# Parameterized stable/unstable manifolds for periodic solutions of implicitly defined dynamical systems

Archana Neupane Timsina * [1] and J.D. Mireles James †[2]

[1,2]Florida Atlantic University, Department of Mathematical Sciences

April 10, 2021

**Abstract**

We develop a multiple shooting parameterization method for studying stable/unstable manifolds attached to periodic orbits of systems whose dynamics is determined by an implicit rule. We represent the local invariant manifold using high order polynomials and show that the method leads to efficient numerical calculations. We implement the method for several example systems in dimension two and three. The resulting manifolds provide useful information about the orbit structure of the implicit system even in the case that the implicit relation is neither invertible nor single-valued.

**Key words.** Implicitly defined dynamical systems, computational methods, invariant manifolds, periodic orbits, parameterization method

## 1 Introduction

The qualitative theory of dynamical systems is built on invariant objects like fixed points, periodic/quasiperiodic orbits, and invariant manifolds of diffeomorphisms and flows. Generalizations of nonlinear dynamics to the setting of relations instead of functions, where neither uniqueness of forward or backward iterates is required, appeared in the early 1990's in the work of Akin [2] and McGehee [43]. The Ph.D. dissertation of Sander generalized stable/unstable manifold theory to the setting of relations [23], and work by Lerman [39] and Wather [53] studied transvers homoclinic/heteroclinic phenomena in the setting of non-invertible dynamical systems, with a view toward applications to semi-flows in infinite dimensions. Further work by Sander [51, 50, 52] studied homoclinic bifurcations for noninvertible maps and relations.

The ideas of the authors just mentioned have been applied to models coming from Population dynamics [3], Iterated difference methods/Numerical algorithms [41, 23], delay differential equations [53], adaptive control [1], discrete variational problems [24, 55], and economic theory [36, 37, 38, 44]. Indeed, this list is far from comprehensive and the interested reader will find a wealth of additional

*Email: aneupanetims2016@fau.edu

†J.D.M.J partially supported by NSF grant DMS 1813501. Email: jmirelesjames@fau.edu

references in the works just cited. We mention also the recent book on dynamical systems which are defined by an implicit rule [42], where many further examples and references are found.

A complementary approach to the study of generalized dynamics, based on functional analytic rather than topological tools, is given by the parameterization method. The idea of the parameterization method is to consider the equations describing a sufficiently recurrent orbit or orbits: for example the stable/unstable manifold attached to a fixed or periodic orbit, or a quasiperiodic family or orbts - that is an invariant torus. The equations describing special solutions may have nicer properties than the Cauchy problem describing a generic orbit. While this observation is important for classical dynamical systems defined by an invertible map, it can be even more useful when studying dynamical systems which are not invertible, are ill posed, or are not even single valued.

The parameterization method was originally developed for studying non-resonant invariant manifolds attached to fixed points of infinite dimensional maps between Banach spaces in a series of papers by Cabré, Fontich, and de la Llave [6, 7, 8], though the approach has roots going back to the Nineteenth Century (see appendix B of [8]). The method has since been extended to the study of parabolic fixed points [4], invariant tori and their stable/unstable fibers [28, 27, 29, 10, 34], for stable/unstable manifolds attached to periodic solutions of ordinary differential equations [33, 13, 47], and to develop KAM arguments without action angle variables [18, 9]. See also the recent book of Haro, Canadell, Figueras, Luque and Mondelo [26] a for much more complete overview.

More recently, the parameterization method has been extended to generalized dynamical systems like those mentioned above. We refer for example to the work of [14, 15] on stable and center manifolds for ill-posed problem, the work of [20, 56] on invariant tori for ill-posed PDEs and state dependent delay differential equations [31, 30], the work of [17, 16] on periodic orbits and their isochrones in state dependent perturbations of ODEs, and the the related work of [12] on computer assisted existence proofs of periodic orbits for the Boussinesq equation.

The work of [19], which studies stable/unstable manifolds attached to fixed points of implicitly defined discrete time dynamical systems, is especially important in the context of the present study. We develop a multiple shooting parameterization method for computing stable/unstable manifolds attached to periodic orbits of implicitly defined dynamical systems. In this sense the present work can also be seen as extending the ideas of [25] to the setting of implicit maps. We illustrate the use of the method to compute some one and two dimensional stable/unstable manifolds attached to fixed and periodic orbits in application problems involving two and three dimensional implicitly defined dynamical systems.

The remainder of the paper is organized as follows. In Section A we review some background material dealing with classical discrete time dynamical systems defined by an explicit rule. Section B deals with implicitly defined maps and develops the multiple shooting functional equations for invariant manifolds attached to periodic orbits. In Section 4 we develop the formal series solutions to the functional equations in several examples and illustrate the computation of high order approximations of the stable/unstable manifolds.

All the MatLab codes discussed in the present work are found at

`http://cosweb1.fau.edu/~jmirelesjames/parmImplicitMaps.html`

## 2 A brief overview of the parameterization methods for maps

We review some basic results about the parameterization method for stable/unstable manifolds attached to fixed points and periodic orbits of discrete time dynamical systems.

### 2.1 Stable/unstable manifolds attached to fixed points

In this section we recall some basic results from the work of [6, 7, 8].

**Lemma 2.1** (Parameterization method)**.** *Suppose that $U \subset \mathbb{R}^d$ is an open set, that $F\colon U \to \mathbb{R}^d$ is a $C^k(U)$ mapping with $k = 1, 2, 3, \ldots, \infty, \omega$, that $x_* \in U$ is a fixed point of $F$, and that $DF(x_*)$ is invertible. Take $d_s = dim(\mathbb{E}^s)$ to be the dimension of the stable eigenspace/the number of stable eigenvalues.*

*Let $\alpha, \beta > 0$ have that*

$$|\lambda| \leq \alpha < 1,$$

*for all $\lambda \in spec_s(x_*)$ and*

$$1 < \beta \leq |\lambda|,$$

*for all $\lambda \in spec_u(x_*)$. Let $L \in \mathbb{N}$ be the smallest natural number with*

$$\alpha^L < \frac{1}{\beta},$$

*and assume that*

$$L + 1 < k.$$

*Then there exists an open set $D_s \subset \mathbb{R}^{d_s}$ with $0 \in D_s$, a polynomial $K\colon D_s \to \mathbb{R}^{d_s}$ of degree not more than $L$, and a $C^k$ mapping $P\colon D_s \to \mathbb{R}^d$ so that*

$$P(0) = x_*,$$

*the columns of $DP(0)$ span $\mathbb{E}^s$ and*

$$F(P(\theta)) = P(K(\theta)), \tag{1}$$

*for all $\theta \in D_s$. Moreover, $P$ is unique up to the choice of the scalings of the columns of $DP(0)$.*

Several additional comments are in order. First, we remark that the columns of $DP(0)$ can be taken as stable eigenvectors of $DF(x_*)$, so that $DP(x_*)$ is unique up to the choice of the scalings of the eigenvectors. Once these scalings are determined, the parameterization $P$ is uniquely determined.

Note also that if $k = \infty$ or $k = \omega$ then $L + 1 < k$ is automatically satisfied. Consider the case when $k = \omega$, that is $F$ (real) analytic at $x_*$, and suppose that the scalings of $DP(0)$ are fixed. Then $P$, and hence it's power series expansion at 0, are uniquely determined. In practice then $K$ and $P$ can be worked out by power matching arguments, leading to useful numerical schemes. In fact, the scalings of the eigenvectors can be chosen so that the power series coefficients of $P$ decay at a desired exponential rate. Numerical schemes for determining the optimal scalings of eigenvectors are developed in [5].

The following is often very useful for determining the polynomial mapping $K$ in the case that some (generic) conditions hold between the stable eigenvalues.

**Lemma 2.2** (Non-resonant eigenvalues implies $K$ linear )**.** *Let $\lambda_1, \dots, \lambda_{d_s} \in \mathbb{C}$ denote the stable eigenvalues of $DF(x_*)$. Assume that for all $(n_1, \dots, n_{d_s}) \in \mathbb{N}^{d_s}$ with*

$$2 \leq n_1 + \dots + n_{d_s} \leq L,$$

*we have that*

$$\lambda_1^{n_1} \dots \lambda_{d_s}^{n_{d_s}} \neq \lambda, \tag{2}$$

*for all $\lambda \in spce_s(x_*)$. Then we can choose $K$ to be the linear mapping*

$$K(\theta) = \Lambda\theta,$$

*where $\theta = (\theta_1, \dots, \theta_{d_s}) \in \mathbb{R}^{d_s}$ and $\Lambda$ is the $n_s \times n_s$ matrix*

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n_s - 1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_{n_s} \end{pmatrix}.$$

*That is, $\Lambda$ is the matrix with the stable eigenvalues on the diagonal entries and zeros in all other entries.*

We say that the stable eigenvalues are non-resonant when the condition given by Equation (2) is satisfied. We say there is a resonance at $(n_1, \dots, n_{d_s}) \in \mathbb{N}^{d_s}$ if

$$\lambda_1^{n_1} \dots \lambda_{d_s}^{n_{d_s}} \in \text{spec}_s(x_0).$$

In this case, the $K$ is required to have a monomial term of the form $c\theta_1^{n_1} \dots \theta_{d_s}^{n_{d_s}}$ with non-zero $c \in \mathbb{R}^{d_s}$. That is, even in the non-resonant case the form of the polynomial $K$ can be determined by examining the resonances between the stable eigenvalues. Numerical procedures for determining $P$ and $K$ in the resonant case are developed in [54].

It is worth remarking that when $F$ is analytic and the stable eigenvalues are non-resonant, then Equation (1) reduces to

$$F(P(\theta)) = P(\Lambda\theta), \qquad \theta \in D_s \subset \mathbb{R}^{d_s}. \tag{3}$$

Note that $P$ is the only unknown in this equation. The equation can be viewed as requiring a conjugacy between the dynamics on the image of $P$ and the diagonal linear map given by the stable eigenvalues. Note that in case Equation (3) holds it is easy to see that $P$ parameterizes a local stable manifold. Indeed, let $\theta \in D_s$, then since $P$ is continuous (in fact $C^k$) we have that

$$\begin{aligned} \lim_{n\to\infty} F^n(P(\theta)) &= \lim_{n\to\infty} F(P(\Lambda^n\theta)) \\ &= F\left(P\left(\lim_{n\to\infty} \Lambda^n\theta\right)\right) \\ &= F(P(0)) \\ &= F(x_*) \\ &= x_*, \end{aligned}$$

so that image$(P) \subset W^s(x_*)$. Noting that image$(P)$ is a $d_s$ dimensional manifold tangent to $\mathbb{E}^s$ at $x_*$ gives equality rather than inclusion.

**Remark 2.3** (Generality)**.** Lemma 2.1 follows trivially from Theorem 1.1 of [6, 7, 8]. In the much more general work just cited $U$ is taken to be an open subset of a Banach space, and the infinite dimensional complications result in more delicate spectral assumptions. The finite dimensional setting of the present work, and the fact the we parameterize the full stable manifold simplify somewhat the statement of Lemma.

**Remark 2.4** (Unstable manifold parameterization)**.** Note that in Lemma 2.1, the assumption that $DF(x_*)$ is invertible implies that $F$ is a local diffeomorphism. Then, in a small enough neighborhood of $x_*$ there is a well defined $C^k$ inverse mapping $F^{-1}$. So, let $\Sigma$ denote the diagonal matrix of unstable eigenvalues of $DF^{-1}(x_*)$, so that $\Sigma^{-1}$ is the matrix of stable eigenvalue of $DF^{-1}(x_*)$. Assume that these stable eigenvalues are non-resonant. Then there exists an open set $D_u$ and a $C^k$ mapping $Q$ so that

$$F^{-1}(Q(\sigma)) = Q(\Sigma^{-1}\sigma), \qquad \sigma \in D_u.$$

Applying $F$ to both sides of the equation and composing with $\Sigma$ leads to the equation

$$Q(\Sigma\sigma) = F(Q(\sigma)), \qquad \sigma \in D_u.$$

In other words, the stable and unstable parameterizations satisfy exactly the same invariance equation when the eigenvalues are non-resonant. Only the conjugating matrix changes.

## 2.2 Stable/unstable manifolds attached to periodic orbits

The material in this section provides a brief review of the techniques developed in [25] for parameterization of stable/unstable manifolds attached to periodic orbits. The main idea is to exploit multiple shooting schemes to avoid function compositions.

Let $x_1, , ..., x_N \in R^d$ be the points along a hyperbolic period $N$ orbit. Let $\lambda_1, \ldots, \lambda_d$ be the multipliers of the periodic orbit, and let

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_{n_s-1} & 0 \\ 0 & 0 & \ldots & 0 & \lambda_{n_s} \end{pmatrix},$$

denote the $d_s \times d_s$ diagonal matrix of stable multipliers (similarly $\Sigma$ denote the $d_u \times d_u$ diagonal matrix of unstable multipliers). The following is paraphrased from Section 3 of [25].

Suppose that $\lambda_1, \ldots, \lambda_{d_s} \in \mathbb{C}$ be the stable multipliers associated with a non-degenerate period $N$ orbit $\{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ for the $C^k$ mapping $F$. For $1 \le j \le d_s$, let $\xi_{j,1}, \ldots, \xi_{j,d_s} \subset \mathbb{C}^d$ denote the eigenvectors of $DF(x_j)$ associated with the eigenvalue $\lambda_j$.

Assume that the stable multipliers are non-resonant, in the sense of Lemma 2.2. Then, by Lemma 2.2, there is an open set $D_s \subset \mathbb{R}^{d_s}$ and are unique $P_1, \dots P_N \colon D_s \to \mathbb{R}^d$ so that

$$\begin{aligned} P_1(0) &= x_1 \\ &\vdots \\ P_N(0) &= x_N \end{aligned}$$

and

$$\begin{aligned} DP_1(0) &= [\xi_{1,1}, \dots, \xi_{1,d_s}] \\ &\vdots \\ DP_N(0) &= [\xi_{N,1}, \dots, \xi_{N,d_s}], \end{aligned}$$

having that

$$\begin{aligned} F^N(P_1(\theta)) &= P_1(\Sigma\theta) \\ &\vdots \\ F^N(P_N(\theta)) &= P_N(\Sigma\theta) \end{aligned}$$

The difficulty with these equations is that they involve the composition mapping $F^N$, which is in general a much more complicated map than $F$. The main result of [25] (see Section 3) is that the parameterizations admit a composition free formulation.

**Lemma 2.5** (Composition free invariance equations)**.** *Under the hypotheses above (non-degenerate periodic orbit and non-resonant multipliers), the functions $P_1, \dots, P_N \colon D_s \to \mathbb{R}^d$ satisfy the system of composition free equations*

$$\begin{aligned} F(P_1(\theta)) &= P_2\left(\tilde{\Lambda}\theta\right) \\ F(P_2(\theta)) &= P_3\left(\tilde{\Lambda}\theta\right) \\ &\vdots \\ F(P_{N-1}(\theta)) &= P_N\left(\tilde{\Lambda}\theta\right) \\ F(P_N(\theta)) &= P_1\left(\tilde{\Lambda}\theta\right) \end{aligned}$$

*where*

$$\tilde{\Lambda} = \begin{pmatrix} \sqrt[N]{\lambda_1} & 0 & \dots & 0 & 0 \\ 0 & \sqrt[N]{\lambda_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sqrt[N]{\lambda_{n_s-1}} & 0 \\ 0 & 0 & \dots & 0 & \sqrt[N]{\lambda_{n_s}} \end{pmatrix},$$

*is the diagonal matrix of $N$-th roots of the multipliers. (Here it is sufficient to choose any branch of the $N$-th root).*

From the perspective of numerical calculations it is much easier to solve simultaneously the system of equations given in Lemma 2.5 than it is to apply the parameterization method directly to the composition mapping $F^N$. This is illustrated by examples in [25]. Note also that the $N$-th roots of the multipliers are the eigenvalues of the derivative of the multiple shooting map, see Equation (20). If the periodic orbit is obtained by finding $G(x_1, \ldots, x_N) = 0$ then by computing the eigenvalues of $DG(x_1, \ldots, x_N)$ we have the entries of $\tilde{\Lambda}$.

