MATH DAY 2019 at FAU

Competition A–Individual

SOLUTIONS

1. How many 3-digit numbers are there which are squares and the sum of whose digits are also squares?

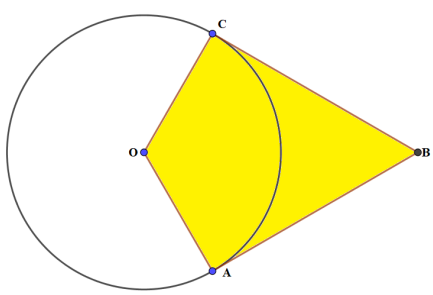
(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** NA

Solution. The 3 digit squares are of the form n^2, where $10 \leq n \leq 31$. We can make a table, denoting by $\sum$ the sum of digits.

n	n^2	$\sum$	square?		n	n^2	$\sum$	square?
10	100	1	T		21	441	9	T
11	121	4	T		22	484	16	T
12	144	9	T		23	529	16	T
13	169	16	T		24	576	18	F
14	196	16	T		25	625	13	F
15	225	9	T		26	676	19	F
16	256	13	F		27	729	18	F
17	289	19	F		28	784	19	F
18	324	9	T		29	841	9	F
29	361	10	F		30	900	9	T
20	400	4	T		31	961	16	T

We have a total of 13 T's. **The correct solution is** $\boxed{\textbf{D}}$

2. In the following diagram, AB and CB are tangent to the circle, and triangle ABC is equilateral. If the radius of the circle is 1, what is the area of the quadrilateral $OABC$?



(A) $\sqrt{3}$ **(B)** $2\sqrt{3}$ **(C)** 3 **(D)** π **(E)** NA

Solution 1. The angles at A and C must be right angles, so the quadrilateral is cyclic. Since the triangle ABC is equilateral, the angle at B is a $60°$ angle, hence the angle at O is $120°$, with $\cos\angle O = -1/2$. By the so called law of cosines $|AB|^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot (-\frac{1}{2}) = 3$ so the sides of triangle ABC are of length $\sqrt{3}$. Thus the quadrilateral has sides of lengths 1, 1, $\sqrt{3}$, $\sqrt{3}$ and, being cyclic, we can compute its area by Brahmagupta's formula. The semi-perimeter is $s = \sqrt{3} + 1$ so that

$$[OABC] = \sqrt{\sqrt{3} \cdot \sqrt{3} \cdot 1 \cdot 1} = \sqrt{3}.$$

Solution 2 One sees as before that $|AB| = \sqrt{3}$. Since OAB is a right triangle of legs OA, AB we have

$$[OAB] = \frac{1}{2}|OA||AB| = \frac{1}{2}\sqrt{3}$$

and

$$[OABC] = 2[OAB] = \sqrt{3}$$

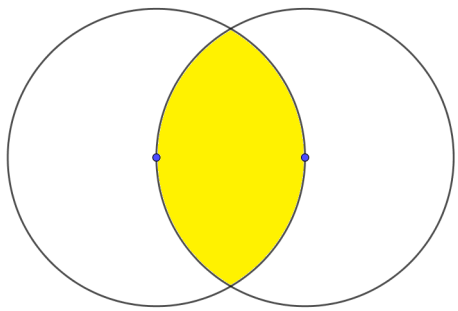
. **The correct solution is** $\boxed{\textbf{A}}$

3. The number of integer solutions of $xy = 3x - 4y$ is

(A) 3 **(B)** 4 **(C)** 6 **(D)** 12 **(E)** 14

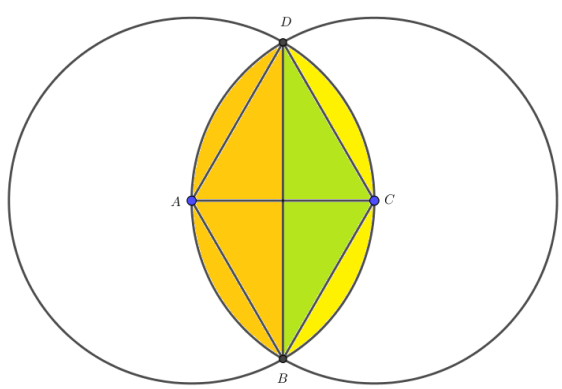
Solution. We can rewrite the equation in the form $(x-3)(y+4) = -12$. Now $-12 = (-1) \cdot 2^2 \cdot 3$ has $2 \cdot 3 \cdot 2 = 12$ integer divisors, (positive and negative) and each one provides a solution to the equation. **The correct solution is** $\boxed{\textbf{D}}$

