(\theta, \omega), \omega]_1 = 1 + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p^2_{mn} \theta^n \omega^m$$

$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{m} a\ p^1_{(m-j,n-k)}p^1_{(j,k)}\theta^n\omega^m, \tag{5.9}$$

and

$$f[P(\theta,\omega),\omega]_2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} bp^1_{mn}\omega^m\theta^n + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} p^1_{(m-1,n)}\omega^m\theta^n \tag{5.10}$$

Now we equate the power series expressions for the left and right hand sides, match like powers, and isolate the highest order terms to obtain the homological equation

$$\begin{bmatrix} -2ap^1_{(00)} - \lambda_0^n & 1 \\ b & \lambda_0^n \end{bmatrix} \begin{bmatrix} p^1_{(m,n)} \\ p^2_{(m,n)} \end{bmatrix} = \begin{bmatrix} s^1_{(m,n)} \\ s^2_{(m,n)} \end{bmatrix} \tag{5.11}$$

where

$$s^1_{(m,n)} = \sum_{j=0}^{m-1} \lambda_{(m-j,n)}p^1_{(j,n)} + \sum_{k=0}^{n}\sum_{j=0}^{m} a\ \bar{p}^1_{(m-j,n-k)}\bar{p}^1_{(j,k)}$$

and

$$s^2_{(m,n)} = -p^1_{(m-1,n)} + \sum_{j=0}^{m-1} \lambda_{(m-j,n)}p^2_{(j,n)}$$

for $n \geq 2, m \geq 1$.

5.1.2. Formal Computation of Polynomial Approximations for the Lorenz System. We illustrate the formal computation for the one parameter branch of two dimensional stable manifolds through the origin of the Lorenz System. Let

$$P(\theta,\omega) = P(\theta_1,\theta_2,\omega) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{m=0}^{\infty} p_{(m,n_1,n_2)}\omega^m\theta_1^{n_1}\theta_2^{n_2}$$

denote the parameterization of the one parameter branch of two dimensional stable manifolds through the origin. Then P satisfies the functional equation

$$f[P(\theta_1,\theta_2,\omega),\omega] = [D_1P(\theta_1,\theta_2,\omega)]\Lambda(\omega)\begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix},$$

where

$$\Lambda(\omega) = \begin{bmatrix} \lambda^1(\omega) & 0 \\ 0 & \lambda^2(\omega)\end{bmatrix} = \sum_{m=0}^{\infty}\begin{bmatrix} \lambda^1_m & 0 \\ 0 & \lambda^2_m\end{bmatrix}\omega^m.$$

Since the origin is a fixed point for all ω the series expansion of $p(\omega)$ is trivial to all orders. Moreover since we take $\beta > 0$, we have that $\lambda^1(\omega) = -\beta$ and $\xi^1(\omega) = (0,0,1)$ are a stable eigenvalue/eigenvector pair for all ω. The remaining unstable eigenvalue/eigenvector pair $\lambda^2(\omega)$ and $\xi_2(\omega)$ do depend on ω and are computed using the techniques of in Section (4.2). We have that $p_{(m,0,0)} = 0$ for all $m \geq 0$, $p_{(m,1,0)} = \xi_m$ for all $m \geq 0$, $p_{0,0,1} = (0,01)$, and $p_{m,0,2} = 0$ for all $m \geq 1$.

The p_{0,n_1,n_2} coefficients are the coefficients for the two dimensional manifold in the $\omega = 0$ system. These are computed using the homological equation (5.2) with

$\alpha = (n_1, n_2)$ a two dimensional multi-index and with the right hand side given by Equation (5.3). What remains is to compute the coefficients $p_{(n_1,n_2,m)}$ for $n_1+n_2 \geq 2$ and $m \geq 1$. As in the previous example for the Hénon map we compute a recursive expression for the remaining coefficients by a power matching scheme which results in

$$\begin{bmatrix} -\sigma - (n_1\lambda_0^1 + n_2\lambda_0^2) & \sigma & 0 \\ \rho - a^3_{(00)} & -1 - (n_1\lambda_0^1 + n_2\lambda_0^2) & -a^1_{(00)} \\ a^2_{(00)} & a^1_{(00)} & -\beta - (n_1\lambda_0^1 + n_2\lambda_0^2) \end{bmatrix} \begin{pmatrix} p^1_{(m,n_1,n_2)} \\ p^2_{(m,n_1,n_2)} \\ p^3_{(m,n_1,n_2)} \end{pmatrix}$$

$$= \begin{pmatrix} s^1_{(m,n_1,n_2)} \\ s^2_{(m,n_1,n_2)} \\ s^3_{(m,n_1,n_2)} \end{pmatrix}, \tag{5.12}$$

where

$$s^1_{(m,n_1,n_2)} = \sum_{k=0}^{m-1} \left[n_1\lambda^1_{m-k} + n_2\lambda^2_{m-k}\right] p^1_{(k,n_1,n_2)}$$

$$s^2_{(m,n_1,n_2)} = -p_{(m-1,n_1,m_2)} + \sum_{k=0}^{m-1} \left[n_1\lambda^1_{m-k} + n_2\lambda^2_{m-k}\right] \; p^2_{(k,n_1,n_2)}$$

$$+ \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{m} \bar{p}^1_{(m-k,n_1-i,n_2-j)}\bar{p}^3_{(kij)}$$

and

$$s^3_{(m,n_1,n_2)} = \sum_{k=0}^{m-1} \left[n_1\lambda^1_{m-k} + n_2\lambda^2_{m-k}\right] \; p^3_{(k,n_1,n_2)} - \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{m} \bar{p}^1_{(m-k,n_1-i,n_2-j)}\bar{p}^2_{(kij)}.$$

5.2. Formal Well-Definedness of $P(\theta, \omega)$ and One Parameter Branches of Non-Resonance Conditions. We note that Equation (5.11), which is the homological equation defining the coefficients of the one parameter branch of chart maps for the stable/unstable manifolds of the Hénon map, has the form

$$[Df(p_0) - \lambda_0^n \mathrm{Id}]a_{(m,n)} = \hat{s}_{(m,n)}$$

where the characteristic matrix on the left-hand side is *exactly* the same matrix as in the left hand side of the homological equation (5.6) for the coefficients of the parameterization for the $\omega = 0$ system. So while the right hand sides of Equations (5.11) and (5.6) are different, we see that the coefficients of $P(\theta, \omega)$ are well defined under precisely the same conditions given in Lemma (5.2). We conclude that if the eigenvalues of the $\omega = 0$ system are non-resonant in the sense of Lemma (5.2), then the formal series for the one parameter branch of parameterizations is well defined to all orders. To put it another way; when we decide to compute a one parameter branch of invariant manifolds *we need impose no extra conditions* in order that the formal solution is well defined to all orders.

Similar comments are seen to apply for the Lorenz system by observing that the matrix on the left-hand-side of the homological Equation (5.12) is the same matrix as on the left-hand-side of the homological Equation for the $\omega = 0$ system of differential equations given by Equation (5.2). So again we see that the one parameter branch of parameterizations is formally well defined under precisely the conditions of Lemma (5.1).

These considerations give rise to an *a-priori* necessary condition on the radius of convergence of the formal series defined above. Namely, for a one parameter branch of invariant manifolds for differential equations we must find a $\tau > 0$ so that

$$\alpha_1 \lambda_1(\omega) + \ldots + \alpha_k \lambda_k(\omega) \neq \lambda_i(\omega)$$

for all $2 \leq |\alpha| \leq \lceil \mu^*/\mu_* \rceil$, $1 \leq i \leq k$, and all $\omega \in B_\tau$. On the other hand, for a one parameter branch of invariant manifolds for diffeomorphisms we must find a $\tau > 0$ so that

$$[\lambda_1(\omega)]^{\alpha_1} \cdot \ldots \cdot [\lambda_k(\omega)]^{\alpha_k} \neq \lambda_i(\omega)$$

for all $2 \leq |\alpha| \leq \lceil \ln(\mu_*)/\ln(\mu^*) \rceil$, $1 \leq i \leq k$, and all $\omega \in B_\tau$.

We focus for the moment on the case of differential equations. Consider the analytic Taylor models

$$\lambda_i(\omega) = \left(\lambda^i_M(\omega), \tau_i, \delta_i\right), \qquad 1 \leq i \leq k$$

for the eigenvalues at an equilibria of a one parameter family of analytic vector fields. Then there is a resonance at $\omega \in B_\tau$ if and only if ω is a solution of one of the equations

$$\alpha_1 \lambda_1(\omega) + \ldots + \alpha_k \lambda_k(\omega) - \lambda_i(\omega) = 0.$$

for $1 \leq i \leq k$ and $2 \leq |\alpha| \leq \lceil \mu^*/\mu_* \rceil$. Assuming that $\omega = 0$ is not a solutions of any of these equations, i.e. under the assumptions of Lemma (5.1), we now want to find a $\tau > 0$ so that if $\omega \in B_\tau$ then there are no solutions for any multi-index α with $2 \leq |\alpha| \leq \lceil \mu^*/\mu_* \rceil$.