# 3 Parameterization methods for implicitly defined maps

Basic definitions for discrete dynamical systems defined by implicit maps are reviewed in Section B. The critical point is that we are interested in the dynamics of a mapping of the form

$$F(x) = y,$$

if and only if

$$T(y, x) = 0.$$

That is, we solve an implicit equation to find the image of $x$ under $F$. This view of the implicit map $F$ is now combined with the parameterization method for $F$. We review the paramterization method for fixed points of implicit maps as introduced in [19], and then extend these ideas via a multiple shooting scheme to periodic orbits of implicit systems.

## 3.1 Stable/unstable manifolds attached to fixed points

The main result of [19] can be paraphrased as follows.

**Theorem 3.1.** *Suppose that $U, V \subset \mathbb{R}^d$ are open sets and that $T \colon U \times V \to \mathbb{R}^d$ is a $C^k$ mapping with fixed point $x_* \in U \cap V$, that is*

$$T(x_*, x_*) = 0.$$

*Assume that*

- *$D_1T(x_*, x_*)$ is invertible.*
- *Let $\lambda_1, \ldots, \lambda_{d_s} \in \mathbb{C}$ denote the stable eigenvalues and $\xi_1, \ldots, \xi_{d_s} \in \mathbb{C}^d$ associated eigenvectors of $D_1T(x_*, x_*)^{-1}D_2T(x_*, x_*)$. Assume that the stable eigenvalues are distinct (otherwise choose the appropriate $\xi_j$ as generalized eigenvectors).*
- *Let*

$$\alpha = \max_{1 \le j \le d_s} |\lambda_j|,$$

$$\beta = \max_{\lambda \in \mathrm{spec}_u(x_*)} \left|\lambda^{-1}\right|,$$

*and $2 \le L$ be the smallest integer so that*

$$\alpha^L \beta < 1.$$

*Assume that $L + 1 \le k$.*

- *Assume that for all* $(n_1, \dots, n_{d_s}) \in \mathbb{N}^{d_s}$ *with* $2 \leq n_1 + \dots + n_{d_s} \leq L$ *we have that*

$$\lambda_1^{n_1} \dots \lambda_{d_s}^{n_{d_s}} \neq \lambda_j$$

  *for* $1 \leq j \leq \lambda_{d_s}$.

*Then there exists an open set* $D_s \subset \mathbb{R}^{d_s}$ *with* $0 \in D_s$*, and a* $C^k$ *mapping* $P \colon D_s \to \mathbb{R}^d$ *so that*

$$P(0) = x_*,$$

$$DP(0) = [\xi_1, \dots, \xi_{d_s}],$$

*and*

$$T(P(\Lambda\theta), P(\theta))) = 0, \qquad \theta \in D_s \tag{4}$$

*where* $\Lambda$ *is the* $d_s \times d_s$ *matrix with the stable eigenvalues on the diagonal entries and zero entries elsewhere.* $P$ *parameterizes a local stable manifold attached to the fixed point* $x_*$ *of the implicitly defined mapping* $F$*.* $P$ *is unique up to the choices of the scalings of the eigenvectors.*

The proof is a simple matter of translating the assumptions about $T$, its derivative, and its eigenvalues/eigenvectors into equivalent statements about $F$, and then applying Lemma 2.1 to the implicitly defined mapping $F$. Recalling for example that $F(x) = y$ if and only if $T(y, x) = 0$, then by letting $y = P(\Lambda\theta)$ and $x = P(\theta)$, Equation (4), is equivalent to

$$F(P(\theta)) = P(\Lambda\theta), \qquad \theta \in D_s,$$

and this is precisely Equation (3).

## 3.2 Stable/unstable manifolds attached to periodic orbits

We now introduce a multiple shooting version of the parameterization method for implicitly defined systems.

**Theorem 3.2.** *Suppose that* $U, V \subset \mathbb{R}^d$ *are open sets and that* $T \colon U \times V \to \mathbb{R}^d$ *is a* $C^k$ *mapping, and that* $x_1, \dots, x_N \in U \cap V$ *have*

$$\begin{aligned} T(x_2, x_1) &= 0 \\ &\vdots \\ T(x_{N-1}, x_N) &= 0 \\ T(x_1, x_N) &= 0 \end{aligned}$$

*Assume that:*

- *the matrices* $D_1T(x_2, x_1), \dots, D_1T(x_N, x_{N-1}), D_1T(x_1, x_N)$ *are invertible.*
- *Let* $\lambda_1, \dots, \lambda_{d_s} \in \mathbb{C}$ *denote the stable multipliers and for* $1 \leq j \leq N$ *let* $\xi_{j,1}, \dots, \xi_{j,N} \in \mathbb{C}^d$ *denote associated eigenvectors. Assume that the stable multipliers are distinct (otherwise choose the appropriate generalized eigenvectors).*

- *Let*
$$\alpha = \max_{1 \le j \le d_s} |\lambda_j|,$$
$$\beta = \max_{\lambda \in \mathrm{spec}_u(x_*)} \left|\lambda^{-1}\right|,$$
*and* $2 \le L$ *be the smallest integer so that*
$$\alpha^L \beta < 1.$$
*Assume that* $L + 1 \le k$.

- *Assume that for all* $(n_1, \ldots, n_{d_s}) \in \mathbb{N}^{d_s}$ *with* $2 \le n_1 + \ldots + n_{d_s} \le L$ *we have that*
$$\lambda_1^{n_1} \ldots \lambda_{d_s}^{n_{d_s}} \ne \lambda_j$$
*for* $1 \le j \le \lambda_{d_s}$.

*Then there exists an open set* $D_s \subset \mathbb{R}^{d_s}$ *with* $0 \in D_s$, *and* $C^k$ *mappings* $P_1, \ldots, P_N : D_s \to \mathbb{R}^d$ *so that*
$$P_1(0) = x_1, \ldots, P_N(0) = x_N$$
$$DP_1(0) = [\xi_{1,1}, \ldots, \xi_{1,d_s}], \ldots, DP_N(0) = [\xi_{N,1}, \ldots, \xi_{N,d_s}],$$
*and*
$$\begin{aligned} T(P_2(\tilde{\Lambda}\theta), P_1(\theta))) &= 0 \\ T(P_3(\tilde{\Lambda}\theta), P_2(\theta))) &= 0 \\ &\vdots \\ T(P_N(\tilde{\Lambda}\theta), P_{N-1}(\theta))) &= 0 \\ T(P_1(\tilde{\Lambda}\theta), P_N(\theta))) &= 0 \end{aligned} \tag{5}$$
*for all* $\theta \in D_s$. *Here* $\tilde{\Lambda}$ *is the* $d_s \times d_s$ *matrix with* $N$*-th roots of the stable eigenvalues on the diagonal entries and zero entries elsewhere.* $P_j$ *parameterizes a local stable manifold attached to the periodic point* $x_j$ *of the implicitly defined mapping* $F$. *The* $P_j$ *are unique up to the choices of the scalings of the eigenvectors.*

The theorem follows by applying Lemma 2.5 to the implicit map $F$ defined by $T(y, x) = 0$. We remark that the knowledge that $P$ exists (or that the $P_j$ exist), and solves the functional equations leads to efficient methods. Indeed, if $T$ is analytic then $P$ (or the $P_j$) will be analytic as well, and it makes sense to look for power series solutions of the functional equations. This topic is pursued in the next section.

# 4 Formal series solutions

We illustrate the computation of the formal series coefficients of parameterizations of several one and two dimensional stable/unstable manifolds attached to fixed and periodic orbits in some polynomial examples in two and three dimensions. It is fairly straight forward to generalize these computations to any polynomial system. Indeed, computations for non-polynomial systems are handled using automatic differentiation for power series. Non-polynomial nonlinearities are discussed in greater detail in [26]. See also [35, 19, 25].

### 4.1 Parameterized stable/unstable manifolds attached to fixed points of the implicit Hénon system

As a first example we derive a formal series solution of the invariance equation given in Equation (4) for the stable/unstable manifold attached to a fixed point if the explicit Hénon system given in Equation 27.

Let $\mathbf{x}_* \in \mathbb{R}^2$ have $T_\epsilon(\mathbf{x}_*, \mathbf{x}_*) = 0$, let $\lambda \in \mathbb{C}$ be the stable eigenvalue and $\xi \in \mathbb{C}^2$ an associated eigenvector. Indeed, note that $\lambda \in \mathbb{R}$ (as the only other eigenvalue is stable), so that we can choose $\xi \in \mathbb{R}^2$. This data is computed numerically following the discussion in Section C.1.

Motivated by Theorem 3.1 we seek $P\colon (-\tau, \tau) \to \mathbb{R}^2$ so that

$$P(0) = \mathbf{x}_*, \qquad P'(0) = \xi,$$

and

$$T_\epsilon(P(\lambda\theta), P(z)) = 0,$$

for $\theta \in (-\tau, \tau)$. Observe that since $\lambda$ is the only stable eigenvalue, the resonance conditions of Theorem 3.1 are automatically satisfied.

Since $T$ is analytic in both variables we look for analytic $P$ of the form

$$P(\theta) = \begin{pmatrix} \sum_{n=0}^\infty a_n \theta^n \\ \sum_{n=0}^\infty b_n \theta^n \end{pmatrix}.$$

Note that

$$T(P(\lambda\theta), P(\theta)) = T\left(\sum_{n=0}^\infty \lambda^n a_n \theta^n, \sum_{n=0}^\infty \lambda^n b_n \theta^n, \sum_{n=0}^\infty a_n \theta^n, \sum_{n=0}^\infty b_n \theta^n\right) = 0$$

has component equations

$$\begin{aligned} \sum_{n=0}^\infty \lambda^n a_n \theta^n - 1 + \alpha \left[\sum_{n=0}^\infty a_n \theta^n\right]^2 - \sum_{n=0}^\infty b_n \theta^n - \epsilon \left[\sum_{n=0}^\infty \lambda^n a_n \theta^n\right]^5 &= 0 \\ \sum_{n=0}^\infty \lambda^n b_n \theta^n - \beta \sum_{n=0}^\infty a_n \theta^n + \epsilon \left[\sum_{n=0}^\infty \lambda^n b_n \theta^n\right]^5 &= 0. \end{aligned} \tag{6}$$

Define the infinite sequence $\{\delta_n\}_{n=0}^\infty$ by

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases},$$

the power series coefficients of the constant function taking the value 1. Then we can rewrite Equation (6) as

$$\begin{aligned} \sum_{n=0}^\infty \left[\lambda^n a_n - \delta_n + \alpha (a * a)_n - b_n - \epsilon \lambda^n (a * a * a * a * a)_n\right] \theta^n &= 0 \\ \sum_{n=0}^\infty \left[\lambda^n b_n - \beta a_n + \epsilon \lambda^n (b * b * b * b * b)_n\right] \theta^n &= 0. \end{aligned}$$

Recalling the notation for the Cauchy "hat products" given in Section D, we observe that

$$(a * a)_n = 2a_0 a_n + (\widehat{a * a})_n,$$

and that

$$(a * a * a * a * a)_n = 5a_0^4 a_n + (a * a \widehat{* a *} a * a)_n,$$

and similarly for the coefficients involving the 5-th power of $b$. Matching like powers of $\theta$ in both sides of (6), and recalling that the first order coefficients $n = 0$ and $n = 1$ are already known, we obtain for $n \geq 2$

$$\begin{aligned} a_n\lambda^n - b_n + 2\alpha a_0 a_n + \alpha(\widehat{a * a})_n - 5\epsilon a_0^4 \lambda^n a_n - \epsilon\lambda^n (a * a \widehat{* a *} a * a)_n &= 0 \\ \lambda^n b_n - \beta a_n + 5\epsilon b_0^4 \lambda^n b_n + \epsilon\lambda^n (b * \widehat{b * b * b} * b)_n &= 0 \end{aligned} \tag{7}$$

and note that the "hat" products depend only on terms of order lower that $n$.

Isolating terms of order $n$ on the left and lower order terms on the right leads to the Homological equations

$$\begin{pmatrix} \lambda^n + 2\alpha a_0 - 5\epsilon a_0^4 \lambda^n & -1 \\ -\beta & 5\epsilon b_0^4 \lambda^n + \lambda^n \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} S_n^1 \\ S_n^2 \end{pmatrix} \tag{8}$$

for $n \geq 2$, where,

$$\begin{aligned} S_n^1 &= -\alpha(\widehat{a * a})_n + \epsilon\lambda^n (a * a \widehat{* a *} a * a)_n \\ S_n^2 &= -\epsilon\lambda^n (b * \widehat{b * b} * b)_n. \end{aligned} \tag{9}$$

This is a linear equation for $(a_n, b_n)$, where the right hand side depends only on terms of lower order. We can solve the homological equations to any desired order, provided that the matrices are invertible.

**Remark 4.1** (Non-resonances and uniqueness)**.** Again, if the fixed point is a saddle, then $\lambda^n$ is never resonant, and Equation (8) has a unique solution for all $n \geq 2$. It follows that the formal power series solution is unique up to the choice of the scaling of the eigenvector. This comment in fact holds generally. See [6].

## 4.2 Parameterized stable/unstable manifolds attached to period two points of the implicit Hénon system

Suppose now that $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$ is a period two point for the implicit Hénon system, which is computed numerically – along with its first order data – as discussed in Section C.1. Motivated by Theorem 3.2, we seek parameterizations $P, Q \colon (-\tau, \tau) \to \mathbb{R}^2$ so that

$$\begin{aligned} T_\epsilon(Q(\lambda\theta), P(\theta)) &= 0 \\ T_\epsilon(P(\lambda\theta), Q(\theta)) &= 0. \end{aligned} \tag{10}$$

Letting

$$P(\theta) = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \theta^n, \qquad Q(\theta) = \sum_{n=0}^{\infty} \begin{pmatrix} c_n \\ d_n \end{pmatrix} \theta^n,$$

Equation (10) becomes

$$
\begin{aligned}
\sum_{n=0}^{\infty} a_n \lambda^n \theta^n - \left(1 - \alpha \left[\sum_{n=0}^{\infty} c_n \theta^n\right]^2 + \sum_{n=0}^{\infty} d_n \theta^n + \epsilon \left[\sum_{n=0}^{\infty} a_n \lambda^n \theta^n\right]^5\right) &= 0 \\
\sum_{n=0}^{\infty} b_n \lambda^n \theta^n - \beta \sum_{n=0}^{\infty} c_n \theta^n + \epsilon \left[\sum_{n=0}^{\infty} b_n \lambda^n \theta^n\right]^5 &= 0 \\
\sum_{n=0}^{\infty} c_n \lambda^n \theta^n - \left(1 - \alpha \left[\sum_{n=0}^{\infty} a_n \theta^n\right]^2 + \sum_{n=0}^{\infty} b_n \theta^n + \epsilon \left[\sum_{n=0}^{\infty} c_n \lambda^n \theta^n\right]^5\right) &= 0 \\
\sum_{n=0}^{\infty} d_n \lambda^n \theta^n - \beta \sum_{n=0}^{\infty} a_n \theta^n + \epsilon \left[\sum_{n=0}^{\infty} d_n \lambda^n \theta^n\right]^5 &= 0
\end{aligned}
\quad . \tag{11}
$$

Expanding the powers as Cauchy products and extracting the terms of order $n$, we have

$$
\sum_{n=0}^{\infty} \begin{pmatrix} \lambda^n a_n - \delta_n + \alpha(c * c)_n - d_n - \epsilon \lambda^n (a * a * a * a * a)_n \\ \lambda^n b_n - \beta c_n + \epsilon \lambda^n (b * b * b * b * b)_n \\ \lambda^n c_n - \delta_n + \alpha(a * a)_n - b_n - \epsilon \lambda^n (c * c * c * c * c)_n \\ \lambda^n d_n - \beta a_n + \epsilon (d * d * d * d * d)_n \end{pmatrix} \theta^n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad . \tag{12}
$$

