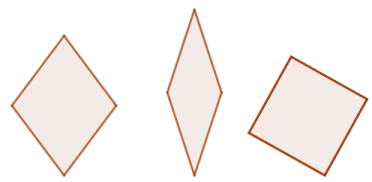
Question 1 – a

A rhombus with diagonals of length 6 and 12 and a rectangle having one side of length 4 have the same area. Find the length of the other side of the rectangle. Enter the answer in the appropriate place on the answer sheet and pass the sheet to your next teammate.

A *rhombus* is a quadrilateral with four equal sides, 3 rhombi are pictured below.



Solution. The rhombus can be described as the quadrilateral one obtains by joining two congruent isosceles triangles of base 6 and altitude $12/2 = 6$ at their bases. The area of the rhombus is therefore $2 \times \left(\frac{1}{2}6 \times 6\right) = 36$. Since one side of the rectangle has length 4, the other sides have length $\boxed{9.}$

Question 1 – b

Let k be the solution of part a) Find the largest number x such that $k + x$ equals $\frac{52}{x}$. Enter the answer in the appropriate place on the answer sheet and pass the sheet to your next teammate.

Solution. We have $k = 9$ so we have to find the largest x such that $9 + x = \frac{52}{x}$. Multiplying by x and rearranging $x^2 + 9x - 52 = 0$. This is a quadratic equation with solutions

$$x = \frac{-9 \pm \sqrt{81 + 208}}{2} = \frac{-9 \pm 17}{2}.$$

The larger one of the two solutions is $(-9 + 17)/2 = 4$.

Question 1 – c

Let k be the answer to part b) Suppose we have a bag containing 5 red and 5 white marbles. If we choose k marbles at random (without replacement), find the probability that at least one of the chosen marbles is red. This probability can be expressed in the form $\frac{m}{n}$ where m, n are positive integers with no common divisor other than 1. Enter $m + n$ in the appropriate place on the answer sheet and pas the answer sheet to your next teammate.

Solution. It is probably easier to consider the opposite situation, none of the chosen marbles is red. From part c, $k = 4$. The number of possible choices of 4 marbles from a set of 10 is $\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} = 210$. To get no red marble we have to choose all the marbles among the white ones; the number of possible choices is $\binom{5}{4} = 5$. The probability of selecting no red marble is $\frac{5}{210} = \frac{1}{42}$. The probability of the opposite event is $\frac{41}{42}$ so that $m = 41, n = 42$ and $m + n = \boxed{83.}$

Question 1 – d

Let k be the solution to part c) and let $m = k - 3$. If x and y are real numbers, what is the least value of $x^4 + y^4$ if x, y satisfy $x^2 + y^2 = m$. Enter your answer in the appropriate place on the answer sheet and hand the sheet to a runner.

Solution. From part c), $k = 83$, $m = k - 3 = 80$, so that $x^4 + y^4 = (x^2 + y^2) - 2x^2y^2 = 80^2 - 2x^2y^2$. The least value of $x^y + y^4$ will thus be assumed when x^2y^2 is largest. Since $x^2 + y^2 = 80$, this happens when $x^2 = y^2 = 80/2 = 40$. Thus

$$x^4 = y^4 = 40^2 = 1600, \quad x^4 + y^4 = \boxed{3200.}$$

Question 2 – a

The arithmetic mean of 4 numbers is k. If 8 is added as a fifth number, the mean is 24. What is k? Enter the answer in the appropriate place on the answer sheet and pass the sheet to your next teammate.
Solution. If the sum of the numbers is s, then $s = 4k$. Once we add 8 to the numbers, the mean is $24 = (4k + 8)/5$. Solving for k we get $k = \boxed{28.}$

Question 2 – b

Let k be the solution of part a) The roots x_1, x_2 of the equation

$$x^2 - ax + 4 = 0$$

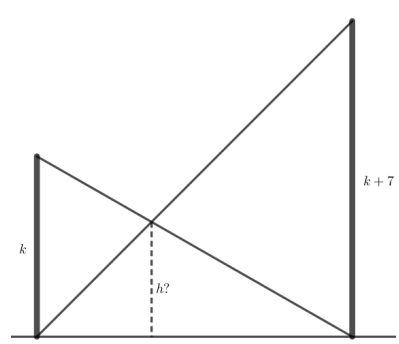
are positive and satisfy $x_1^2 + x_2^2 = k$. What is a? Enter the answer in the appropriate place on the answer sheet and pass the sheet to your next teammate.
Solution. From part a), $k = 28$. We have $x_1 + x_2 = -a$, $x_1x_2 = 4$ so that $a^2 = (x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 = 28 + 8 = 36$, hence $a = \pm 6$. We need $a = +6$ for positive solutions, so the answer is $a = \boxed{6}$.

