COMPUTATION OF A ONE PARAMETER BRANCH OF (UN)STBLE MANIFOLDS WITH RIGOROUS COMPUTER ASSISTED A-POSTERIORI ERROR ANALYSIS

J. D. MIRELES-JAMES* AND KONSTANTIN MISCHAIKOW†

Abstract. In this work we develop a calculus for computation of formal series representations of parameter dependent branches, or sheafs, of stable and unstable manifolds in discrete and continuous time dynamical systems. As an essential first step in this process we must develop formal parameter dependent expansions of the fixed point or equilibria, as well as their associated eigenvalues and eigenvectors. Then we use the fact that the family of invariant manifolds satisfies a functional equation to compute formal expansions of some chart maps of the manifold to arbitrary finite order. We also present a-posteriori theorems which allow the error in the finite approximations to be bound rigorously using validated numerics. We present several example computations, as well applications to manifold visualization and computer assisted proof of the existence of a tangency in a family of diffeomorphisms.

1. Introduction.

2. Parameterization of Invariant Manifolds, Regularity With Respect To Parameters, and one Parameter Branches of Invariant Manifolds.

2.1. Parameterization of (Un)stable Manifolds for a Map at a Fixed Parameter Value. Let $p_0 \in \mathbb{R}^d$. Suppose that $f : \mathbb{R}^d \to \mathbb{R}^d$ is real analytic in a neighborhood of p_0 and that p_0 is a fixed point of f. If zero is not an eigenvalue of $Df(p_0)$ then the differential is invertible, and f is a local diffeomorphism about p_0. If in addition $Df(p_0)$ has no eigenvalues on the unit circle, then there are local stable and unstable manifolds $W^{s,u}_{\text{loc}}(p_0)$ tangent at p_0 to the stable and unstable eigenspaces of $Df(p_0)$.

Let d_s and d_u denote the number of stable and unstable eigenvalues respectively, and note that since p_0 is hyperbolic we have that $d_s + d_u = d$. Then there $\nu_s, \nu_u > 0$ and chart maps, or *parameterizations*, $P : B(0, \nu_u) \subset \mathbb{R}^{d_u} \to \mathbb{R}^d$ and $Q : B(0, \nu_s) \subset \mathbb{R}^{d_s} \to \mathbb{R}^d$ so that

$$P[B_{\nu_u}] = W^u_{\text{loc}}(p_0), \qquad\qquad Q[B_{\nu_s}] = W^s_{\text{loc}}(p_0).$$

[9, 10, 11] develop a general *Parameterization Method* for studying such chart maps. The method is based on the fact that the chart maps solve certain functional equations. More precisely we will assume that $Df(p_0)$ is diagonalizable and let $\Lambda_s \in \text{GL}(\mathbb{R}^{d_s})$ and $\Lambda_u \in \text{GL}(\mathbb{R}^{d_u})$ denote the diagonal matrices with respectively the stable and unstable eigenvalues of $Df(p_0)$ on the diagonal entries and zeros in all other entries. Further, let A_s and A_u denote the $d \times d_s$ and $d \times d_u$ matrices having the stable and unstable eigenvectors of $Df(p_0)$ as columns. Then the chart maps P and Q are solutions of the following functional initial value problems

*Rutgers University, Department of Mathematics, 110 Frelinghuysen Rd, Piscataway, NJ 08854, USA.

†Rutgers University, Department of Mathematics & BioMaPS, 110 Frelinghuysen Rd, Piscataway, NJ 08854, USA.

$$
\begin{aligned}
Q(0) &= p_0 \\
DQ(0) &= A_s \\
f[Q(\theta)] &= Q[\Lambda_s \phi] \qquad \text{for all} \qquad \phi \in B_{\nu_s},
\end{aligned} \tag{2.1}
$$

and

$$
\begin{aligned}
P(0) &= p_0 \\
DP(0) &= A_u \\
f[P(\theta)] &= P[\Lambda_u \theta] \qquad \text{for all} \qquad \theta \in B_{\nu_u}.
\end{aligned} \tag{2.2}
$$

Since f is analytic it can be shown that under generic non-resonance conditions on the eigenvalues the parameterizations P and Q are analytic functions having power series expansions

$$
P(\theta) = \sum_{|\alpha| \geq 0} p_\alpha \theta^\alpha, \qquad Q(\phi) = \sum_{|\beta| \geq 0} q_\beta \phi^\beta
$$

where $\alpha \in \mathbb{N}^{d_u}$, $\beta \in \mathbb{N}^{d_s}$, $\phi \in \mathbb{R}^{d_s}$, $\theta \in \mathbb{R}^{d_u}$, and $q_\beta, p_\alpha \in \mathbb{R}^d$. For more detailed discussion of the analyticity of P and Q, see [11] (a non-constructive proof can be given using the Implicit Function Theorem). For a particular map f linear recurrence equations for the unknown power series coefficients can be developed by standard power matching techniques.

Example: Consider the Hénon mapping

$$
f(x, y) = \left[\begin{array}{c} y + 1 - a x^2 \\ b\, x \end{array} \right],
$$

with $a = 1.4$ and $b = 0.3$ fixed. For these parameters the map has exactly two distinct hyperbolic fixed points, each with a one dimensional stable and unstable manifold. Let p_0, λ and ξ denote respectively a choice of fixed point, eigenvalue, and associated eigenvector. Then the parameterization of the local invariant manifold has power series expansion $P(\theta) \sum_{n=0}^\infty p_n \theta^n$ satisfying

$$
\left[\begin{array}{c} P_2(\theta) + 1 - a[P_1(\theta)]^2 \\ b\, P_1(\theta) \end{array} \right] = \left[\begin{array}{c} P_1(\lambda\theta) \\ P_2(\lambda\theta) \end{array} \right].
$$

(Here we are not specifying whether λ is stable or unstable so we just mean use P for *parameterization*). Of course the coefficient p_0 is equal the fixed point (making p_n a notation consistent with the notation for both the parameterization and the fixed point) and $p_1 = \xi$. To find the remaining coefficients we exploit the functional equation. The right hand side is

$$
\left[\begin{array}{c} P_1(\lambda\theta) \\ P_2(\lambda\theta) \end{array} \right] = \sum_{n=0}^\infty \lambda^n \left[\begin{array}{c} p_n^1 \\ p_n^2 \end{array} \right] \theta^n,
$$

while the left hand side is

$$f[P(\theta)] = \left[\begin{array}{c} 1 + \sum_{n=0}^{\infty} \left[p_n^2 - \sum_{k=0}^{n} a\, p_{n-k}^1 p_k^1\right] \theta^n \\ \sum_{n=0}^{\infty} b\, p_n^1 \theta^n \end{array} \right]$$

Matching like powers of θ and solving for the highest order coefficient p_n in terms of lower order coefficients gives the linear system of equations

$$\left(\begin{array}{cc} -2ap_0^1 - \lambda^n & 1 \\ b & -\lambda^n \end{array} \right) \left[\begin{array}{c} p_n^1 \\ p_n^2 \end{array} \right] = \left[\begin{array}{c} \sum_{k=1}^{n-1} ap_{n-k}^1 p_k^1 \\ 0 \end{array} \right], \tag{2.3}$$

for $n \geq 2$. Equation (2.3) is referred to as the *homological equation*. Note that the homological equation has the form

$$[Df(p_0) - \lambda^n I]p_n = s_n$$

where s_n depends only on terms of order less than n. Also note that the matrix is the characteristic matrix of $Df(p_0)$. Then since $|\lambda| \neq 1$ we have $\lambda^n \neq \lambda$ for all $n \geq 2$ and matrix is always invertible. Then the coefficients p_n are formally well defined to all orders.

Remarks 2.1.

- Generalization
- Resonances
- Numerics/Radius of Convergence

2.2. Parameterization of (Un)stable Manifolds for a Differential Equation at a Fixed Parameter Value. The parameterization method can also be applied to differential equations, as is also shown in [9, 10, 11]. If $p_0 \in \mathbb{R}^d$ is an equilibria of a vector field $f : \mathbb{R}^d \to \mathbb{R}^d$, we suppose that f is real analytic near p_0, that $Df(p_0)$ is diagonalizable, and that p_0 is a hyperbolic equilibria (all eigenvalues have no-zero real part). Let $\lambda_{s,u}$ again denote the diagonal matrices of stable and unstable eigenvalues and $A_{s,u}$ the matrices whose columns are the associated stable and unstable eigenvectors.

The the parameterizations of the local stable and unstable manifolds solve the initial value functional (in this case partial differential) equations

$$\begin{aligned} Q(0) &= p_0 \\ DQ(0) &= A_s \\ f[Q(\theta)] &= DQ(\phi) \cdot \Lambda_s \cdot \phi \qquad \text{for all} \qquad \phi \in B_{\nu_s}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} P(0) &= p_0 \\ DP(0) &= A_u \\ f[P(\theta)] &= DP(\theta) \cdot \Lambda_u \cdot \theta \qquad \text{for all} \qquad \theta \in B_{\nu_u}. \end{aligned} \tag{2.5}$$

with some $\nu_u, \nu_s > 0$. Here the multiplications are matrix-matrix or matrix-vector as appropriate.