Extracting from the Cauchy products terms of order $n$ and matching like powers of $\theta$ leads to the equations

$$
\begin{aligned}
\lambda^n a_n + 2\alpha c_0 c_n + \alpha(\widehat{c * c})_n - d_n - \epsilon \lambda^n 5 a_0^4 a_n - \epsilon \lambda^n (a * a \widehat{* a *} a * a)_n &= 0 \\
\lambda^n b_n - \beta c_n + \epsilon \lambda^n 5 b_0^4 b_n + \epsilon \lambda^n (b * b \widehat{* b *} b * b)_n &= 0 \\
\lambda^n c_n + 2\alpha a_0 a_n + \alpha(\widehat{a * a})_n - b_n - \epsilon \lambda^n 5 c_0^4 c_n - \epsilon \lambda^n (c * c \widehat{* c *} c * c)_n &= 0 \\
\lambda^n d_n - \beta a_n + \epsilon \lambda^n 5 d_0^4 d_n + \epsilon \lambda^n (d * d \widehat{* d *} d * d)_n &= 0
\end{aligned}
$$

for $n \geq 2$. Observing that these equations are linear in $(a_n, b_n, c_n, d_n)$ we isolate the terms of order $n$ on the left and have the homological equations

$$
\begin{pmatrix} \lambda^n - 5\epsilon a_0^4 \lambda^n & 0 & 2\alpha c_0 & -1 \\ 0 & \lambda^n + 5\epsilon b_0^4 \lambda^n & -\beta & 0 \\ 2\alpha a_0 & -1 & \lambda^n - 5\epsilon c_0^4 \lambda^n & 0 \\ -\beta & 0 & 0 & \lambda^n + 5\epsilon d_0^4 \lambda^n \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} \tag{13}
$$

Where

$$
\begin{aligned}
S_1 &= -\alpha(\widehat{c * c})_n + \epsilon \lambda^n (a * a \widehat{* a *} a * a)_n \\
S_2 &= -\epsilon \lambda^n (b * b \widehat{* b *} b * b)_n \\
S_3 &= -\alpha(\widehat{a * a})_n + \epsilon \lambda^n (c * c \widehat{* c *} c * c)_n \\
S_4 &= -\epsilon \lambda^n (d * d \widehat{* d *} d * d)_n.
\end{aligned} \tag{14}
$$

Once the period two point and it's eigenvectors are known, so that we have the first and second order coefficients, we solve the homological equations order by order for $2 \leq n \leq N$ to find the coefficients of the parameterization to order $N$. Indeed, the scheme just described generalizes to manifolds attached to periodic orbits of any period in an obvious way.

## 4.3 Parameterized stable/unstable manifolds attached to fixed points of the implicit Lomelí system

Consider the implicit Lomelí system defined in Equation (31). At the parameter values studied in the present work the Lomelí map has a pair of hyperbolic fixed points. One of the fixed points has 2d unstable and 1d stable manifold, while for the other it is vice versa. For small $\epsilon \neq 0$ these features persist into the implicit system, and we will compute the formal series expansion for the parameterization of a two dimensional stable manifold of the implicit system. We focus on the case of complex conjugate eigenvalues, but the real distinct case is similar.

So, let $\mathbf{x}_* = (x_*, y_*, z_*) \in \mathbb{R}^3$ denote the fixed point, $\lambda_1, \lambda_2 \in \mathbb{C}$ the stable eigenvalues, and $\xi_1, \xi_2 \in \mathbb{C}^3$ be associated stable eigenvectors. Note that $\lambda_2 = \overline{\lambda_1}$ and we choose eigenvectors with the same symmetry. This data is computed numerically as outlined in Section C.2.

Let

$$B_r(0) = \left\{ (\theta_1, \theta_2) \in \mathbb{R}^2 \,|\, \sqrt{\theta_1^2 + \theta_2^2} < 1 \right\}.$$

Motivated again by Theorem 3.1, we seek a smooth function $\mathbf{P} : \mathcal{B}_1(0) \to \mathbb{R}^3$ solving the invariance equation

$$T_\epsilon(\mathbf{P}(\lambda_1\theta_1, \lambda_2\theta_2), \mathbf{P}(\theta_1, \theta_2) = 0, \qquad (\theta_1, \theta_2) \in B_r(0), \tag{15}$$

of the form

$$\mathbf{P}(\theta_1, \theta_2) = \begin{pmatrix} P(\theta_1, \theta_2) \\ Q(\theta_1, \theta_2) \\ R(\theta_1, \theta_2) \end{pmatrix} = \begin{pmatrix} \sum_{m=0}^\infty \sum_{n=0}^\infty u_{mn}\theta_1^m\theta_2^n \\ \sum_{m=0}^\infty \sum_{n=0}^\infty v_{mn}\theta_1^m\theta_2^n \\ \sum_{m=0}^\infty \sum_{n=0}^\infty w_{mn}\theta_1^m\theta_2^n \end{pmatrix}.$$

The first order constraints require that

$$\begin{pmatrix} u_{00} \\ v_{00} \\ w_{00} \end{pmatrix} = \begin{pmatrix} x_* \\ y_* \\ z_* \end{pmatrix}, \quad \text{and that} \quad \begin{pmatrix} u_{10} \\ v_{10} \\ w_{10} \end{pmatrix} = \xi_1 \quad \text{and} \quad \begin{pmatrix} u_{01} \\ v_{01} \\ w_{01} \end{pmatrix} = \xi_2.$$

To work out the higher order terms we plug the power series into the invariance equation and have

$$\begin{pmatrix} P(\lambda_1\theta_1, \lambda_2\theta_2) - \rho - \tau P(\theta_1, \theta_2) - R(\theta_1, \theta_2) - N(\theta_1, \theta_2) + \epsilon H_1(\lambda_1, \theta_1, \lambda_2, \theta_2) \\ Q(\lambda_1\theta_1, \lambda_2\theta_2) - P(\theta_1, \theta_2) + \epsilon\gamma H_2(\lambda_1\theta_1, \lambda_2\theta_2) \\ R(\lambda_1\theta_1, \lambda_2\theta_2) - Q(\theta_1, \theta_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

where

$$N(\theta_1, \theta_2) = aP(\theta_1, \theta_2)^2 + bP(\theta_1, \theta_2)Q(\theta_1, \theta_2) + cQ(\theta_1, \theta_2)^2,$$
$$H_1(\lambda_1\theta_1, \lambda_2\theta_2) = \alpha Q(\lambda_1\theta_1, \lambda_2\theta_2)^5 + \beta R(\lambda_1\theta_1, \lambda_2\theta_2)^5,$$

and

$$H_2(\lambda_1\theta_1, \lambda_2\theta_2) = \gamma R(\lambda_1\theta_1, \lambda_2\theta_2)^5.$$

Define $\{\delta_{mn}\}_{m+n=0}^{\infty}$ by

$$\delta_{mn} = \begin{cases} 1 & m = n = 0 \\ 0 & \text{otherwise} \end{cases},$$

the power series coefficients of the constant function taking value one. Then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left( \begin{array}{c} \lambda_1^m \lambda_2^n u_{mn} - \rho\delta_{mn} - \tau u_{mn} - w_{mn} - N_{mn} + \epsilon H_{mn}^1 \\ \lambda_1^m \lambda_2^n v_{mn} - u_{mn} + \epsilon H_{mn}^2 \\ \lambda_1^m \lambda_2^n w_{mn} - v_{mn} \end{array} \right) \theta_1^m \theta_2^n = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right),$$

where

$$\begin{aligned}
N_{mn} &= a(u*u)_{mn} + b(u*v)_{mn} + c(v*v)_{mn} \\
&= 2au_{00}u_{mn} + a(\widehat{u*u})_{mn} + bu_{00}v_{mn} + bv_{00}u_{mn} + b(\widehat{u*v})_{mn} + 2cv_{00}v_{mn} + c(\widehat{v*v})_{mn} \\
H_{mn}^1 &= \alpha\lambda_1^m\lambda_2^n(v*v*v*v*v)_{mn} + \beta\lambda_1^m\lambda_2^n(w*w*w*w*w)_{mn} \\
&= 5\alpha\lambda_1^m\lambda_2^n v_{00}^4 v_{mn} + \alpha\lambda_1^m\lambda_2^n(v*v\widehat{*v*}v*v)_{mn} \\
&\quad + 5\beta\lambda_1^m\lambda_2^n w_{00}w_{mn} + \beta\lambda_1^m\lambda_2^n(w*w\widehat{*w*}w*w)_{mn} \\
&\text{and} \\
H_{mn}^2 &= 5\gamma\lambda_1^m\lambda_2^n w_{00}w_{mn} + \gamma\lambda_1^m\lambda_2^n(w*w\widehat{*w*}w*w)_{mn}.
\end{aligned}$$

Here again we recall the definition of the Cauchy "hat products" given in Section D. We define

$$\begin{aligned}
\hat{N}_{mn} &= a(\widehat{u*u})_{mn} + b(\widehat{u*v})_{mn} + c(\widehat{v*v})_{mn} \\
\hat{H}_{mn}^1 &= \alpha\lambda_1^m\lambda_2^n(v*v\widehat{*v*}v*v)_{mn} + \beta\lambda_1^m\lambda_2^n(w*w\widehat{*w*}w*w)_{mn} \\
&\text{and} \\
\hat{H}_{mn}^2 &= \gamma\lambda_1^m\lambda_2^n(w*w\widehat{*w*}w*w)_{mn},
\end{aligned}$$

so that

$$N_{mn} = 2au_{00}u_{mn} + bu_{00}v_{mn} + bv_{00}u_{mn} + 2cv_{00}v_{mn} + \hat{N}_{mn},$$
$$H_{mn}^1 = 5\alpha\lambda_1^m\lambda_2^n v_{00}^4 v_{mn} + 5\beta\lambda_1^m\lambda_2^n w_{00}w_{mn} + \hat{H}_{mn}^1,$$

and

$$H_{mn}^2 = 5\gamma\lambda_1^m\lambda_2^n w_{00}w_{mn} + \hat{H}_{mn}^2,$$

are all terms of order $mn$ plus lower order terms. Matching like powers of $\theta$ leads to

$$\begin{aligned}
\lambda_1^m\lambda_2^n u_{mn} - \tau u_{mn} - w_{mn} - 2au_{00}u_{mn} - bu_{00}v_{mn} - bv_{00}u_{mn} - 2cv_{00}v_{mn} - \hat{N}_{mn} \\
+5\epsilon\alpha\lambda_1^m\lambda_2^n v_{00}^4 v_{mn} + 5\epsilon\beta\lambda_1^m\lambda_2^n w_{00}w_{mn} + \epsilon\hat{H}_{mn}^1 = 0 \\
\lambda_1^m\lambda_2^n v_{mn} - u_{mn} + 5\epsilon\gamma\lambda_1^m\lambda_2^n w_{00}w_{mn} + \epsilon\hat{H}_{mn}^2 = 0 \\
\lambda_1^m\lambda_2^n w_{mn} - v_{mn} = 0
\end{aligned}$$

This leads to linear homological equations for $(u_{mn}, v_{mn}, w_{mn})$ when $m + n \geq 2$ of the form

$$A_{mn} \begin{pmatrix} u_{mn} \\ v_{mn} \\ w_{mn} \end{pmatrix} = \begin{pmatrix} S^1_{mn} \\ S^2_{mn} \\ 0 \end{pmatrix}, \tag{16}$$

where

$$A_{mn} = \begin{pmatrix} \lambda_1^m \lambda_2^n - \tau - 2au_{00} - bv_{00} & -bu_{00} - 2cv_{00} + 5\epsilon\alpha v_{00}^4 \lambda_1^m \lambda_2^n & -1 + 5\epsilon\beta w_{00}^4 \lambda_1^m \lambda_2^n \\ -1 & \lambda_1^m \lambda_2^n & 5\epsilon\gamma w_{00}^4 \lambda_1^m \lambda_2^n \\ 0 & -1 & \lambda_1^m \lambda_2^n \end{pmatrix},$$

and the components of the right hand side are given by

$$\begin{aligned} S^1_{mn} &= \hat{N}_{mn} - \epsilon \hat{H}^1_{mn} \\ S^2_{mn} &= -\epsilon \hat{H}^2_{mn}. \end{aligned}$$

Note that the homological equations can be solved order by order for $2 \leq m + n \leq N$ to any desired $N$, as long as the $A_{mn}$ are invertible. Note that, just as in the examples above, the matrix is invertible as long as the non-resonance conditions are met.

### 4.4 Parameterized stable/unstable manifolds attached to period four points of the implicit Lomelí system

Once again consider the implicit Lomelí system defined in Equation (31). We consider the case of a period four orbit with stable saddle-focus stability. That is, we assume that the periodic orbit has that $\lambda_{i1} = \overline{\lambda_{i2}}$ with $|\lambda_{i1}| < 1$ for $i = 1, 2, 3, 4$. Let

$$\tilde{\lambda}_1 = (\lambda_{11})^{1/4}, \qquad \text{and} \qquad \tilde{\lambda}_2 = (\lambda_{12})^{1/4}.$$

Motivated by Theorem 3.2, we seek smooth functions $P_i : \mathcal{B}^2_1(0) \to \mathbb{R}^3$ for $1 \leq i \leq 4$ having

$$P_i(0) = \mathbf{x}^*_i,$$

$$\frac{\partial}{\partial_j} P_i(0) = \xi_{ij},$$

for $1 \leq j \leq 4$ and

$$\begin{aligned} T_\epsilon(P_1(\tilde{\lambda}_1 z_1, \tilde{\lambda}_2 z_2), P_4(z_1, z_2)) &= 0 \\ T_\epsilon(P_2(\tilde{\lambda}_1 z_1, \tilde{\lambda}_2 z_2), P_1(z_1, z_2)) &= 0 \\ T_\epsilon(P_3(\tilde{\lambda}_1 z_1, \tilde{\lambda}_2 z_2), P_2(z_1, z_2)) &= 0 \\ T_\epsilon(P_4(\tilde{\lambda}_1 z_1, \tilde{\lambda}_2 z_2), P_3(z_1, z_2)) &= 0. \end{aligned} \tag{17}$$

We write

$$P_i(z_1,z_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} p^i_{n_1,n_2} z_1^{n_1} z_2^{n_2} = \begin{pmatrix} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} u^i_{n_1,n_2} z_1^{n_1} z_2^{n_2} \\ \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} v^i_{n_1,n_2} z_1^{n_1} z_2^{n_2} \\ \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} w^i_{n_1,n_2} z_1^{n_1} z_2^{n_2} \end{pmatrix},$$

and have that

$$P_i(\lambda_1 z_1, \lambda_2 z_2) = \begin{pmatrix} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u^i_{m,n} \tilde{\lambda}_1^m \tilde{\lambda}_2^n z_1^m z_2^n \\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} v^i_{m,n} \tilde{\lambda}_1^m \tilde{\lambda}_2^n z_1^m z_2^n \\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} w^i_{m,n} \tilde{\lambda}_1^m \tilde{\lambda}_2^n z_1^m z_2^n \end{pmatrix},$$

where

$$p^i_{0,0} = \begin{pmatrix} u^{(i)}_{00} \\ v^{(i)}_{00} \\ w^{(i)}_{00} \end{pmatrix} = \mathbf{x}_i^*, \qquad p^i_{1,0} = \begin{pmatrix} u^{(i)}_{10} \\ v^{(i)}_{10} \\ w^{(i)}_{10} \end{pmatrix} = \xi_{i1}, \qquad p^i_{0,1} = \begin{pmatrix} u^{(i)}_{01} \\ v^{(i)}_{01} \\ w^{(i)}_{01} \end{pmatrix} = \xi_{i2},$$

for $i = 1, 2, 3, 4$.