1 Question 2 – c

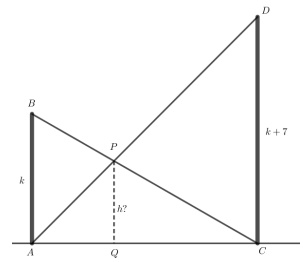
Let k be the answer to part b) Determine the last two digits of $(k + 1)^{2019}$. Write your answer, a two digit number, on the answer sheet and pass the sheet to your next teammate.
Solution. From part b, $k = 6$ so $k + 1 = 7$. If we start multiplying by 7 ignoring all but the last two digits we get 7, 49, 43, 01; after which things repeat. Since 2016 is divisible by 4, the last two digits of 7^{2016} will be 01; the last two of 2^{2017} will be 07, of 2^{2018}, 49, and finally we get $\boxed{43}$ as the last two digits of 7^{2019}.

Question 2 – d

Let k be the answer to part c) Two vertical posts, one of length k, the other one of length $k + 7$ stand some distance apart. A cable stretches from the top of each post to the base of the other one. Find the height h equals a number m/n, where m, n are integers with no common factor other than 1. Enter $m + n$ in the appropriate place in the answer sheet and hand the sheet to a runner.



Solution. Let us give some names to some of the points in the picture.



Notice that $\triangle ADC \sim \triangle APQ$ so that $\dfrac{|PQ|}{|DC|}$; that is, $\dfrac{h}{50} = \dfrac{|AQ|}{|AC|}$. Similarly, $\dfrac{h}{43} = \dfrac{|QC|}{|AC|}$. Using that $|QC| = |AC| - |AQ|$, we get

$$\frac{h}{43} = \frac{|AC| - |AQ|}{|AC|} = 1 - \frac{|AQ|}{|AC|} = 1 - \frac{h}{50}.$$

We cannow solve for h to get $h = \frac{2150}{93}$. Thus $m = 2150, n = 93$ and $m + n = \boxed{2243.}$

Question 3 – a

Janine rides her bike along a straight road from a point A to a point B. She does the first half of the trip pedaling at 4 miles per hour. How fast does she have to pedal in the second half so that her average speed for the whole trip is 6 miles per hour? Write your answer on the appropriate spot on the answer sheet (do **not** enter mph, just the number) and pass the sheet to your next teammate.

Solution 1. Since the length of the road does not seem to matter, we could give it a convenient length, say 4 miles so that Janine got to the half way spot in 1/2 an hour. If now v is her velocity for the second half then the the time T it takes her to complete it is $T = 2/v$. The total time is thus $\dfrac{1}{2} + \dfrac{2}{v}$. The average speed in mph is the total length of the trip (4) divided by the total time so that we have

$$6 = \frac{4}{\left(\frac{1}{2} + \frac{2}{v}\right)};$$

solving for v we get $v = \boxed{12}$.

Solution 2. If we know that the average speed is the *harmonic average* of the two speeds, we can obtain v by solving for v the equation

$$6 = \frac{2 \cdot 4 \cdot v}{4 + v}.$$

Question 3 – b

Let k be the answer to part a) The largest solution of the equation

$$x^3 + \left(\frac{k-2}{2}\right)x^2 + x = \frac{k}{4}$$

has the form $a + \sqrt{b}$, where a and b are integers. Determine $a + b$. Enter the answer in the appropriate place on the answer sheet and pass the sheet to your next teammate.

Solution. The correct value of k is $k = 12$ so the equation is $x^3 + 5x^2 + x - 3 = 0$. It is not too hard to guess that $x = -1$ is a solution. If we divide by $x + 1$ we see that

$$x^3 + 5x^2 + x - 3 = (x + 1)(x^2 + 4x - 3).$$

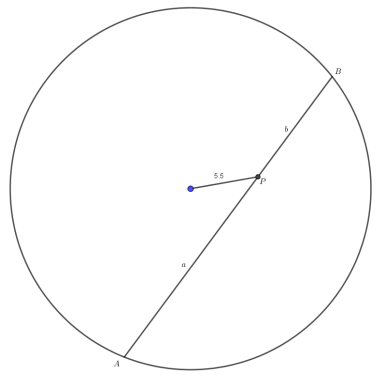
The quadratic equation $x^2 + 4x - 3 = 0$ has the solutions

$$\frac{-4 \pm \sqrt{28}}{2} = -2 \pm \sqrt{7}.$$

Of the three roots $-1, -\sqrt{7} - 2, \sqrt{7} - 2$, the largest is $\sqrt{7} - 2$ so $a = -2, b = 7$ and the answer to enter is $-2 + 7 = \boxed{5}$.