4. The center of each of two discs of radius 1 lies on the circumference of the other. The area of the intersection of the discs is



(A) $\frac{\pi}{3}$ **(B)** $\frac{2\pi}{3}$ **(C)** $\frac{\pi}{3} - \frac{\sqrt{3}}{3}$ **(D)** $\frac{2\pi}{3} - \frac{\sqrt{3}}{3}$ **(E)** $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

Solution. Refer to the figure below, where the region is now in yellow, green and orange.



If we denote the areas of the yellow, green, orange regions by Y, G, O, resepctively, the area we are looking for is $Y + G + O$. We see that triangle ACD and ACB are equilateral so that $\angle ACD = \angle ACB = 60°$, hence $\angle BCD = 120°$ and the circular sector of the circle on the right bounded by the radii CB and CD has area equal to one third of the area of the whole circle, thus $\pi/3$; that is, $G + O = \pi/3$. Now G is clearly equal to the area of the equilateral triangle ACD, of sides equal 1, thus $G = \sqrt{3}/4$ and hence $O = (G + O) - G = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$. Since clearly O equals half of the area we are looking for, the answer is $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$. **The correct solution is** $\boxed{\textbf{E}}$

5. If $f(n+1) = f(n) + n$ for all integers $n > 0$, and if $f(0) = 1$, then $f(100)$ is equal to:

(A) 4950 **(B)** 4951 **(C)** 5050 **(D)** 5051 **(E)** NA

.

Solution. We see that $f(1) = f(0) + 0 = 1$, $f(2) = f(1) + 1 = 1 + 1$, $f(3) = f(2) + 2 = 1 + 1 + 2$, $f(4) = f(3) + 3 = 1 + 1 + 2 + 3$, and so forth. Thus

$$f(100)1 + 1 + 2 + \cdots + 99 = 1 + \frac{99 \cdot 100}{2} = 4951.$$

The correct solution is $\boxed{\textbf{B}}$

6. Let us call each positive whole number which (in base 10) can be written as a string of 1's and 2's *simple*. For example 22121 and 2222 are simple; 1021 is not simple. How many simple numbers are there less than one million?

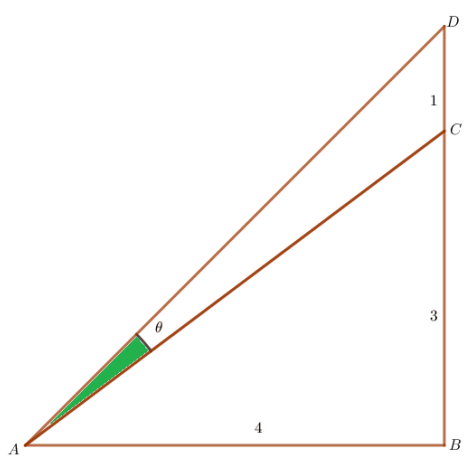
(A) 62 **(B)** 63 **(C)** 126 **(D)** 127 **(E)** 128

Solution. A number less than one million has less than 6 digits. There are 2^k simple numbers of k digits, so the answer is

$$2 + 2^2 + 2^3 + 2^4 + 2^+2^6 = 2(2^6 - 1) = 126.$$

The correct solution is $\boxed{\textbf{C}}$

7. In the picture below, the angle at B is a right angle, $|AB| = 4$, $|BC| = 3$ and $|CD| = 1$. Find $\tan\theta$, the tangent of the angle $\theta = \angle CAD$.