For any $\tau > 0$ we define the quantities

$$b_\alpha(\tau) \equiv \min_{1 \leq i \leq k} \inf_{|\omega| \leq \tau} |\alpha_1 \lambda_1(\omega) + \ldots + \alpha_k \lambda_k(\omega) - \lambda_i(\omega)|$$

Let $\lambda^i_M(\omega) = \sum_{m=0}^M \lambda^i_m \omega^m$, so that λ^i_m are the polynomial coefficients associated with $\lambda_i(\omega)$. Then we have the bound

$$b_\alpha(\tau) \geq \min_{1 \leq i \leq k} \left|\alpha_1 \lambda^1_0 + \ldots + \alpha_k \lambda^k_0 - \lambda^i_0\right| - B_\alpha(\tau)$$

where

$$B_\alpha(\tau) \equiv \tau \sum_{m=1}^M \left|\alpha_1 \lambda^1_m + \ldots + \alpha_k \lambda^k_m - \lambda^i_m\right| \tau^{m-1} + |\alpha_1 \delta_1 + \ldots + \alpha_k \delta_k + \delta_i|. \tag{5.13}$$

If

$$\left|\alpha_1 \lambda^1_0 + \ldots + \alpha_k \lambda^k_0 - \lambda^i_0\right| > \alpha_1 \delta_1 + \ldots + \alpha_k \delta_k + \delta_i, \tag{5.14}$$

for each $1 \leq i \leq k$, then there exists a $\tau > 0$ so that $b_\alpha(\tau) > 0$ for every α. If there is further a $\tau > 0$ for which all of the Equations (5.14) holds for each multi-index α with $2 \leq |\alpha| \leq \lceil \mu^*/\mu_* \rceil$. then there are no resonances on B_τ with this choice of τ.

Since in the present work we consider only the two dimensional Hénon map with one stable and one unstable direction, there are no possible resonances. Then the only restrictions on the parameter domain come from assumptions **A1-A3(Maps)**; namely we must choose a B_τ so that for all $|\omega| \leq \tau$ the differential is invertible and there are no eigenvalue bifurcations. If we were to consider the secondary equilibria (or "eyes") of the Lorenz System near the classical parameters then again there would be no possible resonances, as at the classic parameters the eyes have one stable direction and one complex unstable direction, and again there are no possible resonances.

On the other hand when we consider the stable manifold associated with the equilibria at the origin of the Lorenz System near the classical parameter values then the eigenvalues are real distinct and we must rule out any possible resonances.

Example: Consider the Lorenz System with parameter values $\sigma = 10$, $\beta = 8/3$, and $\rho = 13.9265$ (parameters close to the classical homoclinic tangency). Using IntLab we compute the eigenvalue enclosures

$$\begin{aligned}
\lambda_1 &\in B\left(-18.12992478204046, 3.56 \times 10^{-15}\right), \\
\lambda_2 &\in B\left(-2.66666666666666, 4.45 \times 10^{-16}\right), \\
\lambda_3 &\in B\left(7.12992478204047, 6.22 \times 10^{-15}\right),
\end{aligned}$$

which are clearly real and distinct. Considering only the stable eigenvalues we take

$$\mu_* = 2.6 < \min_{1 \leq i \leq 2}, |\text{real}(\lambda_i)| \qquad \text{and} \qquad \mu^* = 18.13 > \max_{1 \leq i \leq 2}, |\text{real}(\lambda_i)|.$$

We can check that $\lceil \mu^*/\mu_* \rceil = 7$. Then if $n_1 + n_2 > 7$, we have that

$$n_1\lambda_1 + n_2\lambda_2 < -n_1\mu_* - n_2\mu_* < -7\mu_* < -\mu^* < \lambda_1 < \lambda_2,$$

which shows explicitly that $n_1 + n_2 > 7$ implies that $n_1\lambda_1 + n_2\lambda_2 \neq \lambda_i$ for $1 = 1, 2$ and we conclude there are no possible resonances with for multi-indices of order greater or equal to 7. What remains is to check the 33 remaining non-resonance conditions of the form

$$b_{(n_1,n_2)} = \min_{1 \leq i \leq 2} |n_1\lambda_1 + n_2\lambda_2 - \lambda_i| > 0.$$

with $2 \leq n_1 + n_2 \leq 7$. We compute $b_{(n_1,n_2)}$ using interval arithmetic and check that the resulting interval does not contain zero. We tabulate the results and find the the closest the system ever comes to resonance is when $(n_1, n_2) = (0, 7)$, in which case

$$|0\lambda_1 + 7\lambda_2 - \lambda_1| \in [0.53674188462619, 0.53674188462621].$$

This tells us that when we compute the analytic Taylor models which parameterize the branches of stable eigenvalues, we have to take care with the equation

$$7\lambda_2(\omega) - \lambda_1(\omega) = 0.$$

We compute the analytic Taylor models for λ_i, $i = 1, 2$ to order 12, and check the resonance bounds for all $2 \leq n_1 + n_2 \leq 7$. For the multi-indices with $(n_1, n_2) \neq (0, 7)$

we find that for τ as large as 1.3 we have that $b_{(n_1,n_2)}(\tau) > 1$. The difficult multi-index is $(0,7)$ where we only have $b_{(0,7)}(1.3) > 0.01$. However, we also report that when $\tau = 1.4$ we cannot guarantee that $b_{(0,7)}(1.4) > 0$ using interval arithmetic and the bounds given by Equation (5.13). On the other hand if we take $\tau \leq 0.5$ we have $b_{(n_1,n_2)}(0.5) > 1$ for all $2 \leq n_1 + n_2 \leq 7$.

REMARK 5.4 (Range Bounding Using Interval Arithmetic). Since the resonance conditions always involve bounding functions of one variable away from zero we can always try to use other range bounding methods instead of the simple bounds given by Equation (5.13). This could be especially useful for any multi-indices where there is a near resonance in the $\omega = 0$ equation (like the equation associated with the multi-index $(0,7)$ for the homoclinic tangency parameters in Lorenz that we have been discussing). See [27] for a more sophisticated treatment of techniques for obtaining range bounds using interval arithmetic. For the present work Equation (5.13) will be sufficient.

5.3. Validated Truncation Error for Coefficient Tails. Suppose that

$$P_{MN}(\theta,\omega) = \sum_{|\alpha|=0}^{N} \sum_{m=0}^{M} a_{(\alpha,m)} \omega^m \theta^\alpha \tag{5.15}$$

is an NM-order polynomial approximation of a solution of either Equation (1.2) or (1.3) whose coefficients are solutions of either the homological Equations (5.11) in the case of the Hénon family or the homological Equations (5.12) in the case of the Lorenz system. We treat the truncation error associated with this polynomial approximation in two steps. In the present section we bound the truncation error in ω up to all powers of θ with $|\alpha| \leq N$ on a complex disk $|\omega| \leq \tau$. In the next section we treat with the truncation error due to terms of order $|\alpha| > N$.

Then for the purposes of the present discussion it is helpful to think of P_{MN} as a polynomial in the variable θ, whose coefficients are polynomials in the variable ω. So, we write the true solution as a power series in θ whose coefficients depend analytically on the parameter ω;

$$P(\theta,\omega) = \sum_{|\alpha|=0}^{\infty} a_\alpha(\omega)\theta^\alpha,$$

and think of each coefficient as an analytic function

$$a_\alpha(\omega) = \sum_{m=0}^{\infty} a_{(\alpha,m)}\omega^m = \sum_{m=0}^{M} a_{(m,\alpha)}\omega^m + h_\alpha(\omega) = a_\alpha^M(\omega) + h_\alpha(\omega).$$

We note that by computing the formal solution P_{MN} given by Equation (5.15) we have actually computed the M-th order polynomials $a_\alpha^M(\theta)$ for each $0 \leq |\alpha| \leq N$. Now we want to bound the truncation errors h_α on some disk B_τ.

Note that $a_0(\omega) = p(\omega) = p_M(\omega) + h_p(\omega)$ is the one parameter branch of fixed points and we already have an analytic Taylor model $p(\omega) = (p_M(\omega), \tau, \delta_p)$. Then if we write $a_0(\omega) = a_M^0(\omega) + h_0(\omega)$ then we have $a_0^M(\omega) = p_M(\omega)$ and the bound $\|h_0\|_\tau \leq \delta_p \equiv \delta_0$. Similarly, for the first order multi-indices we let $e_i = (0,\ldots,1,\ldots,0)$ be the i-th standard basis vector of $\mathbb{N}^k$ (i.e. e_i is the multi-index with a one in the i-th slot and zeros elsewhere). Then $a_{e_i}(\omega) = \xi_i(\omega)$ and again if we consider

$a_{e_i}(\omega) = a_{e_i}(\omega)^M + h_{e_i}(\omega)$ then we have that $\|h_{e_i}\|_\tau \leq \delta_{\xi_i}$, where δ_{ξ_i} is the truncation error associated with $\xi_i(\omega)^M$. Then we define $\delta_{e_i} = \delta_{\xi_i}$.

What is left is to prove that for $2 \leq |\alpha| \leq N$ we must prove that there exist analytic functions $h_\alpha \colon B_\tau \to \mathbb{C}^n$ so that $a_\alpha(\omega) = a_\alpha^M(\omega) + h_\alpha(\omega)$ and truncation error bounds of the form $\|h_\alpha\|_\tau \leq \delta_\alpha$. We proceed by exploiting the fact that $a_\alpha(\omega)$ is a solution of a homological equation. To be more precise, we recall that $a_\alpha(0)$ are the power series coefficients of the parameterization of the stable/unstable manifold for the $\omega = 0$ system. Then the $a_\alpha(0)$ terms solve the homological equation (5.2) in the case of differential equations and the homological equation (5.5) in the case of diffeomorphisms. (Explicitly the terms of order zero in m are the solutions of the Equation (5.3) for the Lorenz system and Equation (5.6) for Hénon). Allowing ω to vary we see that $a_\alpha(\omega)$ is a solution of the non-constant matrix equation

$$\left(Df[p(\omega), \omega] - \sum_{i=1}^{k} \alpha_i \lambda_i(\omega) \mathrm{Id}_n \right) a_\alpha(\omega) = s_\alpha(\omega), \tag{5.16}$$

for differential equations and

$$\left(Df[p(\omega), \omega] - \prod_{i=1}^{k} [\lambda_i(\omega)]^{\alpha_i} \mathrm{Id}_n \right) a_\alpha(\omega) = s_\alpha(\omega), \tag{5.17}$$

for diffeomorphisms. Since these are linear equations and we already have an approximate solution $a_\alpha^M(\omega)$, we could use Lemma (4.6) in order to bound the remainders. However the following two theorems show that the desired bounds can be obtained by exploiting the a-priori non-resonance bounds already computed in Section (5.2), and the fact that we have already computed the diagonalizing transformations $Q(\omega)$ and $Q^{-1}(\omega)$.