Plugging the power series for $P_i(z_1, z_2)$ and $P_i(\lambda_1 z_1, \lambda_2 z_2)$ into the equation (17), expanding Cauchy products, matching like powers of $z_1, z_2$, extracting the coefficients of order $m, n$ from the Cauchy products, and isolating them from the lower order terms just as in the other formal series calculations above leads to the homological equation for $(m+n)$th term as follows

$$\mathcal{A}_{mn}\mathbf{v}_{mn} = \mathbf{S}_{mn}. \tag{18}$$

The explicit formulas for $\mathcal{A}_{mn}$ and $S_{mn}$ are recorded in Section E, as they are needed in the numerical implementation. Once again, solving these equations order by order gives the coefficients of the parameterizations to any desired accuracy. Moreover, the parameterizations of manifolds attached to longer period orbits are similar.

# 5 Numerical Results

We illustrate the utility of the explicit homological equations derived in the previous section with some example calculations.

## 5.1 Numerical example: stable/unstable manifolds attached to fixed points of the implicit Hénon system

As a first example we consider stable/unstable manifolds attached to fixed points of the implicit Hénon system defined in Equation (27). We compute a fixed point, and it's stable/unstable eigenvalues and eigenvectors as discussed in Section B.1. The results are summarized in Figure 1, and this first order data is what is needed to compute the parameterizations of the Stable/unstable manifolds order by order, solving the homological equations given in Equation (8). This allows us

| First order data: implicit Hénon | | | |
|---|---|---|---|
| parameter | fixed point | eigenvalues | eigenvectors |
| $\epsilon = 0.01$ | $p_0 \approx \begin{pmatrix} 0.6317 \\ 0.1895 \end{pmatrix}$ | $\lambda_u \approx -1.939$ <br> $\lambda_s \approx 0.1559$ | $\xi_u \approx \begin{pmatrix} -0.9882 \\ 0.1529 \end{pmatrix}$ $\xi_s \approx \begin{pmatrix} -0.4612 \\ -0.8873 \end{pmatrix}$ |
| $\epsilon = 0.03$ | $p_0 \approx \begin{pmatrix} 0.6326 \\ 0.1898 \end{pmatrix}$ | $\lambda_u \approx -1.971$ <br> $\lambda_s \approx 0.1559$ | $\xi_u \approx \begin{pmatrix} -0.9886 \\ 0.1505 \end{pmatrix}$ $\xi_s \approx \begin{pmatrix} -0.4613 \\ -0.8873 \end{pmatrix}$ |
| $\epsilon = 0.0315$ | $p_0 \approx \begin{pmatrix} 0.6326 \\ 0.1898 \end{pmatrix}$ | $\lambda_u \approx -1.973$ <br> $\lambda_s \approx 0.1559$ | $\xi_u \approx \begin{pmatrix} -0.9886 \\ 0.1503 \end{pmatrix}$ $\xi_s \approx \begin{pmatrix} -0.4613 \\ -0.8872 \end{pmatrix}$ |
| $\epsilon = 0.04$ | $p_0 \approx \begin{pmatrix} 0.6330 \\ 0.1900 \end{pmatrix}$ | $\lambda_u \approx -1.987$ <br> $\lambda_s \approx 0.1560$ | $\xi_u \approx \begin{pmatrix} -0.9888 \\ 0.1492 \end{pmatrix}$ $\xi_s \approx \begin{pmatrix} -0.4613 \\ -0.8872 \end{pmatrix}$ |

Table 1: **Fixed point/stability data:** the table reports the location and stability of one of the fixed points of the implicit Hénon system as the parameter $\epsilon$ varies. Data is given to four decimal places. More accurate values (approximately machine precision) are obtained by running the programs.

to compute the Taylor coefficients of parameterizations of the manifolds to any desired order. Some results are reported for the unstable manifold in Figure 1.

The results in the Figure illustrate the fact that, while small changes in $\epsilon$ result in small changes in the first order data, the global dynamics are greatly effected. Note also that the scaling of the eigenvector has to be decreased as $\epsilon$ increases. This reflects the fact that the domain of analyticity of the parameterization shrinks as $\epsilon$ increases. See also the remark below. We note that while the parameterized manifold is not terribly large (roughly order one) many terms are needed to conjugate the nonlinear to the linear dynamics.

The program which generates the results discussed here is

```
henonPaperEx_fixedPoint.m
```

**Remark 5.1** (Loss of the hypotheses of the implicit function theorem)**.** Following the discussion in Section C, we see that the implicit Hénon equations define a local diffeomorphism whenever

$$D_1T(x_2, y_2) = \mathrm{Id} + \epsilon \begin{pmatrix} -5\epsilon x^4 & 0 \\ 0 & 5\epsilon y^4 \end{pmatrix},$$

is invertible. For $\epsilon > 0$ the matrix is singular on the vertical line through

$$x_*(\epsilon) = \left(\frac{1}{5\epsilon}\right)^{1/4}.$$

Note that when $\epsilon = 0.01$ we have that

$$x_*(0.01) \approx 2.115,$$

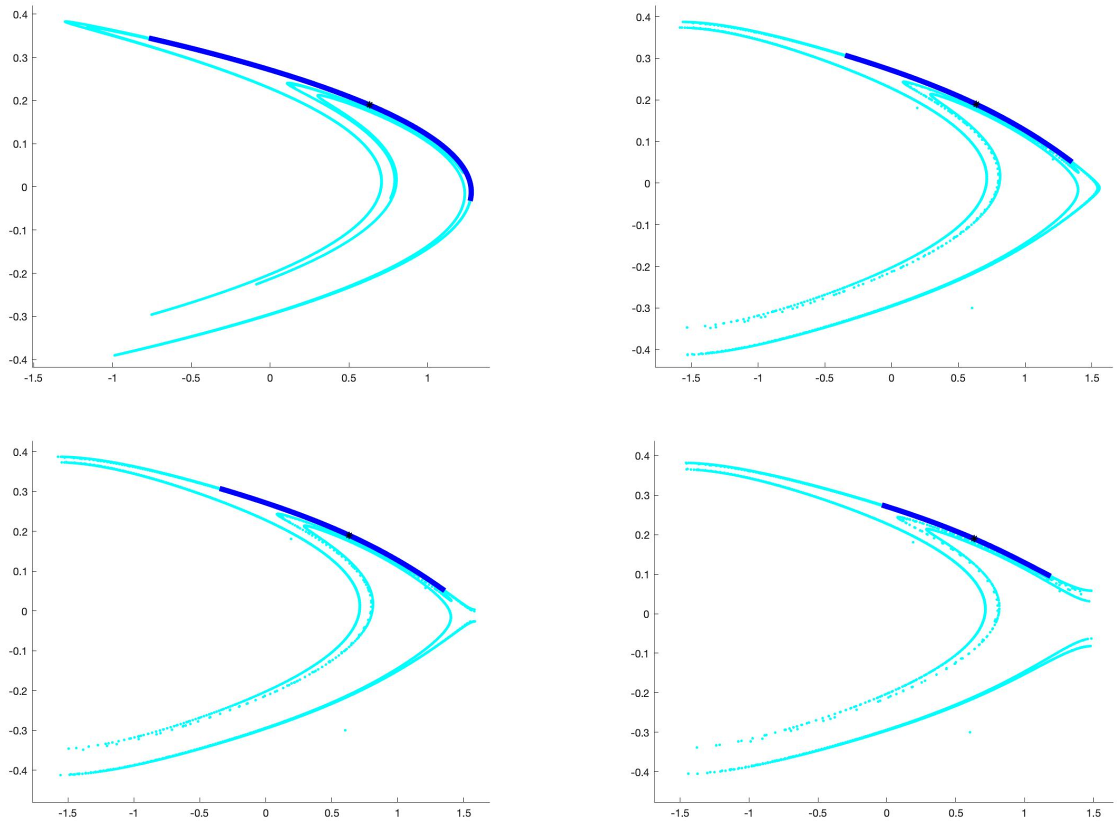


Figure 1: **Implicit Hénon –stable/unstable manifolds attached to fixed points:** four calculations of the local unstable manifold of the fixed point with data as in Table 1. The unstable manifold is colored dark blue, and eight of its forward iterates are lighter. In each case we computed $N = 75$ Taylor coefficients, with the eigenvector scalings as reported below. Top left: $\epsilon = 0.01$. The eigenvector is scaled by $\alpha = 1.0$. Top right: $\epsilon = 0.031$. The eigenvector is scaled by $\alpha = 0.85$. Bottom left: $\epsilon = 0.0315$. The eigenvector is scaled by $\alpha = 0.8$. Bottom right: $\epsilon = 0.4$. The eigenvector is scaled by $\alpha = 0.6$. These scalings insure that the highest order coefficient computed has magnitude on the order of machine epsilon.

and the singular line is far from the attractor. However as $\epsilon$ increases the singular line moves closer to the attractor, disrupting the assymptotic dynamics dramatically. In particular note that

$$x_*(0.0315) \approx 1.59,$$

and

$$x_*(0.04) \approx 1.495,$$

so that the singular line eventually moves into the attractor, creating the jumps, or breaks see in the bottom left and right frames of Figure 1.

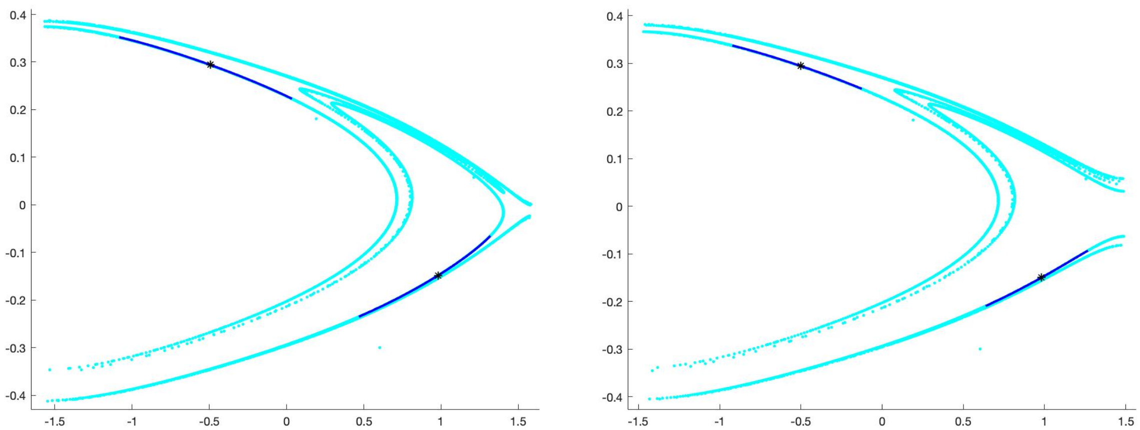


Figure 2: **Implicit Hénon –stable/unstable manifolds attached to period 2 orbits:** two calculations of the local unstable manifolds attached to a period two orbit of the implicit Hénon system. In each case we computed $N = 50$ Taylor coefficients, with eigenvector scalings as reported below. left: $\epsilon = 0.0315$. The eigenvector is scaled by $\alpha = 0.75$. right: $\epsilon = 0.04$. The eigenvector is scaled by $\alpha = 0.5$. These scalings insure that the highest order coefficient computed has magnitude on the order of machine epsilon.

## 5.2 Numerical example: stable/unstable manifolds attached to periodic orbits of the implicit Hénon system

We now illustrate the computation of the stable/unstable manifolds attached a period two point for the implicit Hénon systems. For the period two problem we consider only the two larger values of $\epsilon$. When $\epsilon = 0.0315$ there is a period two orbit located at

$$p_1 \approx \begin{pmatrix} -0.4945 \\ 0.2940 \end{pmatrix} \qquad p_2 \approx \begin{pmatrix} 0.9802 \\ -0.1483 \end{pmatrix}$$

with multipliers

$$\lambda_u \approx= -3.807, \qquad \text{and} \qquad \lambda_s \approx= -0.0279.$$

We choose the square roots

$$\tilde{\lambda}_u \approx 1.951i, \qquad \text{and} \qquad \tilde{\lambda}_s \approx 0.1670i.$$

and eigenvectors

$$\xi_1^u \approx \begin{pmatrix} 0.7868 \\ -0.0919 \end{pmatrix} \qquad \xi_2^u \approx \begin{pmatrix} -0.5982 \\ -0.1210 \end{pmatrix} \qquad \xi_1^s \approx \begin{pmatrix} 0.3958 \\ -0.5076 \end{pmatrix} \quad \text{and} \quad \xi_2^s \approx \begin{pmatrix} 0.2829 \\ 0.7110 \end{pmatrix}.$$

Similarly, when $\epsilon = 0.04$ the data is

$$p_1 \approx \begin{pmatrix} -0.4995 \\ 0.2943 \end{pmatrix} \qquad p_2 \approx \begin{pmatrix} 0.9814 \\ -0.1499 \end{pmatrix}$$

with multipliers

$$\lambda_u \approx= -4.080, \qquad \text{and} \qquad \lambda_s \approx= -0.0274.$$

We choose the square roots

$$\tilde{\lambda}_u \approx 2.020i, \qquad \text{and} \qquad \tilde{\lambda}_s \approx 0.165i.$$

and eigenvectors

$$\xi_1^u \approx \begin{pmatrix} 0.7800 \\ -0.0902 \end{pmatrix} \qquad \xi_2^u \approx \begin{pmatrix} -0.6083 \\ -0.1158 \end{pmatrix} \qquad \xi_1^s \approx \begin{pmatrix} 0.3923 \\ -0.5107 \end{pmatrix} \quad \text{and} \quad \xi_2^s \approx \begin{pmatrix} 0.2821 \\ 0.711 \end{pmatrix}.$$

The results are reported with only four significant figures. More accurate data is obtained by running the computer programs.

In both cases these are taken as initial data for computation of the stable/unstable parameterizations, whose Taylor coefficients for orders $2 \leq n \leq N$ are found by recursive solution or the homological equations defined explicitloy in Equations (13) and (14). The resulting local manifolds and a number of forward iterations are illustrated in Figure 2. See Remark 5.1 for the explication of the "tear" in the attractor.

The programs which generate the results discussed here are

```
more_iteration.m
```

and

```
henonForPaper_per2.m
```

**Remark 5.2** (Heteroclinic/homoclinic connections: infinite forward and backward time orbits)**.** Figure 3 illustrates the stable and unstable local parameterizations attached to the fixed points and the period two orbit when $\epsilon = 0.04$. At this parameter value the singular value has moved into the basin of attraction and strongly disrupts the system. Nevertheless, the intersection of unstable and stable manifolds illustrated in the figure suggest the existence of heteroclinic and homoclinic orbits existing for all forward and backward time. The figure is meant to illustrate that, even though simulating the system for long times is very difficult (as the singular set intersects the attractor) we obtain a great deal of useful information about the global dynamics by studying the parameterized manifolds.

### 5.3 Numerical example: stable/unstable manifolds attached to fixed points of the implicit Lomelí system

In this section we compute and extend the two dimensional local stable/unstable manifolds attached to fixed points of the implicit Lomelí system defined by Equation (31) with parameter values $\rho = 0.344444444$, $\tau = 1.333333333$, $a = 0.5$, $b = 0.5$, $c = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, and $\epsilon = 0.01$.