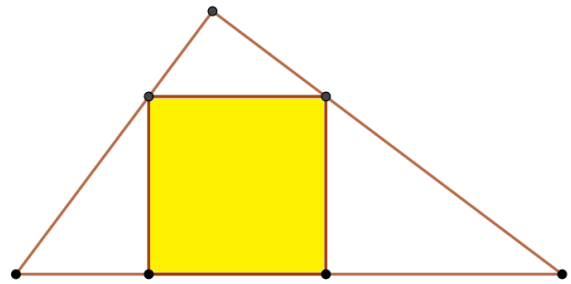
(A) $\frac{1}{7}$ **(B)** $\frac{1}{5}$ **(C)** $\tan 15^\circ$ **(D)** $\frac{1}{3}$ **(E)** NA

Solution. We have

$$\tan\theta = \tan(\angle BAD - \angle BAC) = \frac{\tan\angle BAD - \tan(\angle BAC)}{1 + \tan\angle BAD\tan(\angle BAC)} = \frac{1 - \frac{3}{4}}{1 + \frac{3}{4}} = \frac{1}{7}.$$

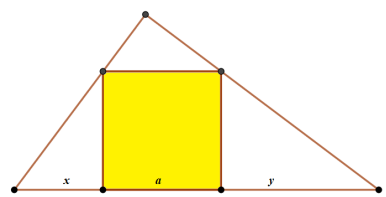
The correct solution is $\boxed{\textbf{A}}$

8. The following figure is built by using little rods all of equal lengths. If the shape of the large triangle is $3:4:5$ and the shaded portion is a square, what is the least amount of little rods to make up the figure?



(A) 120 **(B)** 240 **(C)** 624 **(D)** 836 **(E)** NA

Solution. First of all we notice that if a is the side of the square, then the square decomposes the hypotenuse of the main triangle into segments of length x, a, y, as shown in the picture below.



The little right triangle on the left with shorter leg x and longer leg a, and the right triangle on the right with longer leg y and shorter leg a, are both similar to the $3:4:5$ triangle. Due to this similarity,

$$\frac{x}{a} = \frac{3}{4}, \ \frac{y}{a} = \frac{4}{3}, \ \text{ thus } \ 5 = x + a + y = \left(\frac{3}{4} + 1 + \frac{4}{3}\right) a$$

so that $a = \dfrac{60}{37}$. It is clear that the little rod has to go in an exact number f times in the side of the square; denoting its length by δ, we must have $\dfrac{60}{37} = n\delta$ for some integer n. We could build the square and then the large triangle about it, but the most efficient way is to then use the rod to build the large triangle so that δ must satisfy in addition

$$3 = k\delta, \ 4 = \ell\delta, \ 5 = m\delta,$$

for integers k, ℓ, m and the total number of rods needed will be $k + \ell + m + 3n$. Now $k/\ell = 3/4$ so that $3\ell = 4k$. This implies that ℓ is a multiple of 4, say $\ell = 4s$ for some integer s, and we get that $k = 3s$. Using $m/k = 5/3$ we see that $m = 5s$. This is, of course, just telling us that if we were to build only the triangle, we could use rods of length 1. Now

$$\frac{20}{37} = \frac{\frac{60}{37}}{3} = \frac{a}{3} = \frac{n}{k} = \frac{n}{3s},$$

it follows that $60s = 37n$. The numbers 60 and 37 being relatively prime, we conclude that $n = 60t, s = 37t$ for some integer t. Summarizing we got

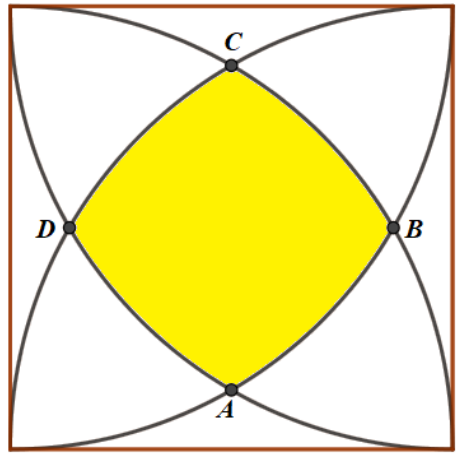
$$k = 3s = 111t, \ \ell = 4s = 148t, \ m = 5s = 185t, n = 60t.$$