Again, an application of the Implicit Function Theorem gives that there are analytic P and Q solving (see for example [11]). Then P and Q have convergent power series expansions and we can try to formally compute the coefficients by power matching.

Example: Consider the differential equation $\dot{x} = f(x)$ given by the vector field

$$f(x, y, z) = \begin{bmatrix} \sigma(y - x) \\ \rho x - xz - y \\ xy - \beta z \end{bmatrix},$$

with the tangency parameters $\sigma = 10$, $\beta = 8/3$, and $\rho = 13.9265$ (the Lorenz system is known to exhibit a homoclinic tangency near this parameter of ρ. See [20]).

The map has three fixed points, each of which is hyperbolic at the specified parameter values. Let p_0 denote one of these fixed points, λ_1 and λ_2 denote two eigenvalues of $Df(p_0)$ with like stability (either both stable or both unstable), and ξ_1, ξ_2 be two associated eigenvectors . We will illustrate the computations of a two dimensional stable or unstable manifold of a fixed point of the Lorenz system (the computation of a one dimensional manifold is similar and has already illustrated for the Hénon map).

We will let P denote the parameterization of the invariant manifold (whether stable or unstable) Λ denote the matrix with λ_1 and λ_2 as diagonal entries. Then in this case the power series is

$$P(\theta) = \sum_{|\alpha| \geq 0} p_\alpha \theta^\alpha = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} p_{(n_1,n_2)} \theta_1^{n_1} \theta_2^{n_2},$$

with $p_{(n_1,n_2)} \in \mathbb{R}^3$ for each $n_1, n_2 \geq 0$. The linear constraints give that $p_{(0,0)} = p_0$, $p_{(0,1)} = \xi_1$, and $p_{(1,0)} = \xi_2$. The coefficients for $n_1 + n_2 \geq 2$ are worked out by considering the functional equation

$$\begin{bmatrix} \sigma(P_2(\theta) - P_1(\theta)) \\ \rho P_1(\theta) - P_1(\theta)P_3(\theta) - P_2(\theta) \\ P_1(\theta)P_2(\theta) - \beta P_3(\theta) \end{bmatrix} = \begin{bmatrix} \theta_1\lambda_1\partial_{\theta_1}P_1(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_1(\theta) \\ \theta_1\lambda_1\partial_{\theta_1}P_2(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_2(\theta) \\ \theta_1\lambda_1\partial_{\theta_1}P_3(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_3(\theta) \end{bmatrix}$$

The right hand side expands as

$$\begin{bmatrix} \theta_1\lambda_1\partial_{\theta_1}P_1(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_1(\theta) \\ \theta_1\lambda_1\partial_{\theta_1}P_2(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_2(\theta) \\ \theta_1\lambda_1\partial_{\theta_1}P_3(\theta) + \theta_2\lambda_2\partial_{\theta_2}P_3(\theta) \end{bmatrix} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (n_1\lambda_1 + n_2\lambda_2) \begin{bmatrix} p^1_{(n_1,n_2)} \\ p^2_{(n_1,n_2)} \\ p^3_{(n_1,n_2)} \end{bmatrix} \theta_1^{n_1}\theta_2^{n_2}$$

while the left hand side is

$$\begin{bmatrix} \sigma(P_2(\theta) - P_1(\theta)) \\ \rho P_1(\theta) - P_1(\theta)P_3(\theta) - P_2(\theta) \\ P_1(\theta)P_2(\theta) - \beta P_3(\theta) \end{bmatrix} =$$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[\begin{array}{c}\sigma\left(p^2_{(n_1,n_2)}-p^1_{(n_1,n_2)}\right)\\ \rho\, p^1_{(n_1,n_2)}-p^2_{(n_1,n_2)}-\sum_{k=0}^{n_2}\sum_{j=0}^{n_1}p^1_{(n_1-j,n_2-k)}p^3_{(j,k)}\\ -\beta\, p^3_{(n_1,n_2)}+\sum_{k=0}^{n_2}\sum_{j=0}^{n_1}p^1_{(n_1-j,n_2-k)}p^2_{(j,k)}\end{array}\right]\theta_1^{n_1}\theta_2^{n_2}.$$

Matching like powers of θ and solving for the highest order terms in terms of the lower order terms gives the homological equation

$$\left(\begin{array}{ccc}\sigma-(n_1\lambda_1+n_2\lambda_2) & \sigma & 0\\ \rho-p^3_{(0,0)} & -1-(n_1\lambda_1+n_2\lambda_2) & -p^1_{(0,0)}\\ p^2_{(0,0)} & p^1_{(0,0)} & -\beta-(n_1\lambda_1+n_2\lambda_2)\end{array}\right)\left[\begin{array}{c}p^1_{(n_1,n_2)}\\ p^2_{(n_1,n_2)}\\ p^3_{(n_1,n_2)}\end{array}\right]$$

$$=\sum_{k=0}^{n_2}\sum_{j=0}^{n_1}\left[\begin{array}{c}0\\ \bar{p}^1_{(n_1-j,n_2-k)}\bar{p}^3_{(j,k)}\\ -\bar{p}^1_{(n_1-j,n_2-k)}\bar{p}^2_{(j,k)}\end{array}\right]$$

where

$$\bar{p}_{(j,k)}=\begin{cases}0 & \text{if either}\quad i=j=0\quad\text{or}\quad i=n_1, j=n_2\\ p_{(i,j)} & \text{otherwise}\end{cases}$$

The homological equation has the form

$$[\,Df(p_0)-(n_1\lambda_1+n_2\lambda_2)I\,]\,p_{(n_1,n_2)}=s_{(n_1,n_2)},$$

with s depending only on lower order terms. Moreover the matrix is a characteristic matrix for $Df(p_0)$ and is invertible as long as $n_2\lambda_1+n_2\lambda_2\neq\lambda_\ell$ for any $n_1+n_2\geq 2$ and either of $\ell=1,2$. Note that since λ_1 and λ_2 have the same stability, this leads to a finite number of *non-resonance* conditions between the eigenvalues. Then the coefficients are well defined to all orders as long as the eigenvalues are non-resonant.

Remarks 2.2.

- Generalization
- Resonance
- Numerics/Radius of Convergence

2.3. Regularity with Respect to a Parameter and Analytic Branches of Parameterizations. Now we consider that $f:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d$ is a one parameter family of maps with $p_0\in\mathbb{R}^d$ a hyperbolic fixed point of $f(x,0)$, so that $f(p_0,\omega)=0$. Moreover we suppose that $f(x,\omega)$ is real analytic jointly in each variable in some neighborhood of $(p_0,0)$. By the implicit function theorem there is a $\tau>0$ and a real analytic function $p:(-\tau,\tau)\subset\mathbb{R}\to\mathbb{R}^d$ so that

$$f(p(\omega),\omega)=p(\omega)\qquad\text{for all}\qquad|\omega|<\tau.$$

Suppose in addition that $Df(p_0,0)$ is diagonalizable and let $\lambda_0^1,\ldots,\lambda_0^{d_u}$, and $\xi_0^1,\ldots,\xi_0^{d_u}$ denote the unstable eigenvalues and associated eigenvectors. Let Λ_0 be the matrix with the λ_0^i as diagonal entries and zeros on the off diagonal entries and A_0^u be the matrix with columns ξ_0^i $(1\leq i\leq d_u)$.

Since the eigenvalues and eigenvectors solve analytic equations the implicit function theorem gives that there is a $\tau > 0$ and analytic functions $\lambda_1, \ldots, \lambda_{d_s} : (-\tau, \tau) \to \mathbb{C}$, $\xi_1, \ldots, \xi_{d_u} : (-\tau, \tau) \to \mathbb{R}^d$ so that $\lambda_i(\omega)$ is an eigenvalue of $Df(p(\omega), \omega)$ with associated eigenvector $\xi_i(\omega)$ for each $|\omega| < \tau$, and having $\lambda_i(0) = \lambda_0^i$ and $\xi_i(0) = \xi_0^i$ for each $1 \leq i \leq d_u$. Moreover, if $|\xi_0^i| = \hat{K}$ then requiring that $|\xi(\omega)| = \hat{K}$ for each $|\omega| < \tau$ determines the branches uniquely. (Here τ is taken to be the smallest number so that all the implicit function arguments go through simultaneously).