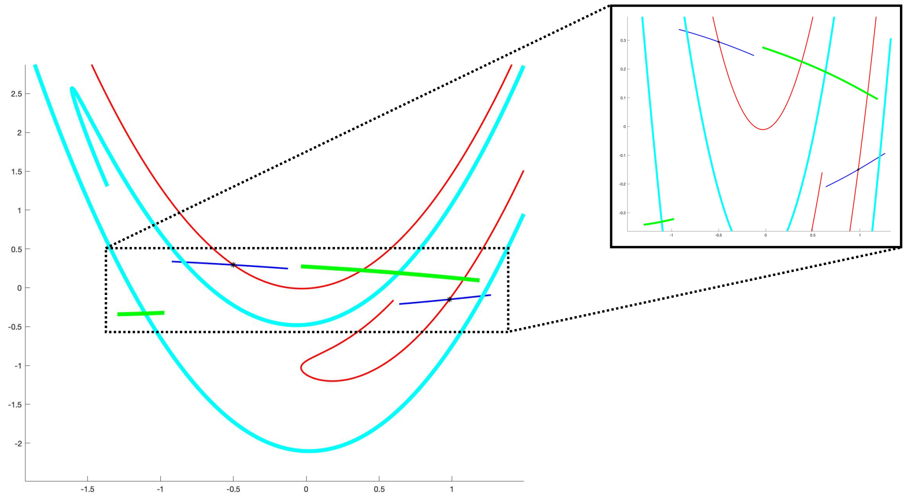


Figure 3: **Implicit Hénon – connecting orbits:** Stable and unstable manifolds when $\epsilon = 0.04$. The green curves represent the unstable manifolds of the two fixed points. The blue curves represent unstable manifolds attached to the period two orbit. Similarly, the cyan curves represent the stable manifolds of the two fixed points, and the red curves the stable manifolds of the period two orbit. We see that the blue and cyan curves intersect, and the green and the red intersect. These intersections provide compelling evidence for the existence of transverse connecting orbits from the period two orbit to the fixed point and from the fixed point to the period two. These connections also appear to be isolated away from the singular set, so that their existence implies the existence of a geometric horseshoe. It also appears that if the blue curve were extended a little it would intersect with the red. This would imply the existence of a homoclinic tangle. We did not iterate the manifold in this picture as the goal is to illustrate how much dynamics can be seen just by computing the parameterized manifolds.

We also compute the two dimensional local stable/unstable manifolds associated with a period four orbit. The results illustrated in Figures 4 and 5 are obtained by solving order by order the homological equations given in Equations (16) and (18) respectively.

The local manifolds in Figures 4 have been iterated (forward for the unstable manifolds and backwards for the stable) and seem to intersect transversally. This suggests that the heteroclinic arcs of the $\epsilon = 0$ system studied in [46] persist into the implicit system at least for small $\epsilon$. Numerical values of the fixed points, period orbits, and their first order data can be found by running the computer programs.

The program generating the results discussed here is

```
TwoD_Manifold_period4.m
```

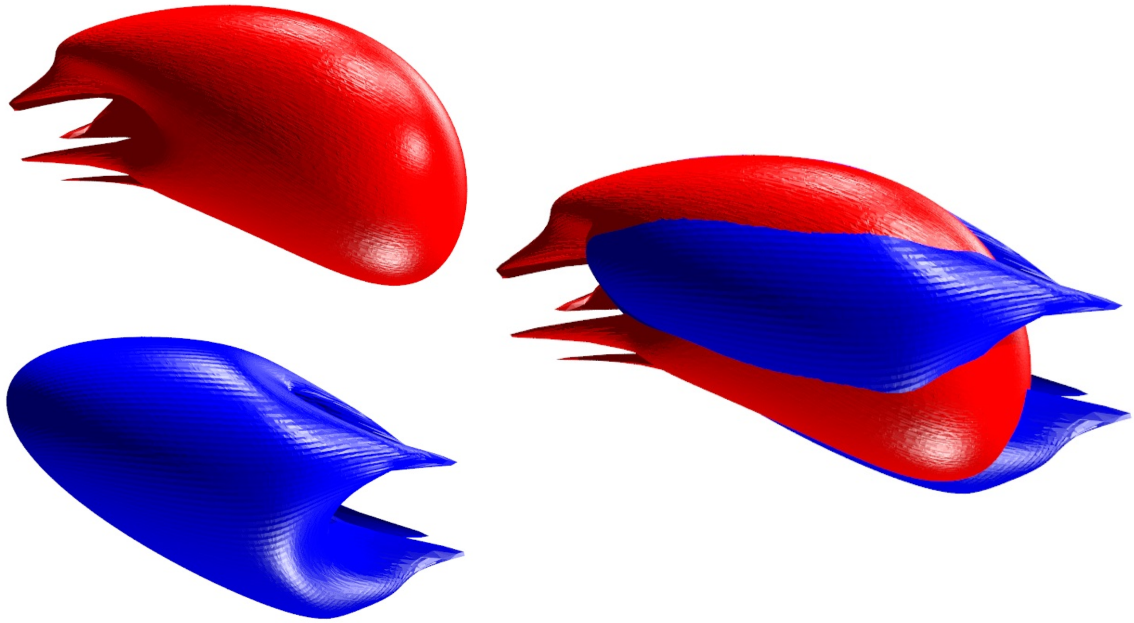

Figure 4: **Implicit Lomelí systems– stable/unstable manifolds attached to fixed points:** the local invariant manifold parameterizations and a number of forward/backward iterations. The image on the right illustrates both manifolds superimposed together, and suggests that the manifolds intersect transversally.

# A Definitions and Background

In this section we review some basic definitions from the qualitative theory of nonlinear dynamical systems. We also review the main results from [6, 7, 8] about the parameterization method for fixed points of local diffeomorphisms, and results from [25] extending these results to periodic orbits. The reader familiar with this material may want to skim or skip this section upon first reading, referring back to it only as needed.

## A.1 Discrete time semi-dynamical systems: Maps

The material in this section is standard, and an excellent reference is [49]. Suppose that $U \subset \mathbb{R}^d$ is an open set and $F\colon U \to U$ is a $C^k(U)$ mapping, with $k = 0, 1, 2, \ldots, \infty, \omega$. For $x_0 \in U$, define the sequence $x_1 = F(x_0)$, $x_2 = F(x_1)$, and in general $x_{n+1} = F(x_n)$ for $n \geq 0$. We refer to the set $\{x_n\}_{n=0}^{\infty}$ as the *forward orbit of* $x_0$ *under* $F$, and write $\text{orbit}(x_0, F)$ to denote this set. Let $F^0(x) = x$, $F^1(x) = F(x)$, $F^2(x) = F(F(x))$ and in general $F^n(x)$ denote the composition of $F$ with itself $n$ times applied to $x$. When $F$ is understood we simply write $\text{orbit}(x_0)$ and talk about the orbit of $x_0$. Then

$$\text{orbit}(x_0) = \bigcup_{n=0}^{\infty} F^n(x_n).$$

A sequence $\{x_n\}_{n=-\infty}^{0} \subset U$ with $F(x_{-1}) = x_0$ and $F(x_n) = x_{n+1}$ for all $n < 0$ is *a backward orbit of* $x_0$ under $F$. The pair $(U, F)$ is referred to as a *semi-dynamical system*, as, while forward orbits are uniquely defined, backwards orbits need not exist and when they do exist they need not be unique.

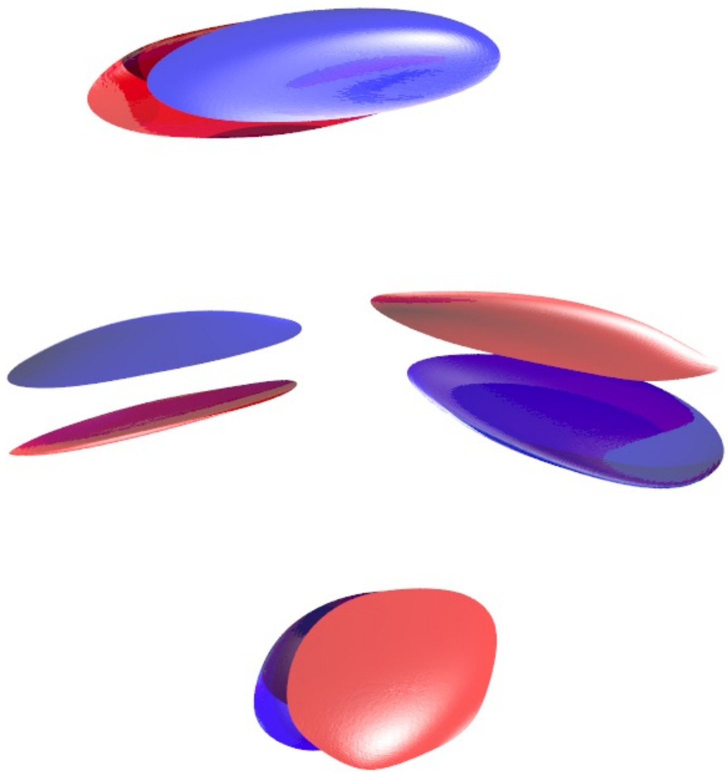

Figure 5: **Implicit Lomelí systems– stable/unstable manifolds attached to a period 4 orbit:** the local invariant manifold parameterizations.

## A.2 Local stable/unstable manifolds for fixed points/periodic orbits

Let $F \in C^k(U)$ with $k \geq 1$ and suppose that $x_* \in U$ is a fixed point, so that

$$F(x_*) = x_*.$$

We write $\mathrm{spec}(x_*) = \{\lambda_1, \ldots, \lambda_d\} \subset \mathbb{C}$ to denote the set of eigenvalues of $DF(x_*)$. Let $\xi_1, \ldots, \xi_d \in \mathbb{C}^d$ be an associated choice of (possibly generalized) eigenvectors. Let $D_1 \subset \mathbb{C}$ denote the open unit disk in the complex plane, and $S_1$ denote the unit circle. Define

$$\begin{aligned}
\mathrm{spec}_s(x_*) &= \mathrm{spec}(x_*) \cap D_1 \\
\mathrm{spec}_c(x_*) &= \mathrm{spec}(x_*) \cap S_1 \\
\mathrm{spec}_u(x_*) &= \mathrm{spec}(x_*) \backslash \left(\mathrm{spec}_s(x_*) \cup \mathrm{spec}_c(x_*)\right),
\end{aligned}$$

and note that $\mathrm{spec}_s(x_*)$ is the set of eigenvalues with complex absolute vale less than one, $\mathrm{spec}_c(x_*)$ is the set of eigenvalues with complex absolute value equal to on, and $\mathrm{spec}_u(x_*)$ is the set of eigenvalues with complex absolute vale greater than one. There are referred to as the stable, center, and unstable eigenvalues respectively, and we note that any of two of these sets could be empty. If $\mathrm{spec}_c(x_*) = \emptyset$ then we say that $x_*$ is a hyperbolic fixed point.

Define the vector spaces

$$\mathbb{E}^s = \{\xi_j \mid \lambda_j \in \mathrm{spec}_s(x_*)\}$$
$$\mathbb{E}^c = \{\xi_j \mid \lambda_j \in \mathrm{spec}_c(x_*)\}$$
$$\mathbb{E}^u = \{\xi_j \mid \lambda_j \in \mathrm{spec}_u(x_*)\}.$$

These are referred to as the stable, center, and unstable eigenspaces of $M$ respectively, they are invariant linear subspaces for the dynamics induced by $M$, and they have corresponding invariant nonlinear manifolds for the dynamics induced by $F$. Let

$$d_s = \dim(\mathbb{E}^s)$$
$$d_c = \dim(\mathbb{E}^c)$$
$$d_u = \dim(\mathbb{E}^u),$$

denote the dimension of the stable/center/unstable eigenspaces, or equivalently the number (counted with multiplicity) of stable/center/unstable eigenvalues.

Define the sets

$$W^s(x_*) = \left\{ x \in U \,:\, \lim_{n\to\infty} F^n(x) = x_* \right\}$$
$$W^u(x_*) = \left\{ x \in U \,:\, \text{there exists a backward orbit } \{x_n\} \text{ of } x \text{ with } \lim_{n\to-\infty} x_n = x_* \right\}.$$

These are referred to as the stable and unstable sets for $x_*$ respectively. In a similar fashion, for any open set $V \subset U$ with $x_* \in V$, define

$$W^s_{\mathrm{loc}}(x_*, V) = \{x \in V : F^n(x) \in V \text{ for all } n \geq 0, \text{ and } F^n(x) \to x_* \text{ as } n \to \infty\}$$
$$W^u_{\mathrm{loc}}(x_*, V) = \left\{ x \in V : \text{there is a backward orbit for } x \text{ in } V \text{ with } \lim_{n\to-\infty} x_n \to x_* \text{ as } n \to -\infty \right\},$$

and note that for any $V \subset U$ we have that $W^s_{\mathrm{loc}}(x_*, V) \subset W^s(x_*)$, and $W^u_{\mathrm{loc}}(x_*, V) \subset W^u(x_*)$.

The following stable manifold theorem says that if $x_*$ is hyperbolic then there exist local stable/unstable sets with especially nice properties.

**Theorem A.1** (Local stable manifold theorem)**.** *Suppose that $x_*$ is a hyperbolic fixed point for $F$. Then there exists an open set $V \subset U$ with $x_* \in V$ so that $W^s_{loc}(x_*, V)$ and $W^u_{loc}(x_*, V)$ are respectively $d_s$ and $d_u$ dimensional embedded disks – as smooth as $F$– and tangent at $x*$ to $\mathbb{E}^s$ and $\mathbb{E}^u$ respectively.*

The theorem gives that the stable/unstable sets are locally smooth manifolds. If $F$ is a diffeomorphism then the full stable/unstable sets are obtained by iterating $F$ and $F^{-1}$, hence the stable/unstable sets are smooth manifolds (which can nevertheless be embedded in $U$ in very complicated ways). However, if $F$ is not invertible the global stable/unstable sets might misbehave in a number of ways.

- **Connectedness:** While the unstable set must be connected ( image of a disk is connected under iteration of a continuous map) the stable set can in general be disconnected. The unstable set can have self intersections.
- **Dimension:** both the stable/unstable sets can increase in dimension outside a neighborhood of $x_*$.
- **Smoothness:** the stable/unstable sets need not be smooth manifolds away from $x_*$. At points where $DF(x)$ has an isolated non-singularity the set can develop corners or cusps.

Examples of each of these phenomena are discussed in [52], and many explicit examples are given. See also [23].

## A.3 Multiple shooting for periodic orbits

With $U \subset \mathbb{R}^d$ an open set, and $F\colon U \to \mathbb{R}^d$ a smooth mapp, suppose that $x_1, \ldots, x_N \in U$ have

$$F(x_1) = x_2$$
$$\vdots$$
$$F(x_{N-1}) = x_N$$
$$F(x_N) = x_1$$

Then $\{x_1, \ldots, x_N\}$ is a periodic orbit orbit for $F$. If the $x_j$, $1 \leq j \leq N$ are distinct, then $N$ is the lest period. We refer to $x_j$, $1 \leq j \leq N$ as a period $N$ point. If $DF(x_j)$ is invertible for each $1 \leq j \leq N$ we say that the periodic orbit is non-degenerate.

Note that $\bar{x} \in U$ is a period $N$ point for $F$ if and only if $\bar{x}$ is a fixed point of the composition $F^N$. If the orbit of $\bar{x}$ is non-degenerate and least period $N$ then $DF^N(\bar{x})$ is invertible. We note that if $\{x_1, \ldots, x_N\}$ is a non-degenerate periodic orbit then the matrices $DF^N(x_j)$, $1 \leq j \leq N$ have the same eigenvalues. These are also referred to as the multipliers of the periodic orbit.

If $DF^N(\bar{x})$ has no eigenvalues on the unit circle we say that the periodic orbit is hyperbolic and Theorem A.1 applies to the composition mapping $F^N$. In particular, there are local stable and unstable manifolds attached to the points of the periodic orbit.

Let $U^N = U \times \ldots \times U \subset \mathbb{R}^{Nd}$ denote the product of $N$ copies of $U$. Define $G \colon U^N \to \mathbb{R}^{Nd}$ by

$$G(x_1, x_2, \ldots, x_{N-1}, x_N) = \begin{pmatrix} F(x_N) \\ F(x_1) \\ F(x_2) \\ \vdots \\ F(x_{N-1}) \end{pmatrix}. \tag{19}$$

and observe that if $(x_1, \ldots, x_N) \in \mathbb{R}^{Nd}$ is a fixed point of $G$ then $\{x_1, \ldots, x_N\}$ is a period $N$ orbit for $F$. We refer to $G$ as a multiple shooting map for a period $N$ orbit of $F$. In practice numerically computing fixed points of $G$ is more stable than computing fixed points of $F^N$.