THEOREM 5.5 (h_α Bounds for Differential Equations). *Assume that $\lambda_i(\omega)$, $1 \leq i \leq k$ are non-resonant on B_τ in the sense of differential equations. Assume that we have analytic Taylor model representations $(\lambda_i^M, \tau, \delta_i)$, $1 \leq i \leq k$, $(s_\alpha^M, \tau, \delta_s)$, and (p_M, τ, δ_p) respectively for the functions λ_i, s_α and the fixed point branch p. Define $\delta_\Lambda = \max_i \delta_i$. Additionally let (A_M, τ, δ_A) be an analytic Taylor model of the differential of f at p having*

$$A(\omega) = Df[p(\omega), \omega] = Df[p_M(\omega), \omega] + H_A(\omega),$$

with $A_M = Df[p_M(\omega), \omega]$ an M-th order polynomial in ω with matrix coefficients, and $\|H_A\|_\tau \leq \delta_A$. Let $Q(\omega)$ be the matrix of eigenvectors for $Df[p(\omega), \omega]$.

Define

$$M_\alpha = \max_{1 \leq i \leq k} \sup_{|\omega| \leq \tau} |\lambda_i(\omega) - \alpha_1 \lambda_1(\omega) - \ldots - \alpha_k \lambda_k(\omega)|^{-1}.$$

Let $a_\alpha^M(\omega)$ be the M-th order solution of Equation (5.16) obtained by solving the homological equations. Define the a-posteriori error polynomial

$$E_M^\alpha(\omega) = s_\alpha^M(\omega) - \left(Df[p_M(\omega), \omega] - (\alpha_1 \lambda_1^M(\omega) + \ldots + \alpha_k \lambda_k^M(\omega) Id_n \right) a_\alpha^M(\omega),$$

and the total a-posteriori error bound

$$\epsilon_\alpha = \|E_M^\alpha\|_\tau + \delta_s + (\delta_A + |\alpha| \delta_\Lambda) \|a_\alpha^M\|_\tau.$$

Then there is a unique analytic M-tail $h_\alpha \colon B_\tau \to \mathbb{C}^n$ so that $a_\alpha^M + h_\alpha = a_\alpha$ is the exact solution of Equation (5.16). Moreover we have the bound

$$\|h_\alpha\|_\tau \leq \|Q\|_\tau \|Q^{-1}\|_\tau M_\alpha \epsilon_\alpha.$$

Proof: Let $a_\alpha(\omega) = a_\alpha^M(\omega) + h_\alpha(\omega)$ where we want to determine the function h_α. We re-write Equation (5.16) as

$$(Df[p_M(\omega) + h_p(\omega), \omega] - \langle \Lambda_M(\omega) + H_\Lambda(\omega), \alpha\rangle \,\mathrm{Id})\left[a_\alpha^M(\omega) + h_\alpha(\omega)\right]$$

$$= s_\alpha^M(\omega) + h_s(\omega).$$

or

$$[Df[p(\omega), \omega] - \langle \Lambda(\omega), \alpha\rangle \,\mathrm{Id}]\, h_\alpha(\omega) =$$

$$E_M^\alpha + h_s(\omega) - [H_A(\omega) - \langle H_\Lambda(\omega), \alpha\rangle \,\mathrm{Id}]\, a_\alpha^M(\omega).$$

Let $\hat{E}(\omega)$ denote the right hand side of this equation and note that $\hat{E}$ is an analytic M-tail with $\|\hat{E}\|_\tau \leq \epsilon_\alpha$. Utilizing the diagonalizing transformation $Q(\omega)$ we have that the equation becomes

$$\left[Q(\omega)\Sigma(\omega)Q^{-1}(\omega) - \langle \Lambda(\omega), \alpha\rangle \,\mathrm{Id}\right] h_\alpha(\omega) = \hat{E}(\omega).$$

We now make the change of variables

$$Q(\omega) w_\alpha(\omega) = h_\alpha(\omega),$$

and re-write the equation as

$$(\Sigma(\omega) - \langle \Lambda(\omega), \alpha\rangle \,\mathrm{Id})\, w_\alpha(\omega) = Q^{-1}(\omega)\hat{E}(\omega).$$

The left hand side is now diagonalized and, under the assumption that the eigenvalues are a non-resonant branch, we obtain the component equations

$$[w_\alpha(\omega)]_j = \frac{1}{\lambda_j(\omega) - \alpha_1\lambda_1(\omega) - \ldots - \alpha_k\lambda_k(\omega)}[Q^{-1}(\omega)\hat{E}(\omega)]_j \qquad \text{for} \quad 1 \leq j \leq n.$$

Then w_α exists and is an analytic M-tail. Now since $h_\alpha = Q w_\alpha$ and we see that h_α is an analytic M-tail as desired. Moreover we have the estimate

$$\|h_\alpha\|_\tau \leq \|Q\|_\tau \max_{1\leq j\leq n} \sup_{|\omega|\leq\tau} \left| \frac{1}{\lambda_j(\omega) - \alpha_1\lambda_1(\omega) - \ldots - \alpha_k\lambda_k(\omega)}[Q^{-1}(\omega)\hat{E}(\omega)]_j \right|$$

$$\leq \|Q\|_\tau \|Q^{-1}\|_\tau M_\alpha \epsilon_\alpha,$$

as desired.

$\square$

Note that the $M_\alpha = 1/b_\alpha(\tau)$, where $b_\alpha(\tau)$ is defined as in Section (5.2). Then the bounds given by Equation (5.13) can be used to estimate the M_α in practice.

THEOREM 5.6 (h_α Bounds for Diffeomorphisms). *Assume that $\lambda_i(\omega)$, $1 \le i \le k$ are non-resonant on B_τ in the sense of diffeomorphisms. Assume that we have analytic Taylor model representations $(\lambda_i^M, \tau, \delta_i)$, $1 \le i \le k$, $(s_\alpha^M, \tau, \delta_s)$, and (p_M, τ, δ_p) respectively for the functions λ_i, s_α and the fixed point branch p. Define $\delta_\Lambda = \max_i \delta_i$. Let (A_M, τ, δ_A) be an analytic Taylor model of the differential of f at p having*

$$A(\omega) = Df[p(\omega), \omega] = Df[p_M(\omega), \omega] + H_A(\omega),$$

with $\|H_A\|_\tau \le \delta_A$. We also assume that $(\Lambda_M^\alpha, \tau, \delta_{\Lambda^\alpha})$ is an analytic Taylor model for the scalar function $\Lambda^\alpha(\omega)$. Let $Q(\omega)$ be the matrix of eigenvectors of $Df[p(\omega), \omega]$.

Define

$$M_\alpha = \max_{1 \le i \le k} \sup_{|\omega| \le \tau} \left|\lambda_i(\omega) - \lambda_1^{\alpha_1}(\omega) \cdot \ldots \cdot \lambda_k^{\alpha_k}(\omega)\right|^{-1}.$$

Let $a_\alpha^M(\omega)$ be the M-th order solution of Equation (5.16) obtained by solving the homological equations. Define the a-posteriori error polynomial

$$E_M^\alpha(\omega) = s_\alpha^M(\omega) - (Df[p_M(\omega), \omega] - \Lambda_M^\alpha(\omega) Id_n)\, a_\alpha^M(\omega),$$

and the total a-posteriori error bound

$$\epsilon_\alpha = \|E_M^\alpha\|_\tau + \delta_s + (\delta_A + \delta_{\Lambda^\alpha})\|a_\alpha^M\|_\tau.$$

Then there is a unique analytic M-tail $h_\alpha \colon B_\tau \to \mathbb{C}^n$ so that $a_\alpha^M + h_\alpha = a_\alpha$ is the exact solution of Equation (5.17). Moreover we have the bound

$$\|h_\alpha\|_\tau \le \|Q\|_\tau \, \|Q^{-1}\|_\tau M_\alpha \epsilon_\alpha.$$

The proof is almost identical to the proof of Theorem (5.5.)

REMARK 5.7 (Computational Cost of Computing δ_α). The theorems say that in order to bound h_α we must compute the a-posteriori error polynomial E_M^α, as well as the sigma-norms of E_M^α and a_α^M. Note that the cost of computing E_M^α in both cases is the cost of a Cauchy product of two polynomials of order M. The cost of evaluating the sigma norms are the cost of an inner product.