Assume the eigenvalues $\lambda_0^1, \ldots, \lambda_0^{d_u}$ are non-resonant. Then there is a $\nu_u > 0$ and a real analytic chart map $\hat{P} : B(0, \nu_u) \subset \mathbb{R}^{d_u} \to \mathbb{R}^d$ satisfying Equatoin (2.2) parameterizing the local unstable manifold of p_0. Let

$$\hat{P}(\theta) = \sum_{|\alpha| \geq 0} \hat{p}_\alpha \theta^\alpha$$

Then the power series coefficients satisfy the homological equation

$$[Df(p_0, 0) - \Lambda_0^\alpha]\hat{p}_\alpha = \hat{s}_\alpha, \tag{2.6}$$

and, as per Remark (SOMETHING), the matrix is invertible fo all α by the non-resonance assumption and for each $\alpha \geq 2$, and the right hand side s_α is an analytic function of only the coefficients of order less than α. Moreover, since the eigenvalues are non-resonant there is a $\tau > 0$ so that $\lambda_1(\omega)$, ..., $\lambda_{d_u}(\omega)$ are non-resonant for each $|\omega| < \tau$.

Let $\Lambda(\omega)$ denote the non-constant matrix having $[\Lambda(\omega)]_{ii} = \lambda_i(\omega)$ for each $1 \leq i \leq d_u$ and $[\Lambda(\omega)_{ij} = 0$ if $i \neq j$, and $A_u(\omega)$ be the non-constant matrix having columns $\xi_i(\omega)$ for each $1 \leq i \leq d_u$. It is shown in [10, 11], again using the analytic Implicit Function Theorem, that there is a branch of real analytic functions $P : B(0, \nu_u) \times (-\tau, \tau) \subset \mathbb{R}^{d_u} \times \mathbb{R} \to \mathbb{R}^d$ through $\hat{P}$ (in other words $P(\cdot, 0) = \hat{P}$) solving the functional initial value problems

$$P(0, \omega) = p(\omega) \tag{2.7}$$

$$DP(0, \omega) = A_u(\omega) \tag{2.8}$$

$$f[P(\theta, \omega), \omega] = P[\Lambda(\omega)\theta, \omega] \qquad \text{for all} \qquad \theta \in B_{\nu_u}, \quad |\omega| < \tau. \tag{2.9}$$

Since P is analytic it has a convergent power series representation

$$P(\theta, \omega) = \sum_{|\alpha| \geq 0} \sum_{m=0}^{\infty} p_{(m,\alpha)} \omega^m \theta^\alpha \sum_{|\alpha| \geq 0} p_\alpha(\omega) \theta^\alpha \qquad \theta \in B_{\nu_s}, \quad |\omega| < \tau, \tag{2.10}$$

where we have defined $p_\alpha(\omega) = \sum_{m=0}^{\infty} p_{(m,\alpha)} \omega^m$. Note that $p_\alpha(\omega)$ is analytic for each α as the series given by Equation (2.10) converges. Because $P(\theta, 0) = \hat{P}(\theta)$ we also have that

$$p_{(0,\alpha)} = p_\alpha(0) = \hat{p}_\alpha \qquad \text{for all} \qquad m, |\alpha| \geq 0. \tag{2.11}$$

We also have that each $\hat{p}_\alpha$ solves the homological equation (Equation 2.6). Since the homological equations are analytic, we apply the Implicit Function Theorem again to obtain real analytic branch functions $\hat{p}_\alpha : (-\tau, \tau) \to \mathbb{R}^d$ so that $\hat{p}(0) = \hat{p}_\alpha$ for each

α. But then by the uniqueness of power series coefficients of an analytic function and the uniqueness provided by the Implicit Function Theorem we have

$$\hat{p}_\alpha(\omega) = p_\alpha(\omega) \qquad \text{for all} \qquad |\alpha| \geq 0.$$

In other words the coefficients of P solve homological equations

$$[Df[p(\omega), \omega] - \Lambda(\omega)^\alpha I] p_\alpha(\omega) = s_\alpha(\omega) \tag{2.12}$$

for all $|\omega| < \tau$, where the matrix is invertible for all $|\omega| < \tau$ by the non-resonance assumption.

Consider the $0 \leq |\alpha| \leq 1$ coefficients. We will denote these by letting $\alpha_0 = (0, \ldots, 0)$ be the zero index, and $e_i = (0, \ldots, 1)$ be the index having a one in the i-th component and zeros elsewhere. Let the parameterization of the analytic branch of fixed points have power series expansion $p(\omega) = \sum_{m=0}^{\infty} p_m^0 \, \omega^m$. Then by Equation (2.7) we have that

$$p_{\alpha_0}(\omega) = p(\omega) \qquad \text{or} \qquad p_{(m,\alpha_0)} = p_m^0 \qquad \text{for all} \qquad m \geq 0. \tag{2.13}$$

So the (m, α_0) terms of P are given by the power series coefficients of the fixed point branch p. Similarly let $\xi_i(\omega) = \sum_{m=0}^{\infty} \xi_m^i \, \omega^m$. Then Equation (2.8) gives that

$$p_{e_i}(\omega) = \xi_i(\omega) \qquad \text{or} \qquad p_{m,e_i} = \xi_m^i \qquad m \geq 0. \tag{2.14}$$

Of course Equaiton (2.11) gives that the $m = 0$ coefficients are determined by the power series expansion of the parameterization $\hat{P}$ of the $\omega = 0$ manifold. The computation of the $m = 1$ coefficients are somewhat more delicate. Considering Equation (2.10) we see that just as the $m = 0$ coefficients are given by the coefficients $p_\alpha(0)$, the $m = 1$ coefficients are given by the coefficients of $\frac{\partial}{\partial \omega} p_\alpha(0)$. Explicitly we have that

$$\frac{\partial}{\partial \omega} P(\theta, 0) = \sum_{|\alpha| \geq 0} \sum_{m=0}^{\infty} p_{(m,\alpha)} \frac{\partial}{\partial \omega}_{|\omega=0} \omega^m = \sum_{|\alpha| \geq 0} p_{1,\alpha} \theta^\alpha$$

On the other hand we have that the coefficients $a_\alpha(\omega)$ solve the parameter dependent homological equation (Equation 2.12). Now since $p_{1,\alpha} = \frac{\partial}{\partial \omega} p_\alpha(0)$ we differentiate both sides of Equatoin (2.12) with respect to ω and evaluate at $\omega = 0$ in order to obtain that $p_{1,\alpha}$ solves the linear equation

$$[Df(p_0, 0) - \Lambda_0^\alpha I] \, p_{(1,\alpha)} = \frac{\partial}{\partial \omega} s_\alpha(0) - \frac{\partial}{\partial \omega}_{|\omega=0} [Df(p(\omega), \omega) - \Lambda^\alpha(\omega) I] \, p_{(0,\alpha)} \tag{2.15}$$

for $|\alpha| \geq 2$. We make no attempt at present to simplify the expressions on the right hand side of Equation (2.15). Rather we will work out the formulas only in the context of specific examples, in which case the expressions may simplify dramatically. The essential fact to note at present is that the matrix on the left hand side of of Equation

(2.15) is none other than the characteristic matrix of $Df(p_0, 0)$, so that the coefficients $p_{(1,\alpha)}$ are well defined for all $|\alpha| \geq 2$ due to the non-resonance assumption. In other words, Equation (2.15) introduces no extra constraints.

Finally we must determine the coefficients $p_{m,\alpha}$ when $m + |\alpha| \geq 2$. This could be done by repeatedly differentiating the ω-dependent homological equation (Equatoin 2.12) and evaluation at $\omega = 0$ to obtain homological equations analogous to Equation (2.15) for all $m \geq 2$. Such expressions become both analytically, and computationally cumbersome. It is in fact preferable in the context of specific applications to substitute the power series form of P directly into Equation (2.9) and match like powers in order to develop the homological equations directly. We give examples in Section SOMETHING.

Similar considerations in the context of differential equations lead us to the fact that the parameterization must satisfy

$$P(0, \omega) = p(\omega) \tag{2.16}$$

$$DP(0, \omega) = A_u(\omega) \tag{2.17}$$

$$f[P(\theta, \omega), \omega] = D_\theta P[\theta, \omega] \cdot \Lambda(\omega) \cdot \theta \qquad \text{for all} \qquad \theta \in B_{\nu_u}, \quad |\omega| < \tau. \tag{2.18}$$

The series for P will be given by Equation (2.10) just as before, however in this case the ω-dependent homological equation must be given by

$$[Df[p(\omega), \omega] - \langle \Lambda(\omega), \alpha \rangle \, I] p_\alpha(\omega) = s_\alpha(\omega) \tag{2.19}$$

Proceeding as above we obtain that Equations (2.13, 2.14, and 2.11) hold exactly as before. However the homological equation for the coefficients $p_{1,\alpha}$ of $\frac{\partial}{\partial \omega} P(\theta, 0)$ are given by the homological equation

$$[Df(p_0, 0) - \langle \Lambda_0, \alpha \rangle \, I] \, p_{(1,\alpha)} = \frac{\partial}{\partial \omega} s_\alpha(0) - \frac{\partial}{\partial \omega}_{|\omega=0} [Df(p(\omega), \omega) - \langle \Lambda(\omega), \alpha \rangle \, I] \, p_{(0,\alpha)}. \tag{2.20}$$

Again the matrix on the left is just the characteristic matrix of $Df[p_0, 0]$ so that the non-resonance assumptions yield that the coefficients are formally well defined. Simplification of Equation (2.20) and the formal computation of $p_{m,\alpha}$ for $m \geq 2$ is carried out only in the context of specific applications, which we consider in Section SOMETHING.