Note also that if

$$DG(x_1, \ldots, x_N) = \begin{pmatrix} 0 & 0 & \ldots & 0 & DF(x_N) \\ DF(x_1) & 0 & \ldots & 0 & 0 \\ 0 & DF(x_2) & \ldots & 0 & 0 \\ 0 & 0 & \ldots & DF(x_{N-1}) & 0 \end{pmatrix} \tag{20}$$

is invertible then the periodic orbit is non-degenerate. In fact, $\lambda \in \mathbb{C}$ is an eigenvalue of $DG(x_1, \ldots, x_N)$ if and only if $\lambda^N$ is an eigenvalue of $DF^N(x_j)$. Moreover, one can check that if $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^{dN}$ is an eigenvector associated with the eigenvalue $\lambda$ of the matrix $DG(x_1, \ldots, x_N)$, then for $1 \leq j \leq N$ we have that $(\lambda^N, \xi_j)$ is an eigenvalue/eigenvector pair for the matrix $DF^N(x_j)$. That is, the multipliers of the periodic orbit and the eigenspaces of $DF^N(x_j)$ are easily recovered from the eigenvalues/eigenvectors of $DG(x_1, \ldots, x_N)$. The interested reader will find a more thorough discussion of the relationship between multiple shooting maps and periodic orbits in [25].

# B Implicitly defined dynamical systems

We now come to the main complication or the present work, which is that the formula for $F$ may not be explicitly given. Rather it may be only implicitly defined via some relation. In the present work we focus on the case that $F(x)$ is the unique solution of some system of equations where we think of $x$ as a parameter. To formalize the discussion let us introduce some notation.

Let $U, V \subset \mathbb{R}^d$ be open sets and suppose that $T \colon U \times V \to \mathbb{R}^d$ is a smooth function. We are interested in the existence of open sets $D \subset U$, $R \subset V$ and a mapping $F \colon D \to R \subset \mathbb{R}^d$ defined by the rule

$$F(x) = y, \tag{21}$$

if and only if for a fixed given input $x \in D$, $y$ is the unique solution of the equation

$$T(y, x) = 0, \tag{22}$$

with $y \in R$. We say that the mapping $F$ is *implicitly defined* by the rule given in Equation (22). Note that $F$ need not be one-to-one or even single valued globally.

Suppose that $x_0 \in D_0 \neq \emptyset$ and let $F(x_0) = x_1 \in R_0 \neq \emptyset$. If $x_1 \in U \cap V$, then it is possible to repeat process and look for the image of $x_1$ under $F$. Should this exists we have $F(x_1) = x_2 \in V$. If $x_2 \in U \cap V$ we can repeat yet again. It is of course an interesting question to ask: "does there exists $x_0 \in U \cap V$ so that $x_0$ has an orbit under $F$? The question is a fundamental to the present work, and will be taken up in a moment.

An more elementary preliminary observation is as follows: if $T(x_1, x_0) = 0$, that is if $F(x_0) = x_1$, then a local condition guaranteeing that $F$ is well defined in a neighborhood of $x_0$ is as follows. Let $D_1 T(y, x)$ and $D_2 T(y, x)$ denote the partial derivatives of $T$ with respect to the first and second variables respectively. (Note that $D_1 T(y, x), D_2 T(y, x)$ are $d \times d$ matrices). If $D_1 T(x_1, x_0)$ is invertible then, by the implicit function theorem [48], there exists an $r > 0$ and a function $F \colon B_r(x_0) \subset U \to \mathbb{R}^d$ so that $F(x_0) = x_1$ and

$$T(F(x), x) = 0, \tag{23}$$

for all $x \in B_r(x_0)$. Moreover, the mapping $F$ is as smooth as $T$ and by differentiating (23) we have that $DF(x_0)$ solves the equation

$$D_1 T(x_1, x_0) DF(x_0) = -D_2 T(x_1, x_2), \tag{24}$$

with $D_1 T(x_1, x_0)$ invertible.

The discussion above motivates the following definition.

**Definition 1.** We say that $\bar{x} \in U$ is a *regular point for* $F$ if there exists a unique $\bar{y} \in V$ such that

$$T(\bar{y}, \bar{x}) = 0,$$

and $D_1 T(\bar{y}, \bar{x})$ is invertible. Note that, by the implicit function theorem as above, $\bar{x}$ is in the interior of $D = \mathrm{dom}(F)$. Moreover, $F$ is a local diffeomorphism of a neighborhood of $\bar{x}$ into a neighborhood of $\bar{y}$.

**Remark B.1** (Numerical evaluation of $F$)**.** Evaluation of $F(x)$ requires solving the nonlinear equation $T(y, x) = 0$ with $x$ given. In practice we use Newton's method as follows. Let $\bar{x}$ be fixed and $y_0$ be an approximate solution in the sense that

$$\|T(y_0, \bar{x})\| \approx 0.$$

For $n \geq 0$, define

$$y_{n+1} = y_n + \Delta_n,$$

where $\Delta_n$ solves the linear equation

$$D_1 T(y_n, \bar{x}) \Delta_n = -T(y_n, \bar{x}).$$

If $\bar{x}$ is a regular point for $F$ and $F(\bar{x}) = \bar{y}$, then $D_1 T(\bar{y}, \bar{x})$ is invertible. Moreover, since invertibility is an open condition, for $y_n$ close enough to $\bar{y}$ the matrices $D_1 T(y_n, \bar{x})$ are invertible as well– and hence the Newton sequence $\{y_n\}_{n=0}^{\infty}$ is well defined. Moreover, by the classic convergence analysis of Newton's method, $y_n \to \bar{y}$ as $n \to \infty$.

## B.1 Fixed and periodic points

Assume that $x_* \in U \cap V \subset \mathbb{R}^d$ is a regular point for $F$ having

$$T(x_*, x_*) = 0.$$

Then there exists an open neighborhood $D \subset U$ of $x_*$ so that $F$ is a local diffeomorphism on $D$ and

$$F(x_*) = x_*.$$

This is the simplest possible case in which the full forward orbit of $x_*$ is defined. Note that we do not rule out the possibilities that either $T(x_*, x) = 0$ or $T(y, x_*) = 0$ have other solutions – that is, $F$ need not be globally one-to-one or even single valued on $U$.

Exploiting the formula for the derivative in Equation (24), we have that

$$DF(x_*) = -D_1 T(x_*, x_*)^{-1} D_2 T(x_*, x_*),$$

where $D_1 T(x_*, x_*)$ is invertible thanks to the assumption that $x_*$ is a regular point for $F$. Assume that $x_*$ is a hyperbolic fixed point, and let $\lambda_1, \ldots, \lambda_{d_s} \in \mathbb{C}$ be the stable eigenvalues of $DF(x_*)$. Let $\xi_1, \ldots, \xi_{d_s} \in \mathbb{C}^d$ be associated stable eigenvectors.

In a similar fashion, suppose instead that $x_1, \ldots, x_N \in U \cap V$ have

$$\begin{aligned} T(x_2, x_1) &= 0 \\ T(x_3, x_2) &= 0 \\ &\vdots \\ T(x_N, x_{N-1}) &= 0 \\ T(x_1, x_N) &= 0 \end{aligned}$$

Then $x_1, \ldots, x_N$ is a periodic orbit for $F$. If the $x_1, \ldots, x_N$ are distinct then $N$ is the least period. We assume that the orbit is non-degenerate, so that the entire periodic orbit is interior to $D = \mathrm{dom}(F)$.

To find the multipliers and eigenvectors, proceed as follows. Recall from Section A.2 that the multipliers are found by computing the eigenvalues and eigenvectors of the derivative of the multiple shooting map. The formula for the derivative is in Equation (20), and exploiting again the formula for the derivative of $F$ given in Equation (24), and the fact that the periodic orbit is non-degenerate, the non-zero entries of $DG(x_1, \ldots, x_N)$ are

$$\begin{aligned} DF(x_1) &= -D_1T(x_2, x_1)^{-1} D_2T(x_2, x_1) \\ DF(x_2) &= -D_1T(x_3, x_2)^{-1} D_2T(x_3, x_2) \\ &\vdots \\ DF(x_{N-1}) &= -D_1T(x_N, x_{N-1})^{-1} D_2T(x_N, x_{N-1}) \\ DF(x_N) &= -D_1T(x_1, x_N)^{-1} D_2T(x_1, x_N). \end{aligned}$$

# C A class of examples: perturbations of explicitly defined maps

We now specify the class of examples with which we work in the remainder of the paper. Suppose that $V \subset \mathbb{R}^d$ is open and let $f \colon U \to \mathbb{R}^d$ be a $C^\omega$ diffeomorphism (or analticomorphism). Then for any and $x_0 \in U$, $f$ defines a dynamical system by the rule

$$f(x_n) = x_{n+1},$$

for $n = 0, 1, 2, \ldots$. Define the function $T \colon \mathbb{R}^d \times U \to \mathbb{R}^d$ by

$$T(y, x) = y - f(x),$$

and note that for a given $\bar{x} \in U$, $\bar{y}$ is the unique solution of the equation $T(y, \bar{x}) = 0$ if and only if

$$\bar{y} = f(\bar{x}).$$

In this case the problem $T(y, x) = 0$ implicitly defines the original dynamical system $f(x) = y$. Now let $V$ be an open subset of $\mathbb{R}^d$ and $H \colon U \times V \to \mathbb{R}^d$ be a $C^k$ function. Consider the one parameter family of problems $T_\epsilon \colon U \times V \times \mathbb{R} \to \mathbb{R}^d$ by

$$T_\epsilon(y, x) = y - f(x) + \epsilon H(x, y), \tag{25}$$

and note that for any $(\bar{y}, \bar{x}) \in V \times U$ we have that

$$D_1 T_\epsilon(\bar{y}, \bar{x}) = \mathrm{Id} + \epsilon D_1 H(\bar{x}, \bar{y}).$$

Note that $D_1 T_0(\bar{y}, \bar{x}) = \mathrm{Id}$, so that –by the implicit function theorem – there is a $\delta > 0$ and a smooth curve $y \colon (-\delta, \delta) \to V \subset \mathbb{R}^d$ so that $y(0) = \bar{y}$ and

$$T_\epsilon(y(\epsilon), \bar{x}) = 0,$$

for all $\epsilon \in (-\delta, \delta)$.

Moreover, for a possibly smaller $\delta > 0$ we have that

$$D_1 T_\epsilon(y(\epsilon), \bar{x}) = \mathrm{Id} + \epsilon DH(y(\epsilon), \bar{x}),$$

is invertible for each $\epsilon \in (-\delta, \delta)$, by the Neumann theorem. Then there exists an $r > 0$ and a family of functions $F_\epsilon \colon B_r(\bar{x}) \times (-\delta, \delta) \to \mathbb{R}^d$ so that

$$F_0(x) = f(x),$$

and

$$T_\epsilon(F_\epsilon(x), x) = 0,$$

for all $x \in B_r(\bar{x})$ and $\epsilon \in (-\delta, \delta)$. The family $F_\epsilon$ depends smoothly on $\epsilon$ and is $C^k$ for each fixed $\epsilon$. Moreover, for small $\epsilon \neq 0$ and $\bar{x} \in U$ we take $\bar{y} = f(\bar{x})$ as an approximate zero for $T_\epsilon(y, \bar{x})$ and apply Newton's method to find $y(\epsilon)$ so that $T_\epsilon(y(\epsilon), \bar{x}) = 0$. That is, for small $\epsilon$ we can compute images of the implicitly defined mapping $F_\epsilon(x)$ using Newton's method. For larger $\epsilon$ we perform numerical continuation from the $\epsilon = 0$ case.

Finally, suppose that $\bar{x} \in U$ is a hyperbolic fixed point of $f$, and recall that $F_0(x) = f(x)$. Since $F_\epsilon$ depends smoothly on $\epsilon$, it follows that for small $\epsilon \neq 0$ the map $F_\epsilon(x)$ has a hyperbolic fixed point near $\bar{x}$ by the usual perturbation argument for maps.

The discussion just given shows that problems of the form given in Equation (25) provide a natural class of examples - perturbations of diffeomorphisms - for which our method applies. The two main specific examples studied below are when $f$ is either the classic Hénon map or it's three dimensional generalization to the Lomelí map. The main purpose of the calculations below is to illustrate the formal series solution of the parameterization method invariance equations in explicit examples.

## C.1 Example 1: the Hénon map

Let $\mathbf{x} = (x_1, y_1), \mathbf{y} = (x_2, y_2)$ denote points in the plane. The Hénon map is actually a two parameter family of quadratic mappings defined by

$$f(\mathbf{x}) = f(x_1, y_1) = \left( \begin{array}{c} 1 + y_1 - \alpha x_1^2 \\ \beta x_1 \end{array} \right) \tag{26}$$

The mapping is a classic example of complex dynamics and was originally introduced in [32]. See also the books of [21, 49]. We define an implicit Hénon system $T_\epsilon : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ given by

$$T_\epsilon(\mathbf{y}, \mathbf{x}) = T_\epsilon(x_2, y_2, x_1, y_1) = \left( \begin{array}{c} x_2 - (1 - \alpha x_1^2 + y_1 + \epsilon x_2^5) \\ y_2 - \beta x_1 + \epsilon y_2^5 \end{array} \right). \tag{27}$$

Here we choose, somewhat arbitrarily, the perturbation term of the form

$$H(x, y) = \left( \begin{array}{c} x^5 \\ y^5 \end{array} \right).$$

The equation for a fixed point is $T_\epsilon(\mathbf{x}, \mathbf{x}) = 0$, or

$$\left( \begin{array}{c} x_1 - (1 - \alpha x_1^2 + y_1 + \epsilon x_1^5) \\ y_1 - \beta x_1 + \epsilon y_1^5 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right).$$

Similarly, the multiple shooting equations for a period two orbit are

$$\begin{aligned} T_\epsilon(\mathbf{x}_2, \mathbf{x}_1) &= 0 \\ T_\epsilon(\mathbf{x}_1, \mathbf{x}_2) &= 0, \end{aligned} \tag{28}$$

or

$$\begin{aligned} x_2 - (1 - \alpha x_1^2 + y_1 + \epsilon x_2^5) &= 0 \\ y_2 - \beta x_1 + \epsilon y_2^5 &= 0 \\ x_1 - (1 - \alpha x_2^2 + y_2 + \epsilon x_1^5) &= 0 \\ y_1 - \beta x_2 + \epsilon y_1^5 &= 0 \end{aligned} \tag{29}$$

The implicit equations for fixed points or periodic orbits are solved using Newton's method. Eigenvalues and eigenvectors we compute using the approach outlined in Section B.1.

For classical parameters $a, b \in \mathbb{R}$ the Hénon map has a pair of hyperbolic fixed points, each with one stable and one unstable eigenvalue. Then for small epsilon the same is true for the perturbation. The numerical value of the unperturbed fixed points serve as initial guesses for the perturbed fixed points in the Newton method. Similar comments hold for periodic orbits.

## C.2 Example 2: the Lomelí map

In this section we consider the five parameter family of maps $f : R^3 \to R^3$ given by

$$f(x, y, z) = \left( \begin{array}{c} z + Q(x, y) \\ x \\ y \end{array} \right) \tag{30}$$

where $Q(x, y) = \rho + \gamma x + a x^2 + b xy + c y^2$ and one usually takes $a + b + c = 1$. The system is known as *the Lomelí map*, and it is a normal form quadratic volume preserving maps with quadratic inverse. In that sense it can be thought of as a three dimensional generalization of the area preserving Hénon map. The map was first introduced in [40], and was subsequently studied by a number of authors including [40, 22, 45, 46, 11, 25].