5.4. A-Posteriori Analysis of The Full Truncation Error. In this section we state and prove the main theorems of the paper; one theorem for flows and one for maps. Throughout the section we take f, ρ, ν, $p(\omega)$, $D_1 f[p(\omega), \omega]$, k, $\lambda_i(\omega)$ and $\xi_i(\omega)$ for $1 \le i \le k$, $\Lambda(\omega)$, and $A(\omega)$ to be as in either **A1-A3-flows** or **A1-A3-maps** from Section (1) depending on wether we are discussing differential equations or diffeomorphisms. In either case we assume that that $P_N \colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ is an N-th order polynomial in $\theta \in B_\nu(0) \subset \mathbb{C}^k$ with coefficients analytic in the variable $\omega \in B_\tau(0) \subset \mathbb{C}$, so that P_N has power series expansion

$$P_N(\theta, \omega) = \sum_{|\alpha|=0}^{N} a_\alpha(\omega)\theta^\alpha = \sum_{|\alpha|=0}^{N} \sum_{m=0}^{\infty} a_{(m,\alpha)} \omega^m \theta^\alpha,$$

convergent on $B_\nu \times B_\tau$. Moreover suppose that P_N satisfies the first order constraints

$$P_N(0,\omega) = p(\omega), \quad \text{and} \quad D_1 P_N(0,\omega) = [\xi_1(\omega)|\dots|\xi_k(\omega)].$$

Suppose that the power series of the differential

$$Df[P_N(\theta,\omega),\omega] = \sum_{|\alpha|=0}^{\infty} \sum_{m=0}^{\infty} A_{(\alpha,m)}\, \omega^m\, \theta^\alpha$$

also converges on $B_\nu \times B_\tau$. Take $Q, Q^{-1}\colon B_\tau \subset \mathbb{C} \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ to be the transformations which diagonalize $Df[p(\omega),\omega]$; i.e. Q is the matrix whose columns are all of the stable and unstable eigenvectors, and Q^{-1} it's inverse.

Finally, assume that P_N is the N-th order formal solution either Equation (1.2) in the case of vector fields, or Equation (1.3) in the case of difeomorphisms; i.e. suppose that the coefficients of P_N are exact solutions of the homological equations for a one parameter family of stable manifolds.

5.4.1. Differential Equations. Define the total a-posteriori error

$$E_N(\theta,\omega) = f[P_N(\theta,\omega),\omega] - D_1 P_N(\theta,\omega)\Lambda(\omega)\theta, \tag{5.18}$$

for the case of vector fields.

Definition 5.1. [Validation Values for a One Parameter Branch of Local Stable Manifolds at an Equilibria of a Vector Field] A set of positive real constants, ϵ, ρ', μ_*, μ^*, M_1, M_2, C_1, and C_2 are called *validation values* for the one parameter branch of stable manifolds problem if

(i): $\|E\|_{\nu,\tau} \leq \epsilon$,

(ii):

$$\min_{1\leq i\leq k} \inf_{|\omega|\leq\tau} \mathrm{real}\,(\lambda_i(\omega)) \leq \mu^* \quad \text{and} \quad \mu_* \leq \max_{1\leq i\leq k} \sup_{|\omega|\leq\tau} \mathrm{real}\,(\lambda_i(\omega))$$

(iii)

$$\|Q\|_\tau \|Q^{-1}\|_\tau \leq C_1,$$

(iv)

$$\sum_{|\alpha|=1}^{\infty} \sum_{m=0}^{\infty} \frac{|A_{(\alpha,m)}|}{\mu_* |\alpha|} \tau^m \nu^{|\alpha|} \leq C_2.$$

(v) $\mathcal{M}$ is the set of multi-indices $(i,\beta) \in \mathbb{N}\times\mathbb{N}^n$ with $1\leq i\leq n$, $|\beta| = 2$, so that $\partial_\beta f_i(z,\omega)$ is not identically zero. M_1 denotes the cadnality of $\mathcal{M}$, and M_2 is any uniform bound of the form

$$\max_{(i,\beta)\in\mathcal{M}} \sup_{z\in B_{\rho'}} \sup_{\omega\in B_\tau} \partial_\beta \|f_i(z,\omega)\| \leq M_2.$$

(vi): ρ' has $0 < \rho' < \rho$, with

$$\sup_{|\theta|\leq\nu} \sup_{|\omega|} \|P_N(\theta,\omega) - p(\omega)\| \leq \rho'.$$

This insures that domain $B_\nu(0) \subset \mathbb{C}^k$ is small enough that the image of P_N is contained in the interior of the domain $B_\rho(p) \times B_\tau(0) \subset \mathbb{C}^n \times \mathbb{C}$ of the family of vector fields f.

THEOREM 5.8 (A-Posteriori Error for a One Parameter Branch of Stable Manifolds for an Equilibria of a Vector Field). *Suppose that ϵ, ρ', μ_*, μ^*, M_1, M_2, C_1, and C_2 are validation values for a one parameter branch of local stable manifolds at an equilibria of a vector field.*

Assume that $N \in \mathbb{N}$ and $\delta > 0$ are such that

- $$(N+1) > \frac{\mu^*}{\mu_*}, \tag{5.19}$$
- $$\delta < e^{-1} \ \min \left\{ \frac{(N+1)\mu_* - \mu^*}{2n\pi M_1 M_2 C_1 e^{C_2}}, \rho - \rho' \right\} \tag{5.20}$$
- *and*
$$\frac{2C_1 e^{C_2}}{(N+1)\mu_* - \mu^*} \epsilon < \delta \tag{5.21}$$

Then there is a unique one parameter family of analytic N-tails $H\colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ with

$$\|H\|_{\nu,\tau} \leq \delta,$$

so that

$$P(\theta, \omega) = P_N(\theta, \omega) + H(\theta, \omega)$$

is the exact solution of Equation (1.2) on $B_\nu \times B_\tau$.

Proof: We seek a one parameter family of bounded analytic N-tails so that

$$f[P_N(\theta,\omega) + H(\theta,\omega), \omega] = D_1[P_N(\theta,\omega) + H(\theta,\omega)]\Lambda(\omega)\theta \tag{5.22}$$

for all $(\theta, \omega) \in B_\nu \times B_\tau$. Image$(P_N) \subset B(p_0, \rho')$ and f is analytic on $B(p_0, \rho)$ so we can Taylor expand the left hand side of Equation (5.22) to second order and obtain

$$f[P_N(\theta,\omega) + H(\theta,\omega), \omega] = f[P_N(\theta,\omega)] + Df[P_N(\theta,\omega)]H(\theta,\omega) + R_{P_N(\theta,\omega)}[H(\theta,\omega), \omega],$$

where R is the quadratic remainder there. Rearranging Equation (5.22) we have

$$D_1 H(\theta,\omega)\Lambda(\omega)\theta - Df[P_N(\theta,\omega), \omega]H(\theta,\omega) = E_N(\theta,\omega) + R_{P_N(\theta,\omega)}[H(\theta,\omega), \omega].$$

Recalling the definition of $\mathfrak{L}_{\text{flow}}$ from Section (3.1) we note that this is

$$\mathfrak{L}_{\text{flow}}[H](\theta,\omega) = E_N(\theta,\omega) + R_{P_N(\theta,\omega)}[H(\theta,\omega), \omega]. \tag{5.23}$$

Letting

$$A(\theta,\omega) = Df[P_N(\theta,\omega), \omega],$$

and considering (SOMETHING) from the definition of Validation Values we see that the conditions of Theorem (3.2) are satisfied and the the $\mathfrak{L}_{\text{flows}}$ on the left hand side of is boundedly invertible. By considering Assumptions (SOMETHING) we see that that Equation (5.23) satisfies the hypotheses of Corollary (3.4), and hence has a unique solution H with $\|H\|_{\nu,\tau} < \delta$.

$\square$

5.4.2. Maps. Define the total a-posteriori error

$$E_N(\theta,\omega) = f[P_N(\theta,\omega),\omega] - P_N(\Lambda(\omega)\theta,\omega), \tag{5.24}$$

for the case of diffeomorphisms.

DEFINITION 5.2. [Validation Values for a One Parameter Branch of Local Stable Manifolds at a Fixed Point of a Diffeomorphism] A set of positive real constants, ϵ, ρ', μ_*, μ^*, M_1, M_2, and C are called *validation values* for the one parameter branch of stable manifolds problem if

(i): $\|E\|_{\nu,\tau} \leq \epsilon$,

(ii):

$$0 < \mu_* \leq \min_{1\leq i\leq k} \inf_{|\omega|\leq\tau} |\lambda_i(\omega)| \qquad \text{and} \qquad \max_{1\leq i\leq k} \sup_{|\omega|\leq\tau} |\lambda_i(\omega)| \leq \mu^* < 1$$

(iii)

$$\sum_{|\alpha|=1}^{\infty} \sum_{m=0}^{\infty} \frac{|A_{(\alpha,m)}|}{\mu_*|\alpha|} \tau^m \nu^{|\alpha|} \leq C.$$

(iv) $\mathcal{M}$ is the set of multi-indices $(i,\beta) \in \mathbb{N}\times\mathbb{N}^n$ with $1\leq i\leq n$, $|\beta| = 2$, so that $\partial_\beta f_i(z,\omega)$ is not identically zero. M_1 denotes the cadnality of $\mathcal{M}$, and M_2 is any uniform bound of the form

$$\max_{(i,\beta)\in\mathcal{M}} \sup_{z\in B_{\rho'}} \sup_{\omega\in B_\tau} \partial_\beta \|f_i(z,\omega)\| \leq M_2.$$

(v) and that there is a $0 < \rho' < \rho$ with

$$\sup_{|\theta|\leq\nu} \sup_{|\omega|} \|P_N(\theta,\omega) - p(\omega)\| \leq \rho',$$

again insuring that domain $B_\nu(0) \subset \mathbb{C}^k$ is small enough that the image of P_N is contained in the interior of the domain $B_\rho(p)\times B_\tau(0) \subset \mathbb{C}^n\times\mathbb{C}$ of the family of diffeomorphisms f.