The number of rods is minimized taking $t = 1$ so that the minimum number of rods needed is

$$111 + 148 + 185 + 3 \cdot 60 = 624.$$

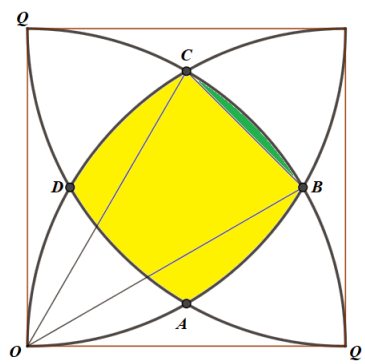
The correct solution is $\boxed{\textbf{C}}$

9. If each side of the square has length 1, what is the area of the curvilinear quadrilateral $ABCD$ bounded by the circular arcs?



(A) $1-\sqrt{3}-\frac{\pi}{3}$ **(B)** $1+\sqrt{3}+\frac{\pi}{3}$ **(C)** $1+\sqrt{3}-\frac{\pi}{3}$ **(D)** $1-\sqrt{3}+\frac{\pi}{3}$ **(E)** $\frac{\pi}{3}-\frac{\sqrt{3}}{2}$

Solution. The three sectors POB, BOC, and COQ are congruent, each containing an angle 30∘.



The length of BC is $2\sin 15°$. The square $ABCD$ has area $4\sin^2 15° = 2(1-\cos 30°) = 2-\sqrt{3}$.

$$\begin{aligned}
\text{Area of region} &= \text{square } ABCD + 4\times\text{green segment}\\
&= \textit{square } ABCD + 4\times(\text{sector } OBC - \text{triangle } OBC)\\
&= 2-\sqrt{3}+4\left(\frac{1}{2}\cdot 1^2\cdot\frac{\pi}{6}-\frac{1}{2}\cdot 2\sin 15°\cos 15°\right)\\
&= \frac{\pi}{3}-\sqrt{3}+1.
\end{aligned}$$

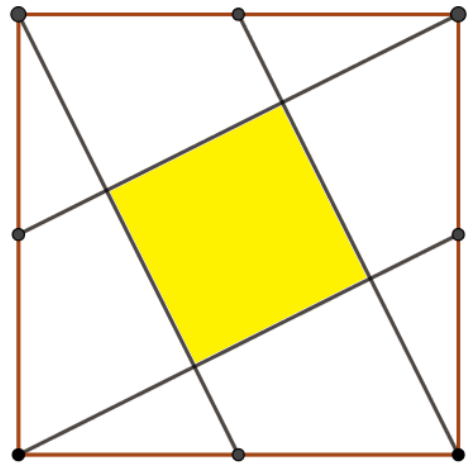
The correct solution is $\boxed{\textbf{D}}$

10. What is the angle between the two hands of a clock at 2:30?

(A) 100° **(B)** 105° **(C)** 110° **(D)** 120° **(E)** 135°

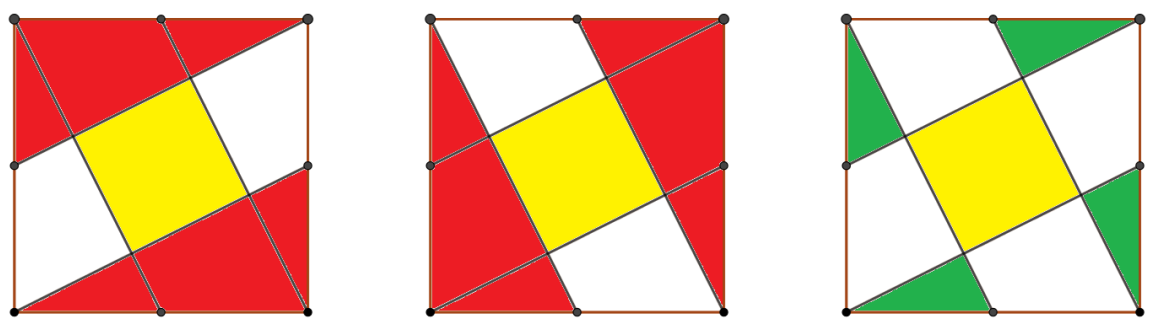
Solution. At $2:30$, the minutes hand is making an angle 180° clockwise with the $12:00$ position, and the hours hand $2\frac{1}{2}\times 30° = 75°$. Therefore, the angle between the two hands is $180°-75° = 105°$. **The correct solution is** $\boxed{\textbf{B}}$

11. The large square has area 1. The inside lines join a vertex of the square to the midpoint of a side as shown. What is the area of the small central square?