Also: note that all the comments made in this section apply equally well to stable manifolds of maps and flows. We have focused on unstable manifolds in order to minimize the proliferation of sub and superscripts.

REMARK 2.1. [$P(\theta, \omega)$**-Algorithm**] The discussion above provides us with a four step meta-algorithm for development of the formal series expansion of a branch of invariant manifolds

Step 1: Compute the parameterization $\hat{P}(\theta)$ of the invariant manifold at $\omega = 0$. This determines the coefficients $p_{0,\alpha}$ of P. This step was discussed in Section SOMETHING for both the Hénon map and the Lorenz system.

Step 2: Compute the power series of the analytic branch functions $p(\omega)$, $\lambda_i(\omega)$, and $\xi_i(\omega)$, $1 \leq i \leq d_{u,s}$ for the fixed point, eigenvalues, and eigenvectors. The coefficients of $p(\omega)$ determine the coefficients p_{m,α_0}, while the coefficients of

$\xi_i(\omega)$ determine the coefficients $p_{(m,e_i)}$. These computations are the subject of Section SOMETHING.

Step 3: Depending on whether f generates a discrete or continuous time dynamical system (maps or differential equations) use either Equation (2.15) or Equation (2.20) along with the specific form of the map f to compute the $p_{1,\alpha}$ coefficients. Examples of this computation are given in Section SOMETHING.

Step 4: Plug the unknown power series given by Equation (2.10) into the either Equation (2.9) if f is a map, or Equation (2.18) if f is a vector field. Expand both sides as power seres, match like powers of $\omega^m \theta^\alpha$, and isolate the highest order coefficients from the lower order coefficients in order to obtain a homological equation for the $p_{m,\alpha}$ coefficients when $m \geq 2$. We illustrate this computation for the Hénon and Lorenz systems in Section (SOMETHING).

3. One Parameter Families of Fixed Points, Equilibria, Eigenvalues, and Eigenvectors. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a one parameter family of real analytic vector fields denoted by $f(x, \omega)$ with $x \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$. Suppose that $x_0 \in \mathbb{R}^n$ is an equilibria for f at $\omega = 0$, so that $f(x_0, 0) = 0$. Then if $D_x f(x_0, 0)$ is non-singular, the implicit function theorem (in the analytic category) gives that the is an analytic branch of equilibria through x_0.

More formally there exists a $\tau > 0$ real analytic function $x : (-\tau, \tau) \to \mathbb{R}^n$ so that $x(0) = x_0$ and

$$f[x(\omega), \omega] = 0 \qquad \text{for all} \qquad \omega \in (-\tau, \tau).$$

We say that x parameterizes the branch of equilibria through x_0. Since x is analytic it has a power series expansion $x(\omega) = \sum_{n=0}^{\infty} x_n \omega^n$ with $x_n \in \mathbb{R}^n$ convergent for $|\omega| < \tau$. In order to exploit this fact in a computational setting we must determine

(I) the coefficients x_n of the power series expansion for the branch of zeros,
(II) the radius of convergence τ of the power series.

Since x solves a (functional) equation, recurrence relations for the coefficients x_n can be computed by the usual power matching schemes. The convergence of the series could be treated in any of several ways. One could for a given problem prove the convergence of the power series directly by the method of majorization. Since we are using these series as inputs into further numerical computations, we pursue a numerical alternative. In any given problem we will compute the coefficients x_n to some finite order $M \in \mathbb{N}$, giving an approximate parameterization $x_M(\omega) = \sum_{n=0}^{M} x_n \omega^n$ of the branch of equilibria. Then we use residual methods based on the Newton-Kantorovich Theorem to prove that the series converge on some finite disk, and to rigorously bound the truncation error of the finite series. The radius of convergence is determined using numerical methods. We discuss the formal computations and the a-posteriori numerical argument in the next two sections respectively.

Finally we note that the comments above apply equally well to parameterizations of fixed points of diffeomorphisms, as well as to parameterizations of eigenvalues and eigenvectors, as in each of these solve functional equations of their own.

3.1. Computation of Formal Series Expansions for Linear Data. We consider the Hénon Family

$$f(x, y, \omega) = \begin{bmatrix} y + 1 - ax^2 \\ (b + \omega)x \end{bmatrix}, \tag{3.1}$$

where we think of a and b as fixed. Let

$$x(\omega) = \sum_{n=0}^{\infty} x_n \omega^n$$

parameterize an analytic branch of the first component of a fixed point of Equation (3.1). Then $x(\omega)$ solves the equaiton

$$a[x(\omega)]^2 + x(\omega)(1 - b - \omega) - 1 = 0. \tag{3.2}$$

Then

$$x_0 = \frac{b - 1 \pm \sqrt{(1-b)^2 + 4a}}{2a},$$

and

$$x_1 = \frac{d}{d\omega} x(0) = \frac{x_0}{2ax_0 - b + 1}.$$

(The expression for x_1 can be obtained by implicit differentiation of Equation 3.2). Matching like powers of ω in equation 3.2 gives that

$$x_n = \frac{1}{2ax_0 - b + 1} \left[x_{n-1} - \sum_{k=1}^{n-1} a\, x_{n-k} x_k \right]. \qquad \text{for} \qquad n \geq 2. \tag{3.3}$$

We note that since the second component of the fixed point is given by $y(\omega) = (b + \omega)x(\omega)$ we also have

$$y_0 = bx_0, \qquad y_1 = bx_1 + x_0 \qquad \text{and} \qquad y_n = bx_n + x_{n-1} \qquad n \geq 2.$$

Similarly, if λ_0 is an eigenvalue of $D_{(x,y)} f(x, y, 0)$ then we let

$$\lambda(\omega) = \sum_{n=0}^{\infty} \lambda_n \omega^n$$

parameterize a branch of eigenvalues passing through λ_0. Then $\lambda(\omega)$ satisfies the equation

$$\lambda(\omega)^2 + 2a\, x(\omega)\, \lambda(\omega) - \omega - b = 0, \tag{3.4}$$

with $\lambda(0) = \lambda_0$. As above we compute that

$$\lambda_0 = ax_0 \pm \sqrt{a^2 x_0^2 + b^2}, \qquad \lambda_1 = \frac{1 - 2ax_1\lambda_0}{2\lambda_0 + 2ax_0}$$

and

$$\lambda_n = \frac{-1}{2\lambda_0 + 2ax_0}\left(\sum_{k=1}^{n-1}\lambda_{n-k}\lambda_k + \sum_{k=0}^{n-1} 2ax_{n-k}\lambda_k\right) \qquad \text{with} \qquad n \geq 2. \tag{3.5}$$

Note that the λ_n are formally well defined as long as $\lambda_0 \neq -ax_0$, i.e. as long as λ_0 is not a repeated eigenvalue, and that the coefficient λ_n depends on the coefficients of $x(\omega)$ recurssively to n-th order.

Now suppose that we choose an eigenvector ξ_0 with $\|\xi\|^2 = \hat{K}$ for some $\hat{K} > 0$, associated with the eigenvalue λ_0. Now denote by

$$\xi(\omega) = \sum_{n=0}^{\infty} \xi_n \omega^n$$

a parameterizatoin of the branch of eigenvectors through ξ_0, where the entire branch is normalized to have length $\sqrt{\hat{K}}$. Then $\xi(\omega)$ satisfies the system of equations

$$\left[\begin{array}{cc} -2ax(\omega) - \lambda(\omega) & 1 \\ b+\omega & -\lambda(\omega) \end{array}\right]\left(\begin{array}{c} \xi_1(\omega) \\ \xi_2(\omega) \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right)$$
$$\xi_1(\omega)^2 + \xi_2(\omega)^2 = \hat{K}$$

but since the rows of the matrix equation are linearly dependent, we have that $\xi(\omega)$ must simultaneously safisfy

$$(b+\omega)\xi_1(\omega) - \lambda(\omega)\xi_2(\omega) = 0 \qquad \text{and} \qquad \xi_1(\omega)^2 + \xi_2(\omega)^2 = \hat{K}.$$

Matching like powers leads to the linear systems

$$\left[\begin{array}{cc} b & -\lambda_0 \\ 2\xi_0^1 & 2\xi_0^2 \end{array}\right]\left(\begin{array}{c} \xi_1^1(\omega) \\ \xi_2^1(\omega) \end{array}\right) = \left(\begin{array}{c} \lambda_1\xi_0^2 - \xi_0^1 \\ 0 \end{array}\right)$$

for the coefficient ξ_1 and

$$\left[\begin{array}{cc} b & -\lambda_0 \\ 2\xi_0^1 & 2\xi_0^2 \end{array}\right]\left(\begin{array}{c} \xi_n^1(\omega) \\ \xi_n^2(\omega) \end{array}\right) = \left(\begin{array}{c} -\xi_{n-1}^1 + \sum_{k=0}^{n-1}\lambda_{n-k}\xi_k^2 \\ -\sum_{k=1}^{n-1}\xi_{n-k}^1\xi_k^1 + \xi_{n-k}^2\xi_k^2 \end{array}\right)$$

for ξ_n when $n \geq 2$. The coefficient ξ_n depends recursively on the coefficients of $\lambda(\omega)$ to n-th order.