THEOREM 5.9 (A-Posteriori Error for a One Parameter Branch of Stable Manifolds for a Fixed Point of a Diffeomorphism). *Suppose that ϵ, ρ', μ_*, μ^*, M_1, M_2, and C are a collection of validation values for a one parameter branch of local stable manifolds at a fixed point of a diffeomorphism.*

Assume that $N\in\mathbb{N}$ and $\delta > 0$ are such that

-
$$(\mu^*)^{N+1} C < 1, \tag{5.25}$$

-
$$\delta < e^{-1} \min\left\{ \frac{1 - C(\mu^*)^{N+1}}{2n\pi C M_1 M_2}, \rho - \rho' \right\} \tag{5.26}$$

- *and*
$$\frac{2C}{1 - C(\mu^*)^{N+1}} \epsilon < \delta \tag{5.27}$$

Then there is a unique one parameter family of analytic N-tails $H\colon B_\nu \times B_\tau \subset \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^n$ with

$$\|H\|_{\nu,\tau} \leq \delta,$$

so that

$$P(\theta,\omega) = P_N(\theta,\omega) + H(\theta,\omega)$$

is the exact solution of Equation (1.3) on $B_\nu \times B_\tau$.
Proof: Now we are looking for a one parameter family of bounded analytic N-tails so that

$$f[P_N + H](\theta,\omega) = [P_N + H](\Lambda(\omega)\theta,\omega) \tag{5.28}$$

for all $(\theta,\omega) \in B_\nu \times B_\tau$. Image$(P_N) \subset B(p_0,\rho')$ and f is analytic on $B(p_0,\rho)$ so we Taylor expand the left hand side of Equation (5.28) to second order and obtain

$$f[P_N + H](\theta,\omega) = f[P_N(\theta,\omega)] + Df[P_N(\theta,\omega)]H(\theta,\omega) + R_{P_N(\theta,\omega)}[H(\theta,\omega),\omega],$$

where R is the quadratic remainder term. Rearranging Equation (5.28) we have

$$H[\Lambda(\omega)\theta,\omega] - Df[P_N(\theta,\omega),\omega]H(\theta,\omega) = E_N(\theta,\omega) + R_{P_N(\theta,\omega)}[H(\theta,\omega),\omega]$$

Recalling the definition of $\mathfrak{L}_{\text{map}}$ from Section (3.1) we note that this is

$$\mathfrak{L}_{\text{map}}[H](\theta,\omega) = E_N(\theta,\omega) + R_{P_N(\theta,\omega)}[H(\theta,\omega),\omega] \tag{5.29}$$

Letting

$$A(\theta,\omega) = Df[P_N(\theta,\omega),\omega],$$

and considering (SOMETHING) from the definition of Validation Values we see that the conditions of Theorem (3.1) are satisfied and the the $\mathfrak{L}_{\text{map}}$ on the left hand side of is boundedly invertible. By considering Assumptions (SOMETHING) we see that that Equation (5.29) satisfies the hypotheses of Corollary (3.4), and hence has a unique solution H with $\|H\|_{\nu,\tau} < \delta$.

□

Remark 5.10. In practice the methods of Section (4) provide us with only analytic Taylor models for the analytic branches of fixed points/equilibria $p(\omega)$, stable eigenvalues $\lambda_i(\omega)$, stable eigenvectors $\xi_i(\omega)$, the inverse transformation $Q^{-1}(\omega)$, and for the case of diffeomorphisms the powers of the stable eigenvalues. Similarly the methods of Section (5.3) provide analytic Taylor models for the coefficients

$$a_\alpha(\omega) = \sum_{m=0}^{\infty} a_{(\alpha,m)}\omega^m.$$

In other words all terms are known up to interval enclosures of the M -th order taylor polynomials, plus a validated error term on the complex parameter disk $B_\tau(0)$. In other words, we don't actually know exactly the polynomial P_N hypothesized in Definitions (5.1) and (5.2), we only have an interval inclosure of it with validated

M	N	τ	$\bar{\tau}$	ν	δ_H	δ	time
1	1	10^{-6}	0.995×10^{-6}	10^{-8}	4.48×10^{-13}	3.22×10^{-11}	0.3 (sec)
3	5	10^{-2}	0.995×10^{-2}	0.1	2.1×10^{-9}	1.95×10^{-6}	1.7 (sec)
6	10	10^{-2}	0.995×10^{-2}	0.1	5.1×10^{-15}	3.28×10^{-12}	6.1 (sec)
10	10	10^{-1}	0.995×10^{-1}	0.5	2.5×10^{-10}	1.04×10^{-6}	10.7 (sec)
20	10	10^{-1}	0.995×10^{-1}	0.5	1.4×10^{-13}	6.12×10^{-10}	28.6 (sec)
20	10	0.2	0.1991	0.75	2.1×10^{-11}	2.46×10^{-7}	28.5 (sec)
20	10	0.25	0.248	0.75	9.36×10^{-9}	1.3×10^{-4}	28.4 (sec)

TABLE 6.1
Branch of Stable Manifold Performance Data for the Hénon Family: *a;slkdfj*

error bounds. However we do know that a polynomial satisfying all the conditions of the theorems is enclosed by our Taylor model. Moreover all the conditions of the theorems can be checked using only the information provided by the Taylor model.

We also remark that in order to obtain the bound C_2 for vector fields, C for diffeomorphisms, and ϵ it is best to take the explicit form of the given system into account, i.e. to exploit the actual formulas for the map or vector field in order to obtain simplifications which depend on the specific form of the given dynamical system f. This is discussed in detail for the Hénon and Lorenz systems in Appendix (A).

6. Numerical Examples. Consider first the Hénon map we we fix the classical parameter values of $a = 1.4$ and $b = 0.3$. There are two fixed points p_1, $p_2 \in \mathbb{R}^2$ with

$$p_1 \in B\left(\left[\begin{array}{c} 0.631354477089505 \\ 0.18940634312685 \end{array}\right], 3.34 \times 10^{-16}\right)$$

and

$$p_2 \in B\left(\left[\begin{array}{c} -1.131354477089504 \\ -0.339406343126851 \end{array}\right], 4.45 \times 10^{-16}\right)$$

The eigenvalues associated with p_1 are

$$\lambda_1^u \in [-1.923738858153408, -1.923738858153406],$$

and

$$\lambda_1^s \in [0.155946322302794, 0.155946322302795,]$$

while the eigenvalues for associated with p_2 are

$$\lambda_2^u \in [3.259822097891451, 3.259822097891454]$$

and

$$\lambda_2^s \in [-0.0920295620408391, -0.0920295620408399].$$

We choose eigenvectors with lengths 1.25. The taylor expansions for the fixed points, eigenvalues, and eigenvectors are computed to order $M = 20$, and validated on an parameter interval of $[0.2, 0.4]$, i.e. an interval of radius 0.1 about the classic parameter value of $b = 0.3$. The validated bounds on the analytic Taylor models for the linear data are $\delta_{p_1} \leq 2.62 \times 10^{-15}$, $\delta_{p_2} \leq 0.747 \times 10^{-15}$, $\delta_{\lambda_1^u} \leq 2.92 \times 10^{-14}$, $\delta_{\lambda_1^s} \leq 2.51 \times 10^{-15}$,

$\delta_{\lambda_2^u} \leq 6.11 \times 10^{-14}$, $\delta_{\lambda_2^s} \leq 2.14 \times 10^{-15}$, and the validated bounds for the eigenvector series expansions are all also less then 8×10^{-14}. The eigenvector series are used to compute an analytic Taylor model for the inverse of the diagonalizing transformation $Q_0(\omega)^{-1}$ to order $M = 20$ with a validated truncation error of less then 3.2×10^{-13}.

Using the methods of Section (4.2) we compute analytic Taylor models for the powers of the stable eigenvalue series $[\lambda_1^s(\omega)]^n$ for $2 \leq n \leq 10$. Each series is computed to order $M = 20$. The truncation errors associated with the powers of the eigenvalues are all bounded by 8×10^{-13}. Since the validation of the powers requires giving up a small portion of the validated parameter domain these bounds are only valid on a range of $|\omega| \leq \bar{\tau} \leq 0.0995$, which we take as the new parameter range for the rest of the computation.

Now we compute the formal series coefficients for the one parameter branch of the stable invariant manifold associated with $p_1(\omega)$ to order $M = 20$ in the parameter variable and order $N = 10$ in the phase space variable. Next we compute the M_α bounds of Section (5.2), and find all to be less than 11. Next we compute the δ_α bounds of Section (5.3). There is one such bound for each coefficient of order greater or equal to two, so nine. All of these bounds are less than 2×10^{-10}. Next we compute that the numerical a-posteriori error is less than 1.1×10^{-14} with $\bar{\tau}$ as in the preceding paragraph and $\nu = 0.75$

Finally we bound the total truncation error. Using the bounds from Lemma (A.1) we compute that $\delta_H \leq 3.1 \times 10^{-13}$, $\delta_N \leq 4.7 \times 10^{-11}$, and $\epsilon_N \leq 2 \times 10^{-9}$. Finally we apply Theorem (5.9) and obtain that the total truncation error is less than 6.7×10^{-9}. Combining this bound with the formal polynomial expansion gives a representation of the one parameter family of invariant manifolds which is accurate to roughly eight significant figures.

Similar validated computations can be done for the other three families of invariant manifolds. The results of these validated computations are plotted in Figure 6.1. For the sake of comparison we also include Figure 6.2, which shows the plots of the same manifolds on a domain of $\tau = 1.5$ and $1 \leq \nu \leq 4$ for each of the manifolds (see the caption under the figure for more details). We note that while the parameterizations cannot be validated for domains of this size using the methods above, the approximations are still "good" in the sense that the residuals are small for each expansion (numerical residual smaller than 10^{-6}). Figure 6.2 gives some idea of what the global dynamics are for parameters near $b = 0.3$ and also highlights that the methods developed here provide numerical insight into the dynamics of the system even in the absence of rigorous proofs.