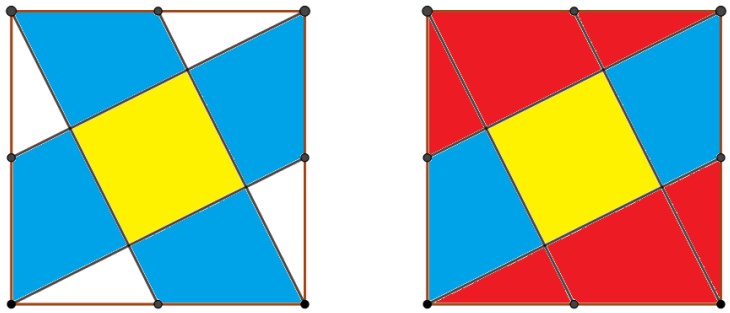


(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{5}$ **(E)** NA

Solution. It is easy to see that the right triangles determined by the lines are either congruent or similar. Specifically, in the pictures below, the red triangles are congruent and the green triangles are also congruent and similar to the red ones.



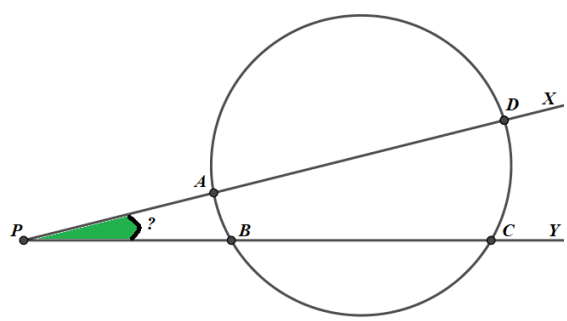
In the large right triangles the shorter leg is half the length of the longer leg, thus the same is true in the small triangles. Since the big square is a 1×1 square, it is clear that the large right triangles have area $\frac{1}{2}(\frac{1}{\times}\frac{1}{2}) = \frac{1}{4}$. The hypotenuse of the small right triangles has length $1/2$; if a is the length of the shorter leg of these triangles, so $2a$ is the length of the larger leg, then $a^2 + (2a)^2 = (\frac{1}{2})^2$, from which it follows that $a^2 = \frac{1}{20}$ and the area of each small right triangle is $\frac{1}{2}(a \times 2a) = \frac{1}{20}$. The blue quadrilaterals in the picture on the left below are obtained by taking away two small triangles from a large triangle, so their area equals $\frac{1}{4} - 2 \cdot \frac{1}{20} = \frac{3}{20}$.



The complement of the little square in the big square is made up of two red triangles and two blue quadrilaterals, as seen in the picture on the right above, thus has area $2 \cdot \frac{1}{4} + 2 \cdot \frac{3}{20} = \frac{4}{5}$. It follows that the area of the little square is $\frac{1}{5}$. **The correct solution is** $\boxed{\textbf{D}}$

12. The rays PX and PY cut off arcs AB and CD of a circle with radius 2. If the length of arc CD is 2 times the length of arc AB and the length of arc CD is $\frac{4}{5}\pi$, then the number of degrees in angle APB is:

(A) 9 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 18



Solution 1., A solution making no assumptions.Let O be the center of the circle, and $\angle OAD = \angle ODA = \theta$. Then $\angle OAP = 180^\circ - \theta$ and $\angle AOD = 180^\circ - 2\theta$. Since the radius of the circle is 2, and the arc CD has length $\frac{4\pi}{5} = 2 \cdot \frac{2\pi}{5}$, $\angle COD = \frac{2\pi}{5}$, or 72°. The length of the arc AB being half of arc CD, $\angle AOB = 36^\circ$. Therefore,