Let $\mathbf{x} = (x_1, y_1, z_1), \mathbf{y} = (x_2, y_2, z_2) \in \mathbf{R}^3$. In the present work we study an implicit Lomelí system $T_\epsilon : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{aligned} T_\epsilon(\mathbf{y}, \mathbf{x}) &= T_\epsilon(x_2, y_2, z_2, x_1, y_1, z_1) \\ &= \begin{pmatrix} x_2 - \rho - \tau x_1 - z_1 - a x_1^2 - b x_1 y_1 - c y_2^2 + \epsilon \left( \alpha y_2^5 + \beta z_2^5 \right) \\ y_2 - x_1 + \epsilon \gamma z_2^5 \\ z_2 - y_1 \end{pmatrix}. \end{aligned} \tag{31}$$

Note that $T_\epsilon$ is analytic in all variables. We remark that the perturbation is chosen so that the system still preserves volume.

Fixed of points of the implicit Lomelí system (31) are obtained as solutions of

$$\begin{pmatrix} x - \rho - \tau x - z - a x^2 - b x y - c y^2 + \boldsymbol{\epsilon}(\alpha y^5 + \beta z^5) \\ y - x + \boldsymbol{\epsilon} \gamma z^5 \\ z - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{32}$$

Similarly, a period four orbit for the Lomelí system solves the equations

$$\begin{aligned} T_\epsilon(\mathbf{x}_4, \mathbf{x}_1) &= 0 \\ T_\epsilon(\mathbf{x}_1, \mathbf{x}_2) &= 0 \\ T_\epsilon(\mathbf{x}_2, \mathbf{x}_3) &= 0 \\ T_\epsilon(\mathbf{x}_3, \mathbf{x}_4) &= 0. \end{aligned} \tag{33}$$

That is

$$\begin{aligned} x_1 - \rho - \tau x_4 - z_4 - a x_4^2 - b x_4 y_4 - c y_4^2 + \boldsymbol{\epsilon}(\alpha y_1^5 + \beta z_1^5) &= 0 \\ y_1 - x_4 + \boldsymbol{\epsilon} \gamma z_1^5 &= 0 \\ z_1 - y_4 &= 0 \\ x_2 - \rho - \tau x_1 - z_1 - a x_1^2 - b x_1 y_1 - c y_1^2 + \boldsymbol{\epsilon}(\alpha y_2^5 + \beta z_2^5) &= 0 \\ y_2 - x_1 + \boldsymbol{\epsilon} \gamma z_2^5 &= 0 \\ z_2 - y_1 &= 0 \\ x_3 - \rho - \tau x_2 - z_2 - a x_2^2 - b x_2 y_2 - c y_2^2 + \boldsymbol{\epsilon}(\alpha y_3^5 + \beta z_3^5) &= 0 \\ y_3 - x_2 + \boldsymbol{\epsilon} \gamma z_3^5 &= 0 \\ z_3 - y_2 &= 0 \\ x_4 - \rho - \tau x_3 - z_3 - a x_3^2 - b x_3 y_3 - c y_3^2 + \boldsymbol{\epsilon}(\alpha y_4^5 + \beta z_4^5) &= 0 \\ y_4 - x_3 + \boldsymbol{\epsilon} \gamma z_4^5 &= 0 \\ z_4 - y_3 &= 0 \end{aligned} \tag{34}$$

Suppose that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbb{R}^3$ is a period 4 orbit and that for $i = 1, 2, 3$, the $\lambda_{i1}, \lambda_{i2}, \lambda_{i3} \in \mathbb{C}$, and $\xi_{i1}, \xi_{i2}, \xi_{i3} \in \mathbb{C}^3$ are the multipliers and eigenvectors for $\mathbf{x}_i$, computed as discussed in Section A.3. The equations for fixed and periodic orbits are amenable to Newton's method, just as in Example 1.

# D Operations on formal power series

Consider two infinite sequences of complex numbers $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty \subset \mathbb{C}$ and the corresponding power series

$$f(z) = \sum_{n=0}^\infty a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^\infty b_n z^n.$$

Suppose that $\lambda \in \mathbb{C}$. Then

$$f(\lambda z) = \sum_{n=0}^\infty \lambda^n a_n z^n.$$

Also, for any $\alpha, \beta \in \mathbb{C}$ the linear combination $\alpha f + \beta g$ has power series

$$\alpha f(z) + \beta g(z) = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) z^n.$$

Moreover, the product of two power series is given by the Cauchy product

$$f(z)g(z) = \sum_{n=0}^{\infty} (a * b)_n z^n,$$

where

$$\begin{aligned}(a * b)_n &= \sum_{k_1 + k_2 = n} a_{k_1} b_{k_2} \\ &= \sum_{k=0}^{n} a_{n-k} b_k.\end{aligned}$$

Higher order products are defined analogously. For example suppose that $f_1, \ldots, f_N$ are power series given by

$$f_i(z) = \sum_{n=0}^{\infty} a_n^i z^n, \qquad 1 \le i \le N.$$

Then

$$f_1(z) \ldots f_N(z) = \sum_{n=0}^{\infty} \left(a^1 * \ldots * a^N\right)_n z^n,$$

where the $N$-th Cauchy product is given by

$$\begin{aligned}(a^1 * \ldots * a^N)_n &= \sum_{k_1 + \ldots + k_N = n} a^1_{k_1} \ldots a^N_{k_N} \\ &= \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \ldots \sum_{k_{N-2}=0}^{k_{N-3}} \sum_{k_{N-1}=0}^{k_{N-2}} a^1_{n-k_1} a^2_{k_1-k_2} \ldots a^{N-1}_{k_{N-2}-k_{N-1}} a^N_{k_{N-1}}.\end{aligned}$$

Note that the first form of the sum is easier to read, but that the second can be easily implemented as a loop.

Another important operation in the formal series calculations below is the extraction of the coefficients of $n$-th order from the $n$-th term of a Cauchy product. For example, we have that

$$(a * b)_n = b_0 a_n + a_0 b_n + \sum_{n=1}^{n-1} a_{n-k} b_k.$$

We write

$$(\widehat{a * b})_n = \sum_{n=1}^{n-1} a_{n-k} b_k,$$

to denote the terms in the Cauchy product depending only on lower order terms. Note that this is

$$(\widehat{a * b})_n = (a * b)_n - a_0 b_n - b_0 a_n = \sum_{\substack{k_1 + k_2 = n \\ k_1, k_2 \neq n}} a_{k_1} b_{k_2}$$

Similarly, define

$$(a^1 \widehat{* \ldots *} a^N)_n = (a^1 * \ldots * a^N)_n - a^1_0 \ldots a^{N-1}_0 a^N_n - \ldots - a^2_0 \ldots a^N_0 a^1_n,$$

which is equivalent to

$$(a^1 \widehat{* \ldots *} a^N)_n = \sum_{\substack{k_1 + \ldots + k_N = n \\ k_1, \ldots, k_N \neq n}} a^1_{k_1} \ldots a^N_{k_N}.$$

The discussion generalizes to power series in any number of variables. We review the case of two variables, as this is what is needed below. So, for

$$f(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} z_1^m z_2^n, \qquad \text{and} \qquad g(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} z_1^m z_2^n,$$

we have that

$$\alpha f(z_1, z_2) + \beta g(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha a_{mn} + \beta b_{mn}) z_1^m z_2^n,$$

$$f(\lambda_1 z_1, \lambda_2 z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_1^m \lambda_2^n a_{mn} z_1^m z_2^n,$$

and

$$f(z_1, z_2) g(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a * b)_{mn} z_1^m z_2^n,$$

where the coefficients of the two variable Cauchy product are given by

$$\begin{aligned}(a * b)_{mn} &= \sum_{\substack{j_1 + j_2 = m \\ k_1 + k_2 = n}} a_{j_1 k_1} b_{j_2 k_2} \\ &= \sum_{j=0}^{m} \sum_{k=0}^{n} a_{m-j, n-k} b_{jk}.\end{aligned}$$

If $f_1, \ldots, f_N$ are power series given by

$$f_i(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}^i z_1^m z_2^n \qquad 1 \leq i \leq N,$$

then

$$f_1(z_1, z_2) \ldots f_N(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a^1 * \ldots * a^N)_{mn} z_1^m z_2^n,$$

where

$$\begin{aligned}(a^1 * \ldots * a^N)_{mn} &= \sum_{\substack{j_1 + \ldots + j_N = m \\ k_1 + \ldots + k_N = n}} a_{j_1 k_1}^1 \ldots a_{j_N k_N}^N \\ &= \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{N-1}=0}^{j_{N-2}} \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{N-1}=0}^{k_{N-2}} a_{m-j_1, n-k_1}^1 \ldots a_{j_{N-1} k_{N-1}}^N\end{aligned}$$

For coefficient extraction define

$$(\widehat{a * b})_{mn} = (a * b)_{mn} - b_{00} a_{mn} - a_{00} b_{mn},$$

and similarly

$$(a^1 \widehat{* \ldots *} a^N)_{mn} = (a^1 * \ldots * a^N)_{mn} - a_{00}^1 \ldots a_{00}^{N-1} a_{mn}^N - \ldots - a_{00}^2 \ldots a_{00}^N a_{mn}^1,$$

to be the Cauchy product of order $m, n$ with the $m, n$-th order coefficients removed.

## E Explicit formulas for the period four homological equations of the implicit Lomelí system

We have that $\mathcal{A}_{mn}$ is the12 by 12 matrix

$$\mathcal{A}_{mn} = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & A_{1,9} & A_{1,10} & A_{1,11} & A_{1,12} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & A_{2,8} & A_{2,9} & A_{2,10} & A_{2,11} & A_{2,12} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} & A_{3,9} & A_{3,10} & A_{3,11} & A_{3,12} \\ B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} & B_{1,5} & B_{1,6} & B_{1,7} & B_{1,8} & B_{1,9} & B_{1,10} & B_{1,11} & B_{1,12} \\ B_{2,1} & B_{2,2} & B_{2,3} & B_{2,4} & B_{2,5} & B_{2,6} & B_{2,7} & B_{2,8} & B_{2,9} & B_{2,10} & B_{2,11} & B_{2,12} \\ B_{3,1} & B_{3,2} & B_{3,3} & B_{3,4} & B_{3,5} & B_{3,6} & B_{3,7} & B_{3,8} & B_{3,9} & B_{3,10} & B_{3,11} & B_{3,12} \\ C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} & C_{1,5} & C_{1,6} & C_{1,7} & C_{1,8} & C_{1,9} & C_{1,10} & C_{1,11} & C_{1,12} \\ C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} & C_{2,5} & C_{2,6} & C_{2,7} & C_{2,8} & C_{2,9} & C_{2,10} & C_{2,11} & C_{2,12} \\ C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} & C_{3,5} & C_{3,6} & C_{3,7} & C_{3,8} & C_{3,9} & C_{3,10} & C_{3,11} & C_{3,12} \\ D_{1,1} & D_{1,2} & D_{1,3} & D_{1,4} & D_{1,5} & D_{1,6} & D_{1,7} & D_{1,8} & D_{1,9} & D_{1,10} & D_{1,11} & D_{1,12} \\ D_{2,1} & D_{2,2} & D_{2,3} & D_{2,4} & D_{2,5} & D_{2,6} & D_{2,7} & D_{2,8} & D_{2,9} & D_{2,10} & D_{2,11} & D_{2,12} \\ D_{3,1} & D_{3,2} & D_{3,3} & D_{3,4} & D_{3,5} & D_{3,6} & D_{3,7} & D_{3,8} & D_{3,9} & D_{3,10} & D_{3,11} & D_{3,12} \end{pmatrix}$$

with entries

$$A_{1,1} = \lambda_1^{n_1}\lambda_2^{n_2} \quad A_{1,2} = 5\epsilon\alpha v_{00}^{(1)}\lambda_1^{n_1}\lambda_2^{n_2} \quad A_{1,3} = 5\epsilon\beta w_{00}^{(1)}\lambda_1^{n_1}\lambda_2^{n_2} \quad A_{1,4} = 0 \quad A_{1,5} = 0 \quad A_{1,6} = 0$$

$$A_{1,7} = 0 \quad A_{1,8} = 0 \quad A_{1,9} = 0 \quad A_{1,10)} = -\gamma - 2\alpha u_{00}^{(4)} - bv_{00}^{(4)} \quad A_{1,11} = -bu_{00}^{(4)} - 2cv_{00}^{(4)} \quad A_{1,12} = -1$$

$$A_{2,1} = 0 \quad A_{2,2} = \lambda_1^{n_1}\lambda_2^{n_2} \quad A_{2,3} = 5\epsilon\gamma w_{00}^{(1)}\lambda_1^{n_1}\lambda_2^{n_2} \quad A_{2,4} = 0 \quad A_{2,5} = 0 \quad A_{2,6} = 0$$
$$A_{2,7} = 0 \quad A_{2,8} = 0 \quad A_{2,9} = 0 \quad A_{2,10} = -1 \quad A_{2,11} = 0 \quad A_{2,12} = 0$$
$$A_{3,1} = 0 \quad A_{3,2} = 0 \quad A_{3,3} = \lambda_1^{n_1}\lambda_2^{n_2} \quad A_{3,4} = 0 \quad A_{3,5} = 0 \quad A_{3,6} = 0$$
$$A_{3,7} = 0 \quad A_{3,8} = 0 \quad A_{3,9} = 0 \quad A_{3,10} = 0 \quad A_{3,11} = -1 \quad A_{3,12} = 0$$

$$B_{1,1} = -\gamma - 2\alpha u_{00}^{(1)} - bv_{00}^{(1)} \quad B_{1,2} = -bu_{00}^{(1)} - 2cv_{00}^{(1)} \quad B_{1,3} = -1 \quad B_{1,4} = \lambda_1^{n_1}\lambda_2^{n_2} \quad B_{1,5} = 5\epsilon\alpha v_{00}^{(2)}\lambda_1^{n_1}\lambda_2^{n_2}$$
$$B_{1,6} = 5\epsilon\beta w_{00}^{(2)}\lambda_1^{n_1}\lambda_2^{n_2} \quad B_{1,7} = 0 \quad B_{1,8} = 0 \quad B_{1,9} = 0 \quad B_{1,10} = 0 \quad B_{1,11} = 0 \quad B_{1,12} = 0$$

$$B_{2,1} = -1 \quad B_{2,2} = 0 \quad B_{2,3} = 0 \quad B_{2,4} = 0 \quad B_{2,5} = \lambda_1^{n_1}\lambda_2^{n_2} \quad B_{2,6} = 5\epsilon\gamma w_{00}^{(2)}\lambda_1^{n_1}\lambda_2^{n_2}$$
$$B_{2,7} = 0 \quad B_{2,8} = 0 \quad B_{2,9} = 0 \quad B_{2,10} = 0 \quad B_{2,11} = 0 \quad B_{2,12} = 0$$
$$B_{3,1} = 0 \quad B_{3,2} = -1 \quad B_{3,3} = 0 \quad B_{3,4} = 0 \quad B_{3,5} = 0 \quad B_{3,6} = \lambda_1^{n_1}\lambda_2^{n_2}$$
$$B_{3,7} = 0 \quad B_{3,8} = 0 \quad B_{3,9} = 0 \quad B_{3,10} = 0 \quad B_{3,11} = 0 \quad B_{3,12} = 0$$