Table (6.1) shows the results of a number of similar computations for the stable manifold at p_1.

We cary out similar computations for the two dimensional stable manifold at the origin of Lorenz System in order to obtain a three variable polynomial approximation to the one parameter family of manifolds. We center the expansion at the parameter set $\sigma = 10$, $\beta = 8/3$, and $\rho = 13.9265$. These parameters put the system near the classic homoclinic bifurcation. We expand the family of manifolds in the ρ parameter.

We note that the origin is an equilibria for the vector field for all values of ρ. Then in this case the power series expansion of the equilibria with respect to parameter is completely trivial. We also note that $\lambda_2^s = -\beta = -8/3$ is a stable eigenvalue for the differential at the origin for all values of ρ, so that one stable eigenvalue and one stable eigenvector have constant power series expansions.

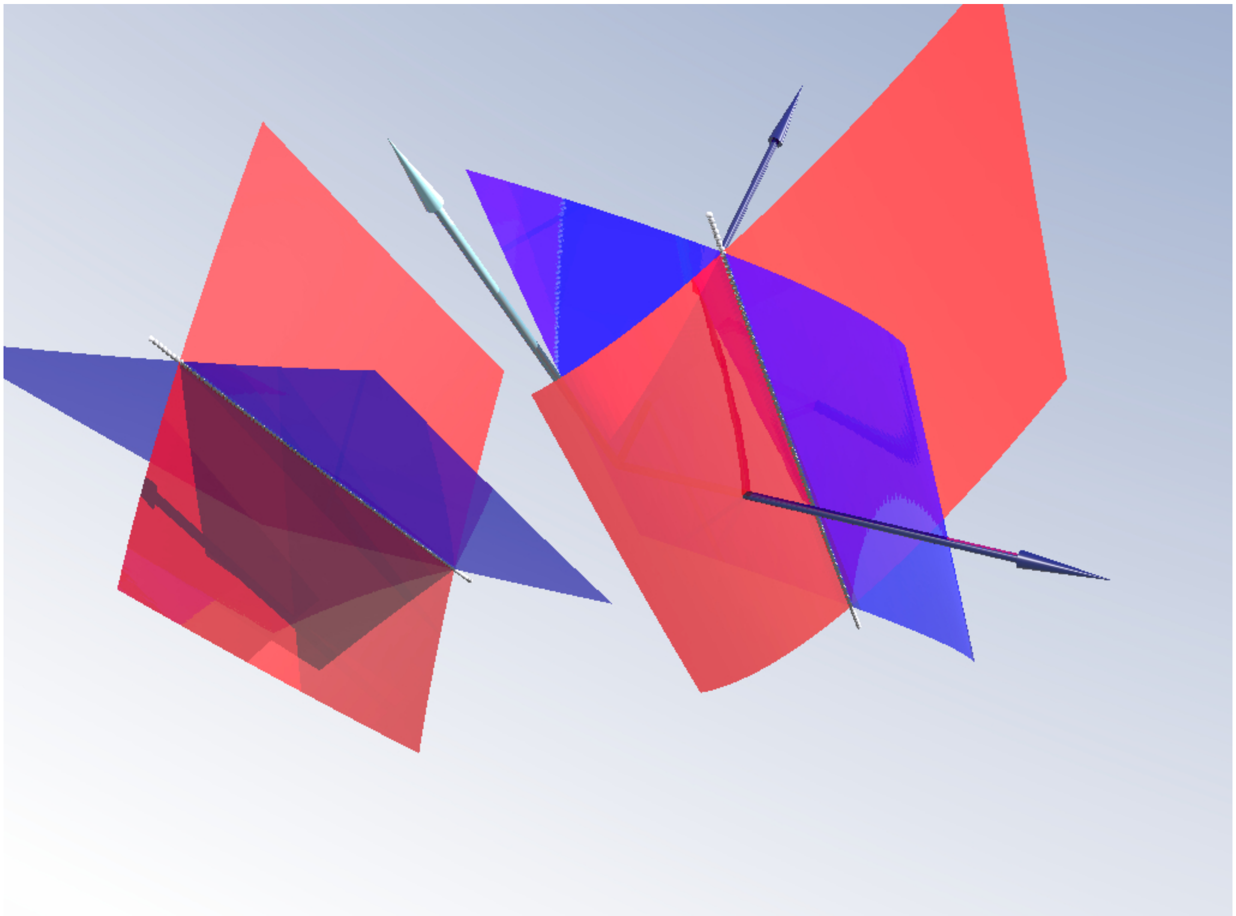

FIG. 6.1. *Some Stuff...*

The remaining stable eigenvalue has

$$\lambda_1^s \in [-18.129924782040472, -18.129924782040465],$$

for the parameters stated above, but does very with ρ. Then we expand this eigenvalue and it's associated eigenvector with respect to the parameter ρ. The results of this computation for several different program inputs are shown in Table (6.2). Since the resulting manifolds are three dimensional we omit graphical results.

Appendix A. Bounding The Error Term E_N and Differential Term $Df[P_N]$ for Hénon and Lorenz.

We will illustrate how the a-posteriori error estimates and the estimates on the differential terms required for Theorems (5.8) and (5.9) in practical applications. In particular we develop explicit formulas for the Hénon and Lorenz systems. We note that even though both the example systems are low dimensional we attempt to proceed in a general manner. The fact that the Hénon and Lorenz systems are quadratic will hold the technical difficulties to a minimum. Nevertheless the following examples illustrate the general procedure.

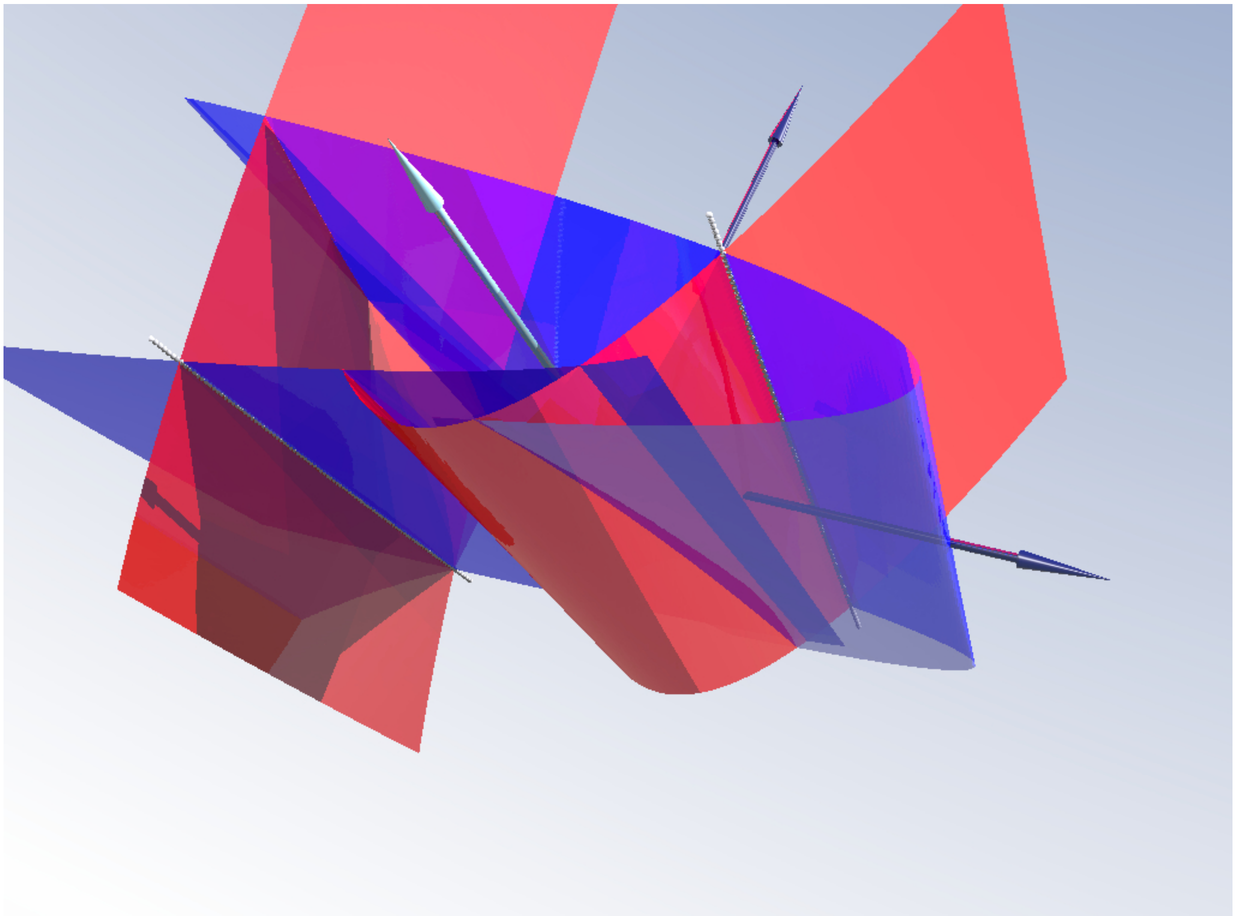

FIG. 6.2. *Some Stuff...*

Example: The Henon Map. We will focus on the stable manifold computation. The unstable is similar. Suppose that

$$\begin{aligned} P_N(\theta, \omega) &= \sum_{n=0}^{N} a_n(\omega)\theta^n \\ &= P_{MN}(\theta, \omega) + H_N(\theta, \omega) \\ &= \sum_{n=0}^{N} \sum_{m=0}^{M} a_{(n,m)}\omega^m \theta^n + \sum_{n=0}^{N} h_n(\omega)\theta^n \end{aligned}$$

with $\|h_n\|_\tau \leq \delta_n$ for $0 \leq n \leq N$ is the validated N-th order formal approximation to a one parameter family of stable manifolds for the Hénon Maps as computed in Section (5.1.1). In order to apply Theorem (5.9) to P_N we have to choose a validation domain $B_\nu \subset \mathbb{C}$ and obtain bounds on both the a-posteriori error term

$$E_N(\theta, \omega) = f[P_N(\theta, \omega)] - P_N[\lambda(\omega)\theta, \omega]$$