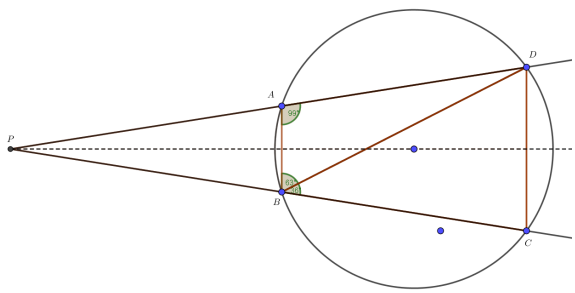
$$\angle BOC = 360^\circ - (180^\circ - 2\theta) - 72^\circ - 36^\circ = 72^\circ + 2\theta,$$

and $\angle OBC = \angle OCB = 54^\circ - \theta$; $\angle OBP = 126^\circ + \theta$.
Finally, in the quadrilateral $OAPB$,

$$\angle PAB = 360^\circ - 36^\circ - (180^\circ - \theta) - (126^\circ + \theta) = 18^\circ.$$

The correct solution is $\boxed{\textbf{E}}$

Solution 2. Since it doesn't seem to matter where the point P is placed, we can place it in such a way so that the bisector of the angle APB goes through the center of the circle. Then the counterclockwise arcs from D to A and from B to C have equal lengths $7\pi/5$. The central angles spanned by these arcs is (dividing by the radius) $7\pi/10$ or 126°. Angle $\angle ABD$ is then half this amount, namely 63°. The central angle of the arc CD measures $2\pi/5$ or 72°, thus angle $\angle DBC$ measures 36° making $\angle CBA$ a $63 + 36 = 99^\circ$ angle. By symmetry angle $\angle BAD$ also measures 99°, thus the angles at A and B of triangle APB measure $180 - 99 = 81^\circ$, hence the angle at P measures $180 - 2cdot81 = 18^\circ$. The figure below, drawn to scale, illustrates this solution. Of course, this solution makes the assumption that it doesn't matter where P is; the first solution implicitly proves this.



13. A polygon has 54 diagonals. How many sides does it have?

(A) 12 **(B)** 20 **(C)** 24 **(D)** 36 **(E)** 40

Solution. A polygon of n sides has $\binom{n}{2} - n = \frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3)$ diagonals. If $\frac{1}{2}n(n-3) = 54$, $n(n-3) = 108$, $n^2 - 3n - 108 = 0$, $(n-12)(n+9) = 0$. Therefore, $n = 12$.

14. A triangle has sides of lengths $a, a+1$, and $a+2$, and one of its altitudes of length $a-1$, where a is a positive integer; $a \geq 2$. The area of this triangle equals

(A) 6 **(B)** 30 **(C)** 56 **(D)** 84 **(E)** NA

Solution. We will try to find the area, rather the square of the area A, by two methods. First by Heron's formula. The semi-perimeter is $s = (3a+3)/2$ so

$$A^2 = \frac{3(a+1)^2(a+3)(a-1)}{16}.$$

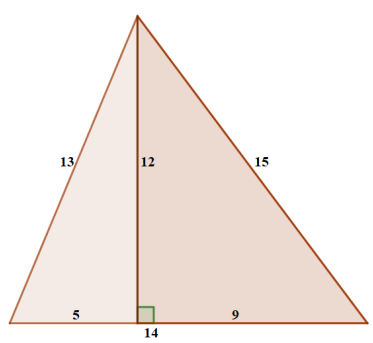
Using the formula area $= \frac{1}{2}$base $\times$ altitude, we see that A is one of $a(a-1)/2$, $(a+1)(a-1)/2$, $(a+2)(a-1)/2$. We thus have

$$\frac{3(a+1)^2(a+3)(a-1)}{16} = \begin{cases} \frac{a^2(a-1)^2}{4} \\ \text{or} \\ \frac{(a+1)^2(a-1)^2}{4} \\ \text{or} \\ \frac{(a+2)^2(a-1)^2}{4}. \end{cases}$$