We consider also the Lorenz System, which is given by the flow of the vector field

$$f(x,y,z,\omega) = \left[\begin{array}{c} \sigma(y-x) \\ (\rho+\omega)x - xz - y \\ xy - \beta z \end{array}\right], \tag{3.6}$$

where we think of σ, ρ, and β as fixed. When $\omega = 0$ the system has equilibria at $p_0 = (0,0,0)^T$ and

$$p_{1,2} = \begin{bmatrix} \pm\sqrt{\beta(\rho-1)} \\ \pm\sqrt{\beta(\rho-1)} \\ \rho - 1 \end{bmatrix}$$

In fact p_0 is fixed for all ω. On the other hand, if we let $p(\omega) = (x(\omega), y(\omega), z(\omega))^T$ be a branch of either $p_{1,2}$, then we can work out that $x_0 = y_0 = \pm\sqrt{\beta(\rho-1)}$, $z_0 = \rho - 1$,

$$x_1 = y_1 = \frac{\pm\beta}{2\sqrt{\beta(\rho-1)}}, \qquad z_1 = \frac{2x_0x_1}{\beta}$$

and

$$x_n = y_n = \frac{-1}{2x_0}\sum_{k=1}^{n-1} x_{n-k}x_k, \qquad z_n = \frac{1}{\beta}\sum_{k=0}^{n} x_{n-k}x_k \qquad n \geq 2.$$

In the applications section we will be more interested in the fixed point at the origin, so we develop the expansions for the eigenvalues and eigenvectors only at p_0. We note that at the origin $-\beta$ is an eigenvalue for all ω. The remaining two eigenvalues do depend on ω and solve the equation

$$\lambda(\omega)^2 + (1+\sigma)\lambda(\omega) - \sigma(\rho + \omega - 1) = 0. \tag{3.7}$$

Denote a parameterization of a branch of these by

$$\lambda(\omega) = \sum_{n=0}^{\infty} \lambda_0 \omega^n.$$

Then

$$\lambda_0 = \frac{-(1+\sigma) \pm \sqrt{(1+\sigma)^2 - 4(\rho-1)}}{2}, \qquad \lambda_1 = \frac{\sigma}{2\lambda_0 + \sigma + 1}$$

and

$$\lambda_n = \frac{-1}{2\lambda_0 + 1 + \sigma}\sum_{k=1}^{n-1} \lambda_{n-k}\lambda_k \qquad n \geq 2.$$

An eigenvector associated with the eigenvalue $-\beta$ is $\xi = (0,0,1)^T$ for all ω. The eigenvectors associated with the solutions of 3.7 lie in the xy-plane for all ω. A computation similar to as in the Hénon case shows that the coefficients of parameterizations of the these planar eigenvectors are given by the solutions of the linear systems

$$\begin{bmatrix} -(\sigma + \lambda_0) & \sigma \\ 2\xi_0^1 & 2\xi_0^2 \end{bmatrix} \begin{pmatrix} \xi_1^1(\omega) \\ \xi_2^1(\omega) \end{pmatrix} = \begin{pmatrix} \lambda_1 \xi_0^1 \\ 0 \end{pmatrix}$$

for the coefficient ξ_1 and

$$\begin{bmatrix} -(\sigma+\lambda_0) & \sigma \\ 2\xi_0^1 & 2\xi_0^2 \end{bmatrix} \begin{pmatrix} \xi_n^1(\omega) \\ \xi_n^2(\omega) \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{n-1} \lambda_{n-k}\xi_k^1 \\ -\sum_{k=1}^{n-1} \xi_{n-k}^1\xi_k^1 + \xi_{n-k}^2\xi_k^2 \end{pmatrix}$$

for ξ_n when $n \geq 2$. Moreover $\xi_n^3 = 0$ for all n.

4. Formal Computation of $P(\theta,\omega)$; Branch of Parameterized Manifolds. In this section we illustrate steps 3-4 of the algorithm stated in Remark (2.1) of Section (SOMETHING). We discuss separately the case of maps and flows.

4.1. Formal Expansion of $P(\theta,\omega)$ for the Hénon Map. Consider again the Hénon Family given by Equation (3.1). At $\omega = 0$ choose p_0 one of the maps two fixed points and λ_0 and ξ_0 an eigenvalue and associated eigenvector of $Df(p_0, 0)$. Using the expansions developed in Section (SOMETHING) we have series

$$\lambda(\omega) = \sum_{m=0}^{\infty} \lambda_m \omega^m \qquad \text{and} \qquad \xi(\omega) = \sum_{m=0}^{\infty} \xi_m \omega^m,$$

where we can compute the coefficients λ_m and ξ_m to any desired order using Equations (SOMETHING) and (SOMETHING). Let $\hat{P}$ be the parameterization of the invariant manifold at $\omega = 0$ associated with λ_0 and having $DP(0) = \xi_0$. Then the coefficients $\hat{p}_n$ of $\hat{P}$ can be computed to any desired order using the homological equation (Equation SOMETHING) given in Section (SOMETHING).

By the discussion in Section (SOMETHING) we know that there is a branch of parameterizations given by

$$P(\theta,\omega) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{(mn)}\theta^n\omega^m,$$

satisfying the functional equation

$$f[P(\theta,\omega),\omega] = P[\lambda(\omega)\theta,\omega]. \tag{4.1}$$

Then $p_{m0} = p_m$, $p_{m1} = \xi_m$ and $p_{0n} = \hat{p}_n$ as discussed in Section SOMETHING. This completes Steps $1-2$ of the algorithm given in Remark (2.1). Now we turn to steps $3-4$.

Step 3: Now we compute the $m = 1$ coefficients for the case of the Hénon map. Recall that the $m = 1$ coefficients solve the homological equation given by Equation (2.15), and we want to simplify the right hand side into a computable form for the specific case of the Hénon map. Then consider that

$$\frac{\partial}{\partial\omega} s_n(\omega) - \frac{\partial}{\partial\omega}\left[Df[p(\omega),\omega] - \Lambda^n(\omega)I\right] p_n(\omega) =$$

$$\frac{\partial}{\partial\omega}\sum_{k=1}^{n-1}\begin{bmatrix} a p_{n-k}^1(\omega)p_k^1(\omega) \\ 0 \end{bmatrix} - \frac{\partial}{\partial\omega}\begin{pmatrix} -2p^1(\omega) - \lambda(\omega)^n & 1 \\ b+\omega & -\lambda(\omega)^n \end{pmatrix}\begin{bmatrix} p_n^1(\omega) \\ p_n^2(\omega) \end{bmatrix}$$

$$= \sum_{k=1}^{n-1} \left[\begin{array}{c} a\left(p_k^1(\omega)\frac{\partial}{\partial\omega}p_{n-k}^1(\omega) + p_{n-k}^1(\omega)\frac{\partial}{\partial\omega}p_k^1(\omega)\right) \\ 0 \end{array} \right]$$

$$- \left(\begin{array}{cc} -2\frac{\partial}{\partial\omega}p^1(\omega) - n\lambda(\omega)^{n-1}\frac{\partial}{\partial\omega}\lambda(\omega) & 0 \\ \omega & -n\lambda(\omega)^{n-1}\frac{\partial}{\partial\omega}\lambda(\omega) \end{array} \right) \left[\begin{array}{c} p_n^1(\omega) \\ p_n^2(\omega) \end{array} \right].$$

Evaluating at $\omega = 0$ gives

$$\frac{\partial}{\partial\omega}s_n(0) - \frac{\partial}{\partial\omega}\left[Df[p(\omega),\omega] - \Lambda^n(\omega)I\right]|_{\omega=0} \left[\begin{array}{c} p_n^1(0) \\ p_n^2(0) \end{array} \right] =$$

$$\sum_{k=1}^{n-1} \left[\begin{array}{c} a\left(p_{(1,n-k)}^1 p_{(0,k)}^1 + p_{(0,n-k)}^1 p_{(1,k)}^1\right) \\ 0 \end{array} \right]$$

$$- \left(\begin{array}{cc} -2p_{(1,0)}^1 - n\lambda_0^{n-1}\lambda_1 & 0 \\ 1 & -n\lambda_0^{n-1}\lambda_1 \end{array} \right) \left[\begin{array}{c} p_{(0,n)}^1 \\ p_{(0,n)}^2 \end{array} \right]. \tag{4.2}$$

Taking Equation (4.2) as the right hand side of Equation (2.15) gives the homological equation for the coefficients $p_{(1,n)}$ in the case of the Hénon map.