M	N	τ	ν	δ_H	δ	time
1	6	10^{-8}	0.0001	4.32×10^{-18}	1.85×10^{-15}	8.5 (sec)
1	10	10^{-8}	0.1	4.1×10^{-12}	9.84×10^{-10}	32.7 (sec)
2	10	10^{-4}	0.1	6.5×10^{-14}	1.6×10^{-12}	39.5 (sec)
3	10	10^{-2}	0.5	7.9×10^{-12}	3.2×10^{-10}	46.3 (sec)
3	10	10^{-2}	1.5	3.4×10^{-10}	1.4×10^{-7}	46.3 (sec)
6	10	0.1	1.5	2.1×10^{-11}	9×10^{-8}	69.7 (sec)
8	15	0.25	1.5	1.3×10^{-8}	6.9×10^{-6}	4.7 (min)
8	20	0.25	2.5	1.4×10^{-8}	4.9×10^{-6}	11.7 (min)
3	30	10^{-4}	3.0	2.6×10^{-8}	1.1×10^{-5}	28.9 (min)

Table 6.2

Branch of Stable Manifold Performance Data for the Lorenz System: *a;slkdfj*

and the inverse of the differential term

$$A(\theta, \omega) = Df[P_N(\theta, \omega), \omega]^{-1}.$$

The complication is that we know the one parameter branch for the stable eigenvalue $\lambda(\omega)$, and the one parameter branches of coefficients $a_n(\omega)$ only up to analytic Taylor Model approximation. That is, the data that we know explicitly are the analytic Taylor models

$$\lambda(\omega) = (\lambda_M(\omega), \tau, \delta_\lambda)\,,$$

$$\lambda^n(\omega) = (\lambda_M^n(\omega), \tau, \delta_{\lambda^n})\,, \qquad \text{for} \quad 2 \leq n \leq N,$$

and

$$a_n(\omega) = \left(a_n^M(\omega), \tau, \delta_n\right) \qquad \text{for} \qquad 0 \leq n \leq N.$$

Given a $\nu > 0$, our goal is to obtain computable bounds on $\|E_N\|_{\nu,\tau}$ and $\|A\|_{\nu,\tau}$ in terms of the known analytic Taylor models. Define

$$\delta_N = \sum_{n=0}^{N} \delta_n \nu^n.$$

We note that for Hénon, the matrix $Df[x, y, \omega]$ is a 2×2 matrix of functions of three variables. We could develop an algorithm and validation theorem similar to those of Section (4.1) for inverting matrices of functions of several variables, however we find that this is usually unnecessary for problems where explicit formulas for f, f^{-1}, their differentials and the inverses of their differential are known.

For Hénon we have that

$$Df(x, y, \omega)^{-1} = \frac{1}{b + \omega} \begin{pmatrix} 0 & 1 \\ b + \omega & 2ax \end{pmatrix}$$

Then

$$\|Df[P_N]^{-1}\|_{\nu,\tau} \leq 1 + \frac{2|a|}{b - \tau} \|P_N\|_{\nu,\tau}$$

where

$$\|P_N\|_{\nu,\tau} \leq \sum_{n=0}^{N} \sum_{m=0}^{M} |a_{(n,m)}| \nu^n \tau^m + \delta_N,$$

a term which can be computed numerically. We must also require that $0 < \tau < b$. This gives a bound on the inverse of the differential in terms of know quantities.

For the a-posteriori error consider

$$E_N(\theta, \omega) = f[P_{MN} + H_N](\theta, \omega) - [P_{MN} + H_N](\lambda(\omega)\theta, \omega).$$

The first term on the right hand side can be expressed explicitly in terms of the known formula for the Hénon mapping. We see that

$$\begin{aligned} f[P_{MN} + H_N](\theta, \omega) &= \left[\begin{array}{c} 1 + P_{MN}^2 + H_N^2 - a(P_{MN}^1)^2 - 2aP_{MN}^1 H_N^1 - a(H_N^1)^2 \\ bP_{MN}^1 + bH_N^1 \end{array} \right] (\theta, \omega) \\ &= f[P_{MN}(\theta, \omega)] + \left[\begin{array}{c} H_N^2 - 2aP_{MN}^1 H_N^1 - a(H_N^1)^2 \\ bH_N^1 \end{array} \right] (\theta, \omega). \end{aligned}$$

For the second term on the right we proceed more generally. Consider

$$[P_{MN} + H_N](\lambda(\omega)\theta, \omega) = P_{MN}\left[(\lambda(\omega)\theta, \omega)\right] + H_N\left[\lambda(\omega)\theta, \omega\right]$$

$$= \sum_{n=0}^{N} \sum_{m=0}^{M} a_{(m,n)} \omega^m \lambda^n(\omega)\ \theta^n + \sum_{n=0}^{N} \sum_{m=M+1}^{\infty} a_{(m,n)} \omega^m \lambda^n(\omega)\ \theta^n.$$

Using the analytic Taylor models of the powers of $\lambda(\omega)$ gives that

$$\sum_{n=0}^{N} \sum_{m=0}^{M} a_{(m,n)} \omega^m \lambda^n(\omega)\ \theta^n = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{(m,n)} \omega^m \left[\lambda_M^n(\omega) + h_{\lambda^n}(\omega)\right]\ \theta^n$$

$$= \sum_{n=0}^{N} \left(\sum_{m=0}^{M} \lambda_m^n \omega^m \right) \left(\sum_{m=0}^{M} a_{(m,n)} \omega^m \right) \theta^n + \sum_{n=0}^{N} \sum_{m=0}^{M} h_{\lambda^n}(\omega) a_{(m,n)} \omega^m \theta^n.$$

Define

$$(P \circ \lambda)_{MN}(\omega, \theta) \equiv \sum_{n=0}^{N} \left(\sum_{m=0}^{M} \lambda_m^n \omega^m \right) \left(\sum_{m=0}^{M} a_{(m,n)} \omega^m \right) \theta^n,$$

and note that this is $2M$-th order polynomial in ω with explicitly known coefficients. On the other hand, the error bounds on the coefficient functions give that

$$\sum_{n=0}^{N} \sum_{m=M+1}^{\infty} a_{(m,n)} \omega^m \lambda^n(\omega)\ \theta^n = \sum_{n=0}^{N} \lambda^n(\omega) h_n(\omega) \theta^n.$$

Let

$$E_{MN}(\theta, \omega) = f[P_{MN}(\theta, \omega), \omega] - (P \circ \lambda)_{MN}[\theta, \omega].$$

We note that this is a composition of only known polynomials and we can numerically bound the quantity $\|E_{MN}\|_{\nu,\tau}$ using the usual sigma norms (the resulting sums are finite). Let ϵ_{MN} be any numerical bound so obtained. We have proven the following Lemma.

LEMMA A.1 (Total A-Posteriori Error for Hénon). *The the validation value a-posteriori error E_N for the Hénon mapping satisfies the following bound;*

$$\|E_N\|_{\nu,\tau} \leq \epsilon_{NM} + \max(\delta_N + 2|a|\|P_{MN}\|_{\nu,\tau}\delta_N + |a|\delta_N^2, |b|\delta_N) + \sum_{n=0}^{N} \delta_{\lambda^n} \sum_{m=0}^{M} |a_{(m,n)}|\tau^m \nu^n + \sum_{n=0}^{N} \delta_n (\mu^*)^n \nu^n.$$

Example: The Lorenz System. This time suppose that

$$\begin{aligned} P_N(\theta, \omega) &= \sum_{|\alpha|=0}^{N} a_\alpha(\omega)\theta^\alpha \\ &= P_{MN}(\theta, \omega) + H_N(\theta, \omega) \\ &= \sum_{|\alpha|=0}^{N} \sum_{m=0}^{M} a_{(\alpha,m)}\omega^m \theta^\alpha + \sum_{|\alpha|=0}^{N} h_\alpha(\omega)\theta^\alpha \end{aligned}$$

with $\|h_\alpha\|_\tau \leq \delta_\alpha$ for $0 \leq |\alpha| \leq N$ is the validated N-th order formal approximation to a one parameter family of two dimensional stable manifolds at the origin of the Lorenz system as computed in Section (5.1.2). We have analytic Taylor models of the stable eigenvalues which we denote by

$$\Lambda(\omega) = \left(\begin{bmatrix} \lambda_M^1(\omega) & 0 \\ 0 & \lambda_M^2(\omega) \end{bmatrix}, \tau, \delta_\Lambda \right),$$

and for the unstable eigenvalue

$$\lambda(\omega) = (\lambda_M(\omega), \tau, \delta_\lambda).$$

Let $\Sigma(\omega) = \mathrm{diag}(\lambda_1(\omega), \lambda_2(\omega), \lambda(\omega))$. We also have analytic Taylor models

$$\xi_i(\omega) = (\xi_M^i(\omega), \tau, \delta_{\xi^i}) \quad \text{for} \quad i = 1, 2, 3$$

for the associated eigenvectors.