We can begin with the most likely one, namely

$$\frac{3(a+1)^2(a+3)(a-1)}{16} = \frac{(a+1)^2(a-1)^2}{4},$$

from which we easily solve for a to get $a = 13$. Then the sides are $13, 14, 15$ and the triangle can be seen to be made up out of two right triangles, one of sides 5,12,13; and one of side 9, 12, 15, as shown below:



Then $s = (13 + 14 + 15)/2 = 21$ and the area is $A = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$, which is one of the provided answers. If there is time, one can check that neither

$$\frac{3(a+1)^2(a+3)(a-1)}{16} = \frac{a^2(a-1)^2}{4} \quad \text{nor} \quad \frac{3(a+1)^2(a+3)(a-1)}{16} = \frac{(a+2)^2(a-1)^2}{4}$$

can work. The first one of these implies $3(a+1)^2(a+3) = 4a^2(a-1)$. Then a^2 divides $3(a+1)^2(a+3)$. This is only possible if $a = 3$ and then the equation becomes $288 = 72$, which is nonsense. If the second rejected possibility were to hold, we'd get $3(a+1)^2(a+3) = 4(a+2)^2(a-1)$, which is obviously impossible since $a+2$ is relatively prime with $a+1$ and $a+3$. and $(a+2)^2$ cannot possibly divide 3. **The correct solution is** $\boxed{\textbf{D}}$

15.* Find the largest value of p for which the there exists a solution (x, y) of the system

$$\begin{array}{rcrcl} (7-p)x & + & 7y & = & 0 \\ 6x & + & (16+p)y & = & 0 \end{array}$$

with not both x and y equal to 0. Find also the ratio y/x for this value of p. This ratio should have the form a/b with a, b integers with no common factor (other than 1), $b > 0$, Enter $p + a + b$ in the appropriate place on the answer sheet.

Solution. Multiply the first equation by $16 + p$, the second by 7 and subtract the second from the first to get $(70 - 90p - p^2)x = 0$. If $70 - 90p - p^2 \neq 0$, then the only possible solution satisfies $x = 0$ and, if $x = 0$ we must also have $y = 0$. So we see that p has to satisfy the quadratic equation $p^2 + 9p - 70 = 0$, which has solutions $p = 5, -14$. The larger one of these is $p = 5$. With $p = 5$ the system has the solution $y = -(2/7)x$, with x arbitrary, so $y/x = -2/7$. Then $5 - 2 + 7 = 10$ so **The correct solution is** $\boxed{\textbf{10}}$

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16.* Find the sum of all 4 digit integers x satisfying: The sum of x and its four digits is 2019. Enter your answer directly onto the answer sheet.

Solution. Suppose the number is $x = abcd = 10^3a+10^2b+10c+d$; then $2019 = x+a+b+c+d = 1001a+101b+11c+2d$. Clearly, $a = 1$ or $a = 2$. Assume first $a = 1$. In this case $101b + 11c + 2d = 1018$. One now sees that b must be 9,thus $11c + 2d = 109$. This implies c must be odd, after which one sees that one must have $c = 9$ and $d = 5$. 1995 is the only solution. Assume next $a = 2$. A similar ana;lysis shows that there is only one solution, namely 2013, and $1995 = 2013 = 4008$. **The correct solution is** $\boxed{\textbf{4008}}$

17.* Of all solutions (x, y, z) of the system

$$\begin{array}{rcrcrcr} 3x & + & 2y & + & 5z & = & 87 \\ 2x & - & 5y & + & 3z & = & -3 \end{array}$$

with x, y, z positive integers, find the one with the largest value of x. Enter the value of $x + y + z$ directly onto the answer sheet.

Solution. One approach is to solve first for x and y in terms of z. We could use Cramer's rule to do this; or any other method. We get

$$x = \frac{429 - 31z}{19}, \qquad y = \frac{183 - z}{19}.$$

For y to be an integer $z - 183$ must be divisible by 19; since $183 = 9 \cdot 19 + 12$, we should have $z = 12 + 19k$ for some integer k, hence $y = 9 - k$ and $x = 3 - 19k$. It is clear that the only value that will give a solution with x, y, z positive is by taking $k = 0$. In his case the solution is $(3, 9, 12)$ and $3 + 9 + 12 = 24$. **The correct solution is** $\boxed{\textbf{24}}$

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18.* Find the number of ordered pairs of integers (x, y) such that $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{36}$. Notice that if (x, y) is such a pair, so is (y, x); and both must be counted. Write your answer directly onto the answer sheet.