Step 4: Finally we obtain the equations for the coefficients $p_{(mn)}$ when $m + n \geq 2$. First we define the coefficients $\lambda_{(m,n)}$ be the series expansion of $\lambda(\omega)^n$. So

$$\lambda(\omega)^n = \sum_{m=0}^{\infty} \lambda_{(m,n)}\omega^m.$$

We expand the right hand side of Equation (4.1) and obtain

$$\begin{aligned} P[\lambda(\omega)\theta,\omega] &= \sum_{n=0}^{\infty} p_n(\omega)[\lambda(\omega)]^n\theta^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} p_{(m,n)}\omega^m\right)\left(\sum_{m=0}^{\infty}\lambda_{(m,n)}\omega^m\right)\theta^n \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\lambda_{(m-k,n)}\left[\begin{array}{c} p_{(k,n)}^1 \\ p_{(k,n)}^2 \end{array} \right]\omega^m\theta^n, \end{aligned} \tag{4.3}$$

Expanding the left hand side of Equation (4.1) as a power series gives

$$f[P(\theta,\omega),\omega] = \left[\begin{array}{c} 1 + P_2(\theta,\omega) - a[P_1(\theta,\omega)]^2 \\ (b+\omega)P_2(\theta,\omega) \end{array} \right]$$

which we expand componentwise to obtain

$$f[P(\theta,\omega),\omega]_1 = 1 + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{mn}^2\theta^n\omega^m$$

$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{m} a\; p^1_{(m-j,n-k)} p^1_{(j,k)} \theta^n \omega^m, \tag{4.4}$$

and

$$f[P(\theta,\omega),\omega]_2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b p^1_{mn}\omega^m\theta^n + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} p^1_{(m-1,n)}\omega^m\theta^n \tag{4.5}$$

Now we equate the power series expressions for the left and right hand sides, match like powers, and isolate the highest order terms to obtain the homological equation

$$\begin{bmatrix} -2ap^1_{(00)} - \lambda_0^n & 1 \\ b & \lambda_0^n \end{bmatrix} \begin{bmatrix} p^1_{(m,n)} \\ p^2_{(m,n)} \end{bmatrix} = \begin{bmatrix} s^1_{(m,n)} \\ s^2_{(m,n)} \end{bmatrix} \tag{4.6}$$

where

$$s^1_{(m,n)} = \sum_{j=0}^{m-1} \lambda_{(m-j,n)} p^2_{(j,n)} + \sum_{k=0}^{n}\sum_{j=0}^{m} a\; \bar{p}^1_{(m-j,n-k)} \bar{p}^2_{(j,k)}$$

and

$$s^2_{(m,n)} = -p^1_{(m-1,n)} + \sum_{j=0}^{m-1} \lambda_{(m-j,n)} p^2_{(j,n)}$$

for $n+m \geq 2$. Again note that the matrix on the left hand side of the homological equation is just the characteristic matrix of $Df(p_0,0)$, so that no new constraints are introduced for computing the branch expansions. The formal series is well defined to all orders by the non-resonance assumption on the eigenvalues at $\omega = 0$.

4.2. Formal Expansion of $P(\theta,\omega)$ for the Lorenz System. Let f be given by Equation (3.6). We consider the equilibria $p_0 = (0,0,0)$ at the origin, and fix σ, ρ, and β. Let λ_0 denote the stable eigenvalue of $Df(0,0)$ and ξ_0 be the associated eigenvector. Then the branch of one dimensional manifolds through $\hat{P}$ is given by

$$P(\theta,\omega) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{(mn)}\omega^m\theta^n.$$

In this case the computations are similar to the case of the Hénon map, except that P satisfies the functional equation

$$f[P(\theta,\omega),\omega] = \lambda(\omega)\theta\frac{\partial}{\partial\theta}P(\theta,\omega) \tag{4.7}$$

where

$$\lambda(\omega) = \sum_{m=0}^{\infty} \lambda_m \omega^m$$

parameterizes the branch of stable eigenvalues of $Df(0,\omega)$ through λ_0. Let

$$\xi(\omega) = \sum_{m=0}^{\infty} \xi_m \omega^m \qquad \text{and} \qquad \hat{P}(\theta) = \sum_{n=0}^{\infty} \hat{p}_n \theta^n,$$

be the series expansions for the stable eigenvector and the parameterization of the stable manifolds associated with λ_0. Explicit formulas for λ_m andξ_m are developed in Section (SOMETHING). The homological equation for the $\hat{p}_n$ is Equation (SOMETHING) developed in Section (SOMETHING).

The remaining computations for Steps $3-4$ are similar to the one dimensional case for maps already studied in detail in Section (SOMETHING). We simply report that when $m=1$ coefficients of P satisfy the homological equation given by

$$[Df(p_0, 0) - n\lambda_0]p_{(1,n)} = s_{(1,n)}$$

where

$$s_{(1,n)} = \left[\begin{array}{c} n\lambda_1 p^1_{(0,n)} \\ n\lambda_1 p^2_{(0,n)} - p^1_{(0,n)} + \sum_{k=1}^{n-1} \left[p^1_{(1,n-k)} p^3_{(0,k)} + p^1_{(0,n-k)} p^3_{(1,k)} \right] \\ n\lambda_1 p^3_{(0,n)} - \sum_{k=1}^{n-1} \left[p^1_{(1,n-k)} p^2_{(0,k)} + p^1_{(0,n-k)} p^2_{(1,k)} \right] \end{array} \right],$$

and for $m \geq 2$ we have the homological equation

$$\left[\begin{array}{ccc} -\sigma - n\lambda_0 & \sigma & 0 \\ \rho - p^3_{(00)} & -1 - n\lambda_0 & -p^1_{(00)} \\ p^2_{(00)} & p^1_{(00)} & -\beta - n\lambda_0 \end{array} \right] \left(\begin{array}{c} p^1_{(mn)} \\ p^2_{(mn)} \\ p^3_{(mn)} \end{array} \right) = \left(\begin{array}{c} s^1_{mn} \\ s^2_{mn} \\ s^3_{mn} \end{array} \right), \tag{4.8}$$

for $n + m \geq 2$, where

$$s^1_{mn} =$$

$$s^2_{mn} =$$

and

$$s^3_{mn} =$$

We illustrate the computation for the branch of two dimensional stable manifolds at the origin in detail. Let

$$P(\theta, \omega) = P(\theta_1, \theta_2, \omega) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m=0}^{\infty} p_{(m,n_1,n_2)} \omega^m \theta_1^{n_1} \theta_2^{n_2}$$

denote the parameterization of the one parameter branch of two dimensional stable manifolds through the origin. Then P satisfies the functional equation

$$f[P(\theta_1, \theta_2, \omega), \omega] = [D_\theta P(\theta, \omega)]\Lambda(\omega)\theta,$$

where we let $\lambda^1(\omega)$, and $\lambda^2(\omega)$ denote the parameterizations of the stable eigenvalues and define

$$\Lambda(\omega) = \begin{bmatrix} \lambda^1(\omega) & 0 \\ 0 & \lambda^2(\omega) \end{bmatrix} = \sum_{m=0}^{\infty} \begin{bmatrix} \lambda^1_m & 0 \\ 0 & \lambda^2_m \end{bmatrix} \omega^m.$$

Since the origin is a fixed point for all ω the series expansion of $p(\omega)$ is trivial to all orders. Moreover since we take $\beta > 0$, we have that $\lambda^1(\omega) = -\beta$ and $\xi^1(\omega) = (0, 0, 1)$ are a stable eigenvalue/eigenvector pair for all ω. The remaining unstable eigenvalue/eigenvector pair $\lambda^2(\omega)$ and $\xi_2(\omega)$ do depend on ω and are computed as in Section (SOMETHING). In addition, let

$$\hat{P}(\theta_1, \theta_2) = \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} \hat{p}_{(n_1,n_2)} \ \theta_1^{n_1} \theta_2^{n_2}$$

be the parameterization of the two dimensional unstable $\omega = 0$ manifold through the origin, where the coefficients $\hat{p}_{(n_1,n_2)}$ solve the homological equation given by Equation (SOMETHING). Then we have that $p_{(0,n_1,n_2)} = \hat{p}_{(n_1,n_2)}$ for all $n_1, n_2 \geq 0$, $p_{(m,0,0)} = 0$ for all $m \geq 0$, $p_{(m,1,0)} = \xi_m$ for all $m \geq 0$, $p_{0,0,1} = (0, 01)$, and $p_{m,0,2} = 0$ for all $m \geq 1$, and can consider Steps $1-2$ of the P-Algorithm complete (see Remark SOMETHING).