$$C_{1,1} = 0 \quad C_{1,2} = 0 \quad C_{1,3} = 0 \quad C_{1,4} = -\gamma - 2\alpha u_{00}^{(2)} - bv_{00}^{(2)} \quad C_{1,5} = -bu_{00}^{(4)} - 2cv_{00}^{(4)} \quad C_{1,6} = 0$$
$$C_{1,7} = \lambda_1^{n_1}\lambda_2^{n_2} \quad C_{1,8} = 5\epsilon\alpha v_{00}^{(3)}\lambda_1^{n_1}\lambda_2^{n_2} \quad C_{1,9} = 5\epsilon\beta w_{00}^{(3)}\lambda_1^{n_1}\lambda_2^{n_2} \quad C_{1,10} = 0 \quad C_{1,11} = 0 \quad C_{1,12} = 0$$
$$C_{2,1} = 0 \quad C_{2,2} = 0 \quad C_{2,3} = 0 \quad C_{2,4} = -1 \quad C_{2,5} = 0 \quad C_{2,6} = 0 \quad C_{2,7} = 0$$
$$C_{2,8} = \lambda_1^{n_1}\lambda_2^{n_2} \quad C_{2,9} = 5\epsilon\gamma w_{00}^{(3)}\lambda_1^{n_1}\lambda_2^{n_2} \quad C_{2,10} = 0 \quad C_{2,11} = 0 \quad C_{2,12} = 0$$
$$C_{3,1} = 0 \quad C_{3,2} = 0 \quad C_{3,3} = 0 \quad C_{3,4} = 0 \quad C_{3,5} = -1 \quad C_{3,6} = 0$$
$$C_{3,7} = 0 \quad C_{3,8} = 0 \quad C_{3,9} = \lambda_1^{n_1}\lambda_2^{n_2} \quad C_{3,10} = 0 \quad C_{3,11} = 0 \quad C_{3,12}$$
$$D_{1,1} = 0 \quad D_{1,2} = 0 \quad D_{1,3} = 0 \quad D_{1,4} = 0 \quad D_{1,5} = 0 \quad D_{1,6} = 0$$
$$D_{1,7} = -\gamma - 2\alpha u_{00}^{(3)} - bv_{00}^{(3)} \quad D_{1,8} = -bu_{00}^{(3)} - 2cv_{00}^{(3)} \quad D_{1,9} = -1 \quad D_{1,10} = \lambda_1^{n_1}\lambda_2^{n_2}$$
$$D_{1,11} = 5\epsilon\alpha v_{00}^{(4)}\lambda_1^{n_1}\lambda_2^{n_2} \quad D_{1,12} = 5\epsilon\beta w_{00}^{(4)}\lambda_1^{n_1}\lambda_2^{n_2}$$

$$D_{2,1} = 0 \quad D_{2,2} = 0 \quad D_{2,3} = 0 \quad D_{2,4} = 0 \quad D_{2,5} = 0 \quad D_{2,6} = 0$$
$$D_{2,7} = -1 \quad D_{2,8} = 0 \quad D_{2,9} = 0 \quad D_{2,10} = 0 \quad D_{2,11} = \lambda_1^{n_1}\lambda_2^{n_2} \quad D_{2,12} = 5\epsilon\gamma w_{00}^{(4)}\lambda_1^{n_1}\lambda_2^{n_2}$$
$$D_{3,1} = 0 \quad D_{3,2} = 0 \quad D_{3,3} = 0 \quad D_{3,4} = 0 \quad D_{3,5} = 0 \quad D_{3,6} = 0$$
$$D_{3,7} = 0 \quad D_{3,8} = -1 \quad D_{3,9} = 0 \quad D_{3,10} = 0 \quad D_{3,11} = 0 \quad D_{3,12} = \lambda_1^{n_1}\lambda_2^{n_2}.$$

Similarly, the right hand side

$$\mathbf{S}_{mn} = (S^1_{mn}, S^2_{mn}, 0, S^3_{mn}, S^4_{mn}, 0, S^5_{mn}, S^6_{mn}, 0, S^7_{mn}, S^8_{mn}, 0)^T$$

has components

$$\begin{aligned}
S^1_{mn} &= a(\widehat{u_4 * u_4})_{mn} + b(\widehat{u_4 * v_4})_{mn} + c(\widehat{v_4 * v_4})_{mn} \\
&\quad - \alpha\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(v_1 * v_1 \widehat{* v_1 *} v_1 * v_1)_{mn} - \beta\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_1 * w_1 \widehat{* w_1 *} w_1 * w_1)_{mn} \\
S^2_{mn} &= \gamma\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_1 * w_1 \widehat{* w_1 *} w_1 * w_1)_{mn} \\
S^3_{mn} &= a(\widehat{u_1 * u_1})_{mn} + b(\widehat{u_1 * v_1})_{mn} + c(\widehat{v_1 * v_1})_{mn} \\
&\quad - \alpha\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(v_2 * v_2 \widehat{* v_2 *} v_2 * v_2)_{mn} - \beta\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_2 * w_2 \widehat{* w_2 *} w_2 * w_2)_{mn} \\
S^4_{mn} &= \gamma\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_2 * w_2 \widehat{* w_2 *} w_2 * w_2)_{mn} \\
S^5_{mn} &= a(\widehat{u_2 * u_2})_{mn} + b(\widehat{u_2 * v_2})_{mn} + c(\widehat{v_2 * v_2})_{mn} \\
&\quad - \alpha\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(v_3 * v_3 \widehat{* v_3 *} v_3 * v_3)_{mn} - \beta\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_3 * w_3 \widehat{* w_3 *} w_3 * w_3)_{mn} \\
S^6_{mn} &= \gamma\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_3 * w_3 \widehat{* w_3 *} w_3 * w_3)_{mn} \\
S^7_{mn} &= a(\widehat{u_3 * u_3})_{mn} + b(\widehat{u_3 * v_3})_{mn} + c(\widehat{v_3 * v_3})_{mn} \\
&\quad - \alpha\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(v_4 * v_4 \widehat{* v_4 *} v_4 * v_4)_{mn} - \beta\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_4 * w_4 \widehat{* w_4 *} w_4 * w_4)_{mn} \\
S^8_{mn} &= \gamma\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_4 * w_4 \widehat{* w_4 *} w_4 * w_4)_{mn}.
\end{aligned}$$

Note that while the formulas defining the components of $\mathcal{A}_{mn}$ are correct, we have suppressed the dependence on $(m, n)$ in the variables names themselves.

# References


[1] R. A. Adomaitis and I. G. Kevrekidis. Noninvertibility and the structure of basins of attraction in a model adaptive control system. *J. Nonlinear Sci.*, 1(1):95–105, 1991.

[2] E. Akin. *The general topology of dynamical systems*, volume 1 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1993.

[3] D. G. Aronson, M. A. Chory, G. R. Hall, and R. P. McGehee. Bifurcations from an invariant circle for two-parameter families of maps of the plane: a computer-assisted study. *Comm. Math. Phys.*, 83(3):303–354, 1982.

[4] I. Baldomá, E. Fontich, R. de la Llave, and P. Martí n. The parameterization method for one-dimensional invariant manifolds of higher dimensional parabolic fixed points. *Discrete Contin. Dyn. Syst.*, 17(4):835–865, 2007.

[5] M. Breden, J.-P. Lessard, and J. D. Mireles James. Computation of maximal local (un)stable manifold patches by the parameterization method. *Indag. Math. (N.S.)*, 27(1):340–367, 2016.

[6] X. Cabré, E. Fontich, and R. de la Llave. The parameterization method for invariant manifolds. I. Manifolds associated to non-resonant subspaces. *Indiana Univ. Math. J.*, 52(2):283–328, 2003.

[7] X. Cabré, E. Fontich, and R. de la Llave. The parameterization method for invariant manifolds. II. Regularity with respect to parameters. *Indiana Univ. Math. J.*, 52(2):329–360, 2003.

[8] X. Cabré, E. Fontich, and R. de la Llave. The parameterization method for invariant manifolds. III. Overview and applications. *J. Differential Equations*, 218(2):444–515, 2005.

[9] R. C. Calleja, A. Celletti, and R. de la Llave. A KAM theory for conformally symplectic systems: efficient algorithms and their validation. *J. Differential Equations*, 255(5):978–1049, 2013.

[10] M. Canadell and A. Haro. Computation of quasi-periodic normally hyperbolic invariant tori: algorithms, numerical explorations and mechanisms of breakdown. *J. Nonlinear Sci.*, 27(6):1829–1868, 2017.

[11] M. J. Capiński and J. D. Mireles James. Validated computation of heteroclinic sets. *SIAM J. Appl. Dyn. Syst.*, 16(1):375–409, 2017.

[12] R. Castelli, M. Gameiro, and J.-P. Lessard. Rigorous numerics for ill-posed PDEs: periodic orbits in the Boussinesq equation. *Arch. Ration. Mech. Anal.*, 228(1):129–157, 2018.

[13] R. Castelli, J.-P. Lessard, and J. D. M. James. Parameterization of invariant manifolds for periodic orbits i: Efficient numerics via the floquet normal form. *SIAM Journal on Applied Dynamical Systems*, 14(1):132–167, 2015.

[14] H. Cheng and R. de la Llave. Stable manifolds to bounded solutions in possibly ill-posed PDEs. *J. Differential Equations*, 268(8):4830–4899, 2020.

[15] R. de la Llave. A smooth center manifold theorem which applies to some ill-posed partial differential equations with unbounded nonlinearities. *J. Dynam. Differential Equations*, 21(3):371–415, 2009.

[16] R. de la Llave, G. Gimeno, and J. Yang. Numerical computation of periodic orbits and isochrons for state dependent delay perturbations of an ode in the plane. (Submitted) 2020.

[17] R. de la Llave, G. Gimeno, and J. Yang. Parameterization method for state-dependent delay perturbation of an ordinary differential equation. (Submitted) 2020.

[18] R. de la Llave, A. González, À. Jorba, and J. Villanueva. KAM theory without action-angle variables. *Nonlinearity*, 18(2):855–895, 2005.

[19] R. de la Llave and H. E. Lomelí. Invariant manifolds for analytic difference equations. *SIAM J. Appl. Dyn. Syst.*, 11(4):1614–1651, 2012.

[20] R. de la Llave and Y. Sire. An a posteriori KAM theorem for whiskered tori in Hamiltonian partial differential equations with applications to some ill-posed equations. *Arch. Ration. Mech. Anal.*, 231(2):971–1044, 2019.

[21] R. L. Devaney. *An introduction to chaotic dynamical systems*. Studies in Nonlinearity. Westview Press, Boulder, CO, 2003. Reprint of the second (1989) edition.

[22] H. R. Dullin and J. D. Meiss. Quadratic volume-preserving maps: invariant circles and bifurcations. *SIAM J. Appl. Dyn. Syst.*, 8(1):76–128, 2009.

[23] C. E. Frouzakis, L. Gardini, I. G. Kevrekidis, G. Millerioux, and C. Mira. On some properties of invariant sets of two-dimensional noninvertible maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 7(6):1167–1194, 1997.

[24] C. Golé. *Symplectic twist maps*, volume 18 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2001. Global variational techniques.

[25] J. Gonzalez and J. D. Mireles James. High-order parameterization of stable/unstable manifolds for long periodic orbits of maps. *SIAM Journal on Applied Dynamical Systems*, 16(3):1748–1795, 2017. `https://doi.org/10.1137/16M1090041`.

[26] A. Haro, M. Canadell, J.-L. s. Figueras, A. Luque, and J.-M. Mondelo. *The parameterization method for invariant manifolds*, volume 195 of *Applied Mathematical Sciences*. Springer, [Cham], 2016. From rigorous results to effective computations.

[27] À. Haro and R. de la Llave. A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: numerical algorithms. *Discrete Contin. Dyn. Syst. Ser. B*, 6(6):1261–1300 (electronic), 2006.

[28] A. Haro and R. de la Llave. A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: rigorous results. *J. Differential Equations*, 228(2):530–579, 2006.

[29] A. Haro and R. de la Llave. A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: explorations and mechanisms for the breakdown of hyperbolicity. *SIAM J. Appl. Dyn. Syst.*, 6(1):142–207 (electronic), 2007.

[30] X. He and R. de la Llave. Construction of quasi-periodic solutions of state-dependent delay differential equations by the parameterization method II: Analytic case. *J. Differential Equations*, 261(3):2068–2108, 2016.

[31] X. He and R. de la Llave. Construction of quasi-periodic solutions of state-dependent delay differential equations by the parameterization method I: Finitely differentiable, hyperbolic case. *J. Dynam. Differential Equations*, 29(4):1503–1517, 2017.

[32] M. Hénon. A two-dimensional mapping with a strange attractor. *Comm. Math. Phys.*, 50(1):69–77, 1976.

[33] G. Huguet and R. de la Llave. Computation of limit cycles and their isochrons: fast algorithms and their convergence. *SIAM J. Appl. Dyn. Syst.*, 12(4):1763–1802, 2013.

[34] G. Huguet, R. de la Llave, and Y. Sire. Computation of whiskered invariant tori and their associated manifolds: new fast algorithms. *Discrete Contin. Dyn. Syst.*, 32(4):1309–1353, 2012.

[35] À. Jorba and M. Zou. A software package for the numerical integration of ODEs by means of high-order Taylor methods. *Experiment. Math.*, 14(1):99–117, 2005.

[36] J. Kennedy. Inverse limits, economics, and backward dynamics. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 102(1):39–73, 2008.

[37] J. Kennedy, D. R. Stockman, and J. A. Yorke. Inverse limits and an implicitly defined difference equation from economics. *Topology Appl.*, 154(13):2533–2552, 2007.

[38] J. A. Kennedy and D. R. Stockman. Chaotic equilibria in models with backward dynamics. *J. Econom. Dynam. Control*, 32(3):939–955, 2008.

[39] L. M. Lerman and L. P. Shilnikov. Homoclinic structures in infinite-dimensional systems. *Sibirsk. Mat. Zh.*, 29(3):92–103, 220, 1988.

[40] H. E. Lomelí and J. D. Meiss. Quadratic volume-preserving maps. *Nonlinearity*, 11(3):557–574, 1998.

[41] E. N. Lorenz. Computational chaos—a prelude to computational instability. *Phys. D*, 35(3):299–317, 1989.

[42] A. C. J. Luo. *Discretization and implicit mapping dynamics.* Nonlinear Physical Science. Higher Education Press, Beijing; Springer, Heidelberg, 2015.

[43] R. McGehee. Attractors for closed relations on compact Hausdorff spaces. *Indiana Univ. Math. J.*, 41(4):1165–1209, 1992.

[44] R. Michener and B. Ravikumar. Chaotic dynamics in a cash-in-advance economy. *J. Econom. Dynam. Control*, 22(7):1117–1137, 1998.

[45] J. D. Mireles James. Quadratic volume-preserving maps: (un)stable manifolds, hyperbolic dynamics, and vortex-bubble bifurcations. *J. Nonlinear Sci.*, 23(4):585–615, 2013.

[46] J. D. Mireles James and H. Lomelí. Computation of heteroclinic arcs with application to the volume preserving Hénon family. *SIAM J. Appl. Dyn. Syst.*, 9(3):919–953, 2010.

[47] J. D. Mireles James and M. Murray. Chebyshev-Taylor parameterization of stable/unstable manifolds for periodic orbits: implementation and applications. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 27(14):1730050, 32, 2017.

[48] J. R. Munkres. *Analysis on manifolds.* Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1991.

[49] R. C. Robinson. *An introduction to dynamical systems: continuous and discrete.* Pearson Prentice Hall, Upper Saddle River, NJ, 2004.

[50] E. Sander. *Hyperbolic sets for noninvertible maps and relations.* ProQuest LLC, Ann Arbor, MI, 1996. Thesis (Ph.D.)–University of Minnesota.

[51] E. Sander. Hyperbolic sets for noninvertible maps and relations. *Discrete Contin. Dynam. Systems*, 5(2):339–357, 1999.

[52] E. Sander. Homoclinic tangles for noninvertible maps. *Nonlinear Anal.*, 41(1-2, Ser. A: Theory Methods):259–276, 2000.

[53] H. Steinlein and H.-O. Walther. Hyperbolic sets, transversal homoclinic trajectories, and symbolic dynamics for $C^1$-maps in Banach spaces. *J. Dynam. Differential Equations*, 2(3):325–365, 1990.

[54] J. B. van den Berg, J. D. Mireles James, and C. Reinhardt. Computing (un)stable manifolds with validated error bounds: non-resonant and resonant spectra. *J. Nonlinear Sci.*, 26(4):1055–1095, 2016.

[55] A. P. Veselov. Integrable mappings. *Uspekhi Mat. Nauk*, 46(5(281)):3–45, 190, 1991.

[56] F. Wang and R. de la Llave. Response solutions to quasi-periodically forced systems, even to possibly ill-posed PDEs, with strong dissipation and any frequency vectors. *SIAM J. Math. Anal.*, 52(4):3149–3191, 2020.