Solution. Obviously $x \neq 0 \neq y$ and for $x \neq 0 \neq y$ the equation is equivalent to $xy - 36x - 36y = 0$. Adding 36^2 to both sides, it becomes $36^2 - 36x - 36y + xy = 36^2$ or $(36 - x)(36 - y) = 36^2$. It follows that $36 - x$ must be a divisor d of 36^2 and then $36 - y = 36^2/d$. In other words, (x, y) solves the equation if and only if $x = 36 - d$, $y = 36 - 36^2/d$, where d divides 36^2; except if $d = 36$ (which would give $x = y = 0$). Now $36^2 = 2^4 \cdot 3^4$ so that 36^2 has $(4 + 1)(4 + 1) = 25$ positive divisors, hence 50 divisors in all (positive and negative). Discarding the divisor 36, we are left with 49 solutions. **The correct solution is** $\boxed{\textbf{49}}$

19.* The numbers from 1 upwards are written consecutively

$$123456789101112131415\ldots$$

What digits will be in positions 2,000,000 and 2,000,001? Your answer, to be written on the answer sheet, should consist of two digits; for example 23 (Hint: 23 is **not** the right answer).

Solution. There are 9 one digit numbers, 90 two one digit numbers, 900 three digit numbers, and so forth. We can make a table.

Digits written out	Last number written down
9	9
$9 + 2 \times 90 = 189$	99
$189 + 3 \times 900 = 2,889$	999
$2,889 + 4 \times 9,000 = 38,889$	9,999
$38,889 + 5 \times 90,000 = 488,889$	99,999

The next row in the table takes us past two millions so that the digits we are looking for appear in a six digit number, one that is $2,000,000 - 488,889 = 1,511,111$ digits away. Now $1,511,111 = 251,851 \times 6 + 5$ so we are $251,851$ numbers past 99,999; and because the remainder is 5 we have to go five digits into $100,000 + 251,851 = 351,851$. **The correct solution is** $\boxed{\mathbf{51}}$

20.* Solve the following system of equations.

$$\begin{array}{rcrcrcr} 6x & + & 6y & + & 6z & = & 1 \\ 36x^2 & + & 36y^2 & + & 36z^2 & = & 325 \\ 9x^2yz & + & 9xy^2z & + & 9xyz^2 & = & 5 \end{array}$$

The solutions are all rational numbers. Write the solution satisfying $x < y < z$ in the appropriate place on the answer sheet. Rational numbers MUST be written in the form a/b where a, b are integers without a common divisor other than 1 and $b > 0$ (except, of course, if $b = 1$; in which case one writes simply a).

Solution. The fact that the left hand side of every equation is symmetric in x, y, z might suggest something. Let $P(x) = x^3 - ax^2 + bx - c$ be the monic polynomial of degree three having x, y, z as zeroes. Then by the so called relations of Viète,

$$\begin{aligned} a &= x + y + z = \frac{1}{6} \\ b &= xy + yz + zx = \frac{1}{2}\left((x+y+z)^2 - (x^2+y^2+z^2)\right) = \frac{1}{2}\left(\frac{1}{36} - \frac{325}{36}\right) = -\frac{9}{2} \\ c &= xyz = \frac{x^2yz + xy^2z + xyz^2}{x+y+z} = \frac{6 \cdot 5}{9} = \frac{10}{3}. \end{aligned}$$

Thus (after multiplying by 6) x, y, z satisfy $6x^3 - x^2 - 27x - 20 = 0$. Knowing that the roots are rational one knows that the roots are of the form $x = m/n$ where m, n are integers with m dividing 20 and n dividing 6. That limits the possibilities. But it is also easy to guess one of the roots, namely $x = -1$. Dividing out by $x+1$ one gets the quadratic equation $6x^2 - 7x - 20 = 0$ which has solutions $-4/3$ and $5/2$. In order **The correct solution is** $\boxed{x = -1, y = -4/3, z = 5/2.}$