Step 3: For the Lorenz equations we consider the right hand side of Equation (2.20), first for ω free, and see that this simplifies to

$$\frac{\partial}{\partial \omega} s_\alpha(\omega) - \frac{\partial}{\partial \omega} \left[Df(p(\omega), \omega) - \langle \Lambda(\omega), \alpha \rangle \, I \right] p_\alpha(\omega) =$$

$$\sum_{k=0}^{n_2} \sum_{j=0}^{n_1} \begin{bmatrix} 0 \\ \bar{p}^3_{(j,k)}(\omega) \partial_\omega \bar{p}^1_{(n_1-j,n_2-k)}(\omega) + \bar{p}^1_{(n_1-j,n_2-k)}(\omega) \partial_\omega \bar{p}^3_{(j,k)}(\omega) \\ -\bar{p}^2_{(j,k)}(\omega) \partial_\omega \bar{p}^1_{(n_1-j,n_2-k)}(\omega) - \bar{p}^1_{(n_1-j,n_2-k)}(\omega) \partial_\omega \bar{p}^2_{(j,k)}(\omega) \end{bmatrix}$$

$$- \begin{pmatrix} -(n_1 \partial_\omega \lambda_1(\omega) + n_2 \partial_\omega \lambda_2(\omega)) & 0 & 0 \\ 1 & -(n_1 \partial_\omega \lambda_1(\omega) + n_2 \partial_\omega \lambda_2(\omega)) & 0 \\ 0 & 0 & -(n_1 \partial_\omega \lambda_1(\omega) + n_2 \partial_\omega \lambda_2(\omega)) \end{pmatrix} \begin{bmatrix} p^1_{(n_1,n_2)}(\omega) \\ p^2_{(n_1,n_2)}(\omega) \\ p^3_{(n_1,n_2)}(\omega) \end{bmatrix}.$$

Evaluating at $\omega = 0$ gives and setting equal to the left hand side of Equation (SOMETHING) gives the homological equation

$$[Df(0, 0) - (n_1 \lambda^1_1 + n_2 \lambda^2_1) I] p_{(1,n_1,n_2)} = s_{(1,n_1,n_2)}$$

(where of course $\lambda^2_1 = 0$) and

$$s_{(1,n_1,n_2)} = \begin{pmatrix} (n_1\lambda_1 - n_2\lambda_2) p^1_{(0,n_1,n_2)} \\ -p^1_{(0,n_1,n_2)} + (n_1\lambda_1 - n_2\lambda_2) p^2_{(0,n_1,n_2)} + \sum_{k=0}^{n_2} \sum_{j=0}^{n_1} \bar{p}^1_{(1,n_1-j,n_2-k)} \bar{p}^3_{(0,j,k)} + \bar{p}^1_{(0,n_1-j,n_2-k)} \bar{p}^3_{(1,j,k)} \\ (n_1\lambda_1 - n_2\lambda_2) p^3_{(0,n_1,n_2)} - \sum_{k=0}^{n_2} \sum_{j=0}^{n_1} \bar{p}^1_{(1,n_1-j,n_2-k)} \bar{p}^2_{(0,j,k)} + \bar{p}^1_{(0,n_1-j,n_2-k)} \bar{p}^2_{(1,j,k)} \end{pmatrix}.$$

Step 4: Since

$$\vdots$$

Then we can work out that for $n_1+n_2+m \geq 2$ the coefficients of P are solutions of the homological equation

$$\begin{bmatrix} -\sigma-(n_1\lambda_0^1+n_2\lambda_0^2) & \sigma & 0 \\ \rho-a^3_{(00)} & -1-(n_1\lambda_0^1+n_2\lambda_0^2) & -a^1_{(00)} \\ a^2_{(00)} & a^1_{(00)} & -\beta-(n_1\lambda_0^1+n_2\lambda_0^2) \end{bmatrix} \begin{pmatrix} p^1_{(m,n_1,n_2)} \\ p^2_{(m,n_1,n_2)} \\ p^3_{(m,n_1,n_2)} \end{pmatrix} \tag{4.9}$$

$$= \begin{pmatrix} s^1_{(m,n_1,n_2)} \\ s^2_{(m,n_1,n_2)} \\ s^3_{(m,n_1,n_2)} \end{pmatrix},$$

where

$$s^1_{(m,n_1,n_2)} = \sum_{k=0}^{m-1} \left[n_1\lambda^1_{m-k}+n_2\lambda^2_{m-k}\right] p^1_{(k,n_1,n_2)}$$

$$s^2_{(m,n_1,n_2)} = -p_{(m-1,n_1,m_2)} + \sum_{k=0}^{m-1} \left[n_1\lambda^1_{m-k}+n_2\lambda^2_{m-k}\right]\ p^2_{(k,n_1,n_2)}$$

$$+\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{m} \bar{p}^1_{(m-k,n_1-i,n_2-j)}\bar{p}^3_{(kij)}$$

and

$$s^3_{(m,n_1,n_2)} = \sum_{k=0}^{m-1} \left[n_1\lambda^1_{m-k}+n_2\lambda^2_{m-k}\right]\ p^3_{(k,n_1,n_2)} - \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{m} \bar{p}^1_{(m-k,n_1-i,n_2-j)}\bar{p}^2_{(kij)}.$$

4.3. A-Posteriori Validation for Formal Expansions Linear Data.

5. A-Posteriori Validation for the Formal Expansion of a Branch of Invariant Manifolds.

5.1. Validation Theorem for the Case of Diffeomorphisms.

5.2. Validation Theorem for the Case of Vector Fields.

6. Applications.

6.1. Visualization of Validated Sheafs of Invariant Manifolds.

6.2. Computer Assisted Proof of the Existence of Tangencies for Families of Diffeomorphisms.

REFERENCES

[1] L. V. Ahlfors. Complex Analysis. An Introduction to the Theory of Analytic Functions of One Complex Variable. Third Edition. International Series in Pure and Applied Mathematics. *McGraw-Hill Book Co.*, New York, 1978.

[2] G. Arioli. Periodic Orbits, Symbolic Dynamics and Topological Entropy for the Restricted 3-Body Problem. *Comm. Math. Phys.* 231 (2002), no. 1, 1-24.

[3] G. Arioli, and P. Zgliczyński. Symbolic dynamics for the Hénon-Heiles Hamiltonian on the Critical Level. J. Differential Equations 171 (2001), no.1, 173-202.

[4] G. Arioli, and P. Zgliczyński. The Hénon-Helies Hamiltonian Near the Critical Energy Level-Some Rigorous Results. *Nonlinearity* 16 (2003), no. 5, 1833-1852.

[5] I. Baldomá, E. Fontich, and R.de la Llave. The Parameterization Method for One-Dimensional Invariant Manifolds of Higher Dimensional Parabolic Fixed Ponts *Discrete and Continuous Dynamical Systems* Vol 17 (4) , pp. 835–865, (2007).

[6] J.B. van den Berg, J.P. Lessard, K. Mischaikow, and J.D. Mireles James. Rigorous numerics for symmetric connecting orbits: even homoclinics of the Gray-Scott. *To appear: SIAM J. on Math. Analysis* Preprint available at: www.math.rutgers.edu /˜jmireles / grayScottPage.html

[7] W. Beyn, and J. Kleinkauf. The Numerical Computation of Homoclinic Orbits for Maps. *SIAM J. Numer. Anal.* 34 (1997), no.3, 1207-1236.

[8] W. Beyn, and J. Kleinkauf. Numerical Approximation of Homoclinic Chaos. Dynamical Numerical Analysis (Atlantl, GA, 1995). *Numer. Algorithms* 14 (1997), no. 1-3, 25-53.

[9] X. Cabré, E. Fontich, and R. de la Llave. The Parameterization Method for Invariant Manifolds. I. Manifolds Associated to Non-resonant Subspaces. *Indiana Univ. Math. J.*, 52(2):283-328, (2003).

[10] X. Cabré, E. Fontich, and R. de la Llave. The Parameterization Method for Invariant Manifolds. II. Regularity with Respect to Parameters. *Indiana Univ. Math. J.*, 52(2):329-360, (2003).

[11] X. Cabré, E. Fontich, and R. de la Llave. The Parameterization Method for Invariant Manifolds. III. Overview and applications. *J. Differential Equations*, 218(2):444-515, (2005).

[12] R. Calleja, and R. de la Llave. Fast Numerical Computation of Quasi-Periodic Equilibrium States in 1D Statistical Mechanics, Including Twist Maps. *Nonlinearity* 22 (2009), no. 6, 1311-1336.

[13] R. Calleja, and R. de la Llave. A Numerically Accessible Criterion for the Breakdown of Quasi-Periodic Solutions and its Rigorous Justification. *Nonlinearity* 23 (2010), no. 9, 2029-2058.

[14] M. Capinski. Covering Relations and the Existence of Topologically Normally Hyperbolic Invariant Sets. *Discrete Contin. Dyn. Syst.* 23 (2009), no. 3, 705-725.