An analytic Taylor model for Q is given by

$$Q(\omega) = ([\xi_M^1|\xi_M^2|\xi_M^3], \tau, \delta_Q),$$

where $\delta_Q = \max(\delta_\Lambda, \delta_\lambda)$. We assume that we also have a validated branch of Q^{-1}, represented by the analytic Taylor model

$$Q^{-1}(\omega) = (Q_M^{-1}, \tau, \delta_{Q^{-1}}),$$

obtained using Lemma (4.5).

Now consider that

$$Df(x,y,z,\omega) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho+\omega-z & -1 & -x \\ y & x & -\beta \end{pmatrix}.$$

Then

$$Df[P_N(\theta,\omega),\omega] = Q(\omega)\Sigma(\omega)Q^{-1}(\omega) + \sum_{|\alpha|=1}^{N} A_\alpha(\omega)\theta^\alpha$$

where

$$A_\alpha(\omega) = \sum_{m=0}^{M} A_{(\alpha,m)}\omega^m + H_\alpha(\omega)$$

with

$$A_{(\alpha,m)} = \begin{pmatrix} 0 & 0 & 0 \\ a^3_{(\alpha,m)} & 0 & -a^1_{(\alpha,m)} \\ a^2_{(\alpha,m)} & a^1_{(\alpha,m)} & 0 \end{pmatrix}$$

for $1 \leq |\alpha| \leq N$, $0 \leq m \leq M$, and

$$\|H_\alpha\|_\tau \leq 2\delta_\alpha.$$

Then take C_1 to be a numerically computed constant having

$$\|Q\|_\tau \|Q^{-1}\| \leq \left(\|Q_M\|_\tau + \delta_Q\right)\left(\|Q_M^{-1}\|_\tau + \delta_{Q^{-1}}\right) \leq C_1$$

and C_2 to be any numerically computed constant having

$$\sum_{|\alpha|=1}^{N}\sum_{m=0}^{M} \frac{|A_{(\alpha,m)}|}{\mu_*|\alpha|}\tau^m \nu^{|\alpha|} + \sum_{|\alpha|=1}^{N} \frac{\delta_\alpha}{|\alpha|\mu_*}\nu^{|\alpha|} \leq C_2.$$

Now consider the a-posteriori error

$$E_N(\theta,\omega) = f[P_{MN}(\theta,\omega) + H_N(\theta,\omega),\omega] - D_1 P_{MN}(\theta,\omega)\Lambda(\omega)\theta - D_1 H_N(\omega)\Lambda(\omega)\theta.$$

Again for the first term on the right hand side it is advantageous to exploit the explicit formula for the Lorenz field and obtain that

$$f[P_{MN}(\theta,\omega)+H_N(\theta,\omega),\omega] = f[P_{MN}(\theta,\omega),\omega] + Df[P_{MN}(\theta,\omega),\omega]H_N(\theta,\omega) + \begin{bmatrix} 0 \\ H_N^1 H_N^3 \\ H_N^1 H_N^2 \end{bmatrix}.$$

For the second term on the left we have that

$$D_1 P_{MN}(\theta,\omega)\Lambda(\omega)\theta = D_1 P_{MN}(\theta,\omega)\Lambda_M(\omega)\theta + D_1 P_{MN}(\theta,\omega)h_\Lambda(\omega)\theta,$$

while the third term on the right is

$$D_1 H_N(\omega)\Lambda_\omega\theta = \left(\sum_{|\alpha|=0}^{N} h_\alpha(\omega) D_1\theta^\alpha\right)\Lambda(\omega)\theta.$$

Again we define the term

$$E_{MN}(\theta,\omega) = f[P_{MN}(\theta,\omega),\omega] - D_1 P_{MN}(\theta,\omega)\Lambda_M(\omega)\theta$$

which is the explicitly polynomial part of the a-posteriori error, and which can be easily bound numerically. Let ϵ_{MN} be any numerically computed constant with $\|E_{MN}\|_{\nu,\tau} \leq \epsilon_{MN}$. We now have that

Lemma A.2 (Total A-Posteriori Error for Lorenz). *The validation value a-posteriori error for the Lorenz system satisfies*

$$\|E_N\|_{\nu,\tau} \leq \epsilon_{NM} + \|Df[P_{MN}]\|_{\nu,\tau}\delta_N + \delta_N^2 + \|D_1 P_{MN}\|_{\nu,\tau}\delta_\Lambda \nu + \sum_{|\alpha|=0}^{N} |\alpha|\delta_\alpha \mu^* \nu^{|\alpha|}.$$

REFERENCES

[1] L. V. Ahlfors. Complex Analysis. An Introduction to the Theory of Analytic Functions of One Complex Variable. Third Edition. International Series in Pure and Applied Mathematics. *McGraw-Hill Book Co.*, New York, 1978.

[2] D. Ambrosi, G. Arioli, and H. Koch. A Homoclinic Solution for Excitation Waves on a Contractile Substratum. (In Preperation)

[3] Arai Z, Kalies W, Kokubu H, Mischaikow K, Oka H, Pilarczyk P (2009) A database schema for the analysis of global dynamics of multiparameter systems. SIAM J Appl Dyn Syst 8(3):757–789,

[4] van den Berg JB, Lessard JP, Mischaikow K (2010) Global smooth solution curves using rigorous branch following. Math Comp 79(271):1565–1584,

[5] J.B. van den Berg, J.P. Lessard, K. Mischaikow, and J.D. Mireles James. Rigorous numerics for symmetric connecting orbits: even homoclinics of the Gray-Scott. SIAM J. Math. Anal. 43 (2011), no. 4, 1557-1594.

[6] M. Berz, and K. Makino. Verified Integration of ODE's and Flows Using Differential Algebraic Methods on High-Order Taylor Models. Reliable Computing. Vol. 4 (1998), No. 4, 361-369.

[7] M. Berz. COSY INFINITY Version 8 reference manual. Technical Report MSUCL-1088, National Superconducting Cyclotron Laboratory, Michigan State University, East Lansing, MI 48824, 1997.

[8] X. Cabré, E. Fontich, and R. de la Llave. The Parameterization Method for Invariant Manifolds. I. Manifolds Associated to Non-resonant Subspaces. *Indiana Univ. Math. J.*, 52(2):283-328, (2003).

[9] X. Cabré, E. Fontich, and R. de la Llave. The Parameterization Method for Invariant Manifolds. II. Regularity with Respect to Parameters. *Indiana Univ. Math. J.*, 52(2):329-360, (2003).

[10] X. Cabré, E. Fontich, and R. de la Llave. The Parameterization Method for Invariant Manifolds. III. Overview and applications. *J. Differential Equations*, 218(2):444-515, (2005).

[11] Collet P, Eckmann JP (2009) Iterated maps on the interval as dynamical systems. Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA,

[12] Collet P, Eckmann JP, Lanford OE III (1980) Universal properties of maps on an interval. Comm Math Phys 76(3):211–254,

[13] Eckmann JP, Koch H, Wittwer P (1984) A computer-assisted proof of universality for area-preserving maps. Mem Amer Math Soc 47(289):vi+122

[14] Gameiro M, Lessard JP (2011) Rigorous computation of smooth branches of equilibria for the three dimensional Cahn-Hilliard equation. Numer Math 117(4):753–778.

[15] J. Hoefkens, and M. Berz. Controlling the Wrapping Errect in the Solution of ODEs for Asteroids. Reliab. Comput. Vol 9 (2003), No. 1, pg. 21-41.

[16] W. M Hirsch, and C.C. Pugh. Stable manifolds and hyperbolic sets. Global Analysis (Berkeley 1960), Proc. Sympos. Pure Math Vol 14, pp 133–163, 1970.

[17] Johnson T, Tucker W (2011) A note on the convergence of parametrised non-resonant invariant manifolds. Qual Theory Dyn Syst 10(1):107–121,

[18] Lanford OE III (1982) A computer-assisted proof of the Feigenbaum conjectures. Bull Amer Math Soc (NS) 6(3):427–434.

[19] Lanford OE III (1984) Computer-assisted proofs in analysis. Phys A 124(1-3):465–470.
[20] J.P. Lessard, J.D. Mireles James, and C. Reinhardt. Computer Assisted Proof of Transverse Saddle-to-Saddle Connecting Orbits for First Order Vector Fields. (Submitted).
[21] K. Makino and M. Berz. Remainder Differential Algebras and Their Applications. *Computational Differentiation: Techniques, Applications, and Tools*. Philadelphia, 1996, SIAM, pg 63-74.
[22] K. Makino. Rigorous Analysis of Nonlinear Motion in Particle Accelerators. PhD thesis, Michigan State University, East Lansing, Michigan, USA, 1998.
[23] J.D. Mireles James, and K. Mischaikow. Rigorous A Posteriori Computation of (Un)Stable Manifolds and Connecting Orbits for Analytic Maps. (submitted SIADS).
[24] J.M. Ortega. The Newton-Kantorovich Theorem. *Amer. Math. Monthly* 75 1968 658-660.
[25] N. Revol, K. Makino, and M. Berz. Taylor models and Floating-Point Arithmetic: Proof that Arithmetic Operations are Validated in COSY. J. Log. Algebr. Program. Vol 64 (2005), No. 1, 135-154.
[26] W. Rudin. *Functional Analysis*, International Series in Pure and Applied Mathematics, Second Edition, McGraw-Hill Inc, New York, 1991
[27] S. Rump. Verification Methods: Rigorous Results Using Floating-Point Arithmetic. *Acta Numer.* 19 (2010), 287-449.
[28] S.M. Rump. INTLAB - INTerval LABoratory, *Developments in Reliable Computing*, Tibor Csendes. Kluwer Academic Publishers, Dordrecht, 1999, 77–104.