[15] M. Capinski, and P. Roldan. Existence of a Center Manifold in a Practical Domain Around L_1 in the Restricted Three Body Problem. (In Preparation: *http://arxiv.org/abs/1103.1970v1*)

[16] A. Celletti, and L. Chierchia. KAM Stability and Celestial Mechanics. *Mem. Amer. Math. Soc.* 187 (2007), no. 878, viii+134 pp.

[17] B. Coomes, H. Koçak, and K. Palmer. Homoclinic Shadowing. *J. Dynam. Differential Equations* 17 (2005), no.1, 175-215.

[18] S. Day, J.P. Lessard, and K. Mischaikow. Validated Continuation for Equilibria of PDEs. *SIAM J. Numer. Anal.* 45 (2007), no. 4, 1398-1424.

[19] S. Day, R. Frongillo, and R. Treviño. Algorithms for Rigorous Entropy Bounds and Symbolic Dynamics. *SIAM J. Appl. Dyn. Syst.* 7 (2008), no. 4, 1477-1506.

[20] E. J. Doedel, B. Krauskopf, and H. M. Osinga. Global Bifurcations of the Lorenz Manifold. *Nonlinearity* 19(12) (2006), 29472972.

[21] H. Dullin, and J. Meiss. Quadratic Volume-Preserving Maps: Invariant Circles and Bifurcations. *SIAM J. Appl. Dyn. Syst.* 8 (2009), no. 1, 76-128.

[22] E. Fontich, R. de la Llave, and Y. Sire. A Method for the Study of Whiskered Quasi-Periodic and Almost-Periodic Solutoins in Finite and Infinite Dimensional Hamiltonian Systems. *Electronic Research Announcements in Mathematical Sciences* Vol 16, pp. 9-22, (2009).

[23] J.E. Fornæss, and E.A. Gavosto, Existence of generic homoclinic tangencies for Hénon mappings, *The Journal of Geometric Analysis*, Vol 2, pp. 429–444, (1992).

[24] F. Gabern, Á. Jorba, and U. Locatelli. On the Construction of the Kolmogorov Normal Form for the Trojan Asteroids. *Nonlinearity* 18 (2005), no. 4, 1705-1734.

[25] M. Gidea, and P. Zgliczyński. Covering Relations for Multidimensional Dynamical Systems. *J. Differential Equations* 202 (2004), no.1, 59-80.

[26] M. Gidea and P. Zgliczyński. Covering Relations for Multidimensional Dynamical Systems. II.

J. Differential Equations, 202(1):59–80, 2004.

[27] C. Grebogi, S. Hammel, J.A. Yorke, and T. Sauer. Shadowing of Physical Trajectories in Chaotic Dynamics: Containment and Refinement. *Phys. Rev. Lett.* 65 (1990), no. 13, 1527-1530.

[28] A. Guillamon and G. Huguet. A Computational and Geometric Approach to Phase Resetting Curves and Surfaces. *SIAM Journal on Applied Dynamical Systems*, Vol 8, Issue 3, pp. 1005-1042 (2009).

[29] Á. Haro, and R. de la Llave. A Parameterization Method for the Computation of Invariant Tori and Their Whiskers in Quasi-Periodic Maps: Numerical Algorithms. *Discrete Contin. Dyn. Syst. Ser. B* 6 (2006), no. 6, 1261-1300.

[30] Á. Haro, and R. de la Llave. A Parameterization Method for the Computation of Invariant Tori and their Whiskers in Quasi-Periodic Maps: Rigorous Results. *J. Differential Equations* 288 (2006), no. 2, 530-579.

[31] W. M Hirsch, and C.C. Pugh. Stable manifolds and hyperbolic sets. Global Analysis (Berkeley 1960), Proc. Sympos. Pure Math Vol 14, pp 133–163, 1970.

[32] L. V. Kantorovič. Functional Analysis and Applied Mathematics. (Russian) *Vestnik Leningrad. Univ.* 3, (1948). no. 6, 3-18.

[33] L.V. Kantorovič. Functional Analysis in Normed Spaces, Moscow, 1959. Translated from the Russian by D.E. Brown. Edited by A.P. Robertson. *International Series of Monographs in Pure and Applied Mathematics*, Vol. 46. *The Macmillan Co.*, New York, 1964.

[34] J. Kennedy, S. Koçak, and J.A. Yorke. A Chaos Lemma. *Amer. Math. Monthly* 108 (2001), no. 5, 411-423.

[35] J. Kennedy, and J.A. Yorke. Topological Horseshoes. *Trans. Amer. Math. Soc.* 353 (2001), no. 6, 2513-2530.

[36] R. de la Llave, A. González, A. Jorba, and J. Villanueva. KAM theory without action-angle variables, *Nonlinearity*, 18(2) pp. 855-895, (2005).

[37] R. de la Llave. A tutorial on KAM theory, Smooth ergodic theory and its applications (Seattle, WA, 1999), *Proc. Sympos. Pure Math.*, Vol 69, pp. 175-292, 2001.

[38] R. de la Llave, J.D. Mireles James. Parameterization of Invariant Manifolds by Reducibility for Volume Preserving and Symplectic Maps, (Submitted)

[39] H. Lomelí, and J. Meiss. Quadratic Volume-Preserving Maps. *Nonlinearity* 11 (1998), no. 3, 557-574.

[40] J.D. Mireles James, and Hector Lomelí. Computation of Heteroclinic Arcs with Application to the Volume Preserving Hénon Family. *SIAM J. Applied Dynamical Systems*, Volume 9, Issue 3, pp 919-953, 2010.

[41] K. Mischaikow, and M. Mrozek. Chaos in the Lorenz Equations: A Computer Assisted Proof. II. Details. *Math. Comp.* 67 (1998), no. 223, 1023-1046.

[42] K. Mischaikow, M. Mrozek, A. Szymczak. Chaos in the Lorenz Equations: A Computer Assisted Proof. III. Classical Parameter Values. Special Issue in Celebration of Jack K. Hale's 70th Birthday, Part 3. (Atlanta, GA/Lisbon, 1998). *J. Differential Equations* 169 (2001), no. 1, 17-56.

[43] A. Neumaier, and T. Rage. Rigorous Chaos Verification in Discrete Dynamical Systems, *Physica D* 67 (1994), no. 4, pp 327-346.

[44] S. Newhouse, M. Berz, J. Grote, and K. Makino. On the Estimation of Topological Entropy on Surfaces. Geometric and probabilistic structures in dynamics, 243-270, *Contemp. Math.*, 469, *Amer. Math. Soc., Providence, RI,* 2008.

[45] J.M. Ortega. The Newton-Kantorovich Theorem. *Amer. Math. Monthly* 75 1968 658-660.

[46] K. J. Palmer. Exponential Dichotomies, the Shadowing Lemma and Transversal Homoclinic Points. Dynamics reporten, Vol. 1, 265-306, *Dynam. Report. Ser. Dynam. Systems Appl.*, 1, Wiley, Chichester, 1988.

[47] S. Rump. Verification Methods: Rigorous Results Using Floating-Point Arithmetic. *Acta Numer.* 19 (2010), 287-449.

[48] T. Sauer, and J.A. Yorke. Rigorous Verrification of Trajectories for the Computer Simulation of Dynamical Systems. *Nonlinearity* 4 (1991), no. 3, 961-979.

[49] S. Smale. Diffeomorphisms with Many Periodic Points. 1965 Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse) pp. 63-80 *Princeton Univ. Press*, Princeton, N.J.

[50] D. Stoffer, and K. Palmer. Rigorous Verification of Chaotic Behavior of Maps Using Validated Shadowing. *Nonlinearity* 12 (1999), no. 6, 1683-1698.

[51] A. Szymczak. The Conley Index for Decompositions of Isolated Invariant Sets. *Fund. Math.* 148 (1995), no. 1, 71-90.

[52] W. Tucker. A Rigorous ODE Solver and Smale's 14th Problem. *Found. Coumpt. Math.* 2

(2002), no. 1, 53-117.

[53] D. Wilczak. Abundancs of Heteroclinic and Homoclinic Orbits for the Hyperchaotic Rössler System. *Discrete Contin. Dyn. Syst. Ser. B 11* (2009), no. 4, 1039-1055.

[54] A. Wittig, M. Berz, J. Grote, K. Makino, and S. Newhouse. Rigorous and Accurate Enclosure of Invariant Manifolds on Surfaces. *Regul. Chaotic Dyn.* 15 (2010), no. 2-3, 107-126.

[55] G. Zbigniew, and P. Zgliczyński. Abundance of Homoclinic and Heteroclinic Orbits and Rigorous Bounds for the Topological Entropy for the Hénon Map. *Nonlinearity* 14 (2001), no. 5, 909-932.

[56] P. Zgliczyński. Covering Relations, Cone Conditions and the Stable Manifold Theorem. *J. Differential Equations*, 246(5):1774–1819, 2009.