Question 3 – c

Let k be the answer to part b. A point P at a distance of 5.5 from the center of a circle divides the chord AB into two parts of lengths $a = 2k$ and $b = 3(k + 1)$. Determine the diameter of the circle. Write your answer in the appropriate place on the answer sheet and pass the sheet to your next teammate.



Solution. $k = 5$ so that $a = 10$ and $b = 18$. We use the concept of *power of a point.* The power of P with respect to the circle is $ab = 180$. If we draw the diameter through P and measure the power of P using the diameter, the power works out to $(r + 5.5)(r - 5.5) = r^2 - 30.25 = r^2 - \frac{121}{4}$, where r is the radius of the circle. Equating the two powers, $180 = r^2 - \frac{121}{4}$, hence

$$r^2 = \frac{841}{4} = \frac{29^2}{4}.$$

Thus $r = 29/2$ and the diameter is $\boxed{29.}$

Question 3 – d

Let k be the answer to part c) What is the smallest integer L so that the equation

$$kx + (2k + 1)y = L$$

has a solution (x, y) with $x > 0, y > 0$? Write your answer on the answer sheet, raise your hand with the sheet so a runner picks it up.

Solution. From c), $k = 29$ so the equation is $29x + 59y = L$. Using the fact that $29 \cdot (-2) + 59 \cdot 1 = 1$ we see that $x = -2L, y = L$ solves the equation and that means that the general solution can be written in the form $x = -2L + 59k$, $y = L - 29k$ where $k = 0, \pm 1, \pm 2, \ldots$. For $x > 0, y > 0$ we need $59k > 2L$ or $k > 2L/59$, and $L > 29k$ or $k < L/29$. It now reduces to figuring out which is the smallest integer L so there is an integer k with

$$\frac{2L}{59} < k < \frac{L}{29}.$$

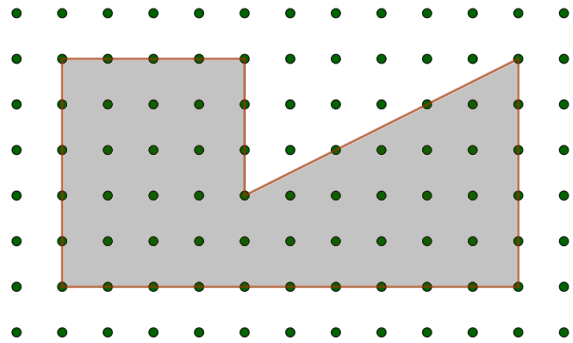
For such an integer to exist we must have

$$1 \leq \frac{L}{29} - \frac{2L}{59} = \frac{L}{29 \cdot 59}$$

so $L \geq 29 \cdot 59$. If $L = 29 \cdot 59$, then the inequality for k becomes $58 < k < 59$; and there is no such k. Increasing L by one; i.e., taking $L = 29 \cdot 59 + 1 = 1712$, the inequality becomes $58 + \frac{2}{59} < k < 59 + \frac{1}{29}$, clearly satisfied by $k = 59$. The answer is $\boxed{1712.}$

Question 4 – a

The distance between a gridpoint and its horizontal or vertical neighbor is 1 foot. Find the area of the shaded figure. Enter the number of square feet (just the number) in the appropriate place of the answer sheet and pass the sheet to your next teammate.



Solution. The figure can be divided into a 4×5 rectangle and a trapezoid of bases of lengths 2 and 5, and altitude 6. The total area is thus

$$(4 \times 5) + \frac{1}{2}(2+5) \times 6 = 41$$

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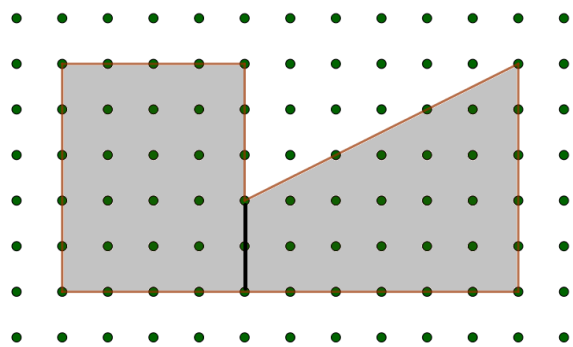

Question 4 – b

Let k be the answer to part a) and let $n = 10k$. Find the number of trailing zeros of $k!$. Enter the answer in the appropriate place on the answer sheet and pass the sheet to your next teammate.

(If n is a positive integer one defines $n!$ to be the product if all positive integers from 1 to n:

$$1! = 1,\ 2! = 2 \cdot 1 = 2,\ 3! = 3 \cdot 2 \cdot 1 = 6; etc.$$

The number of trailing zeros is the number of zeros at the end. Thus, for example, $5! = 120$ has one trailing zero. A number like 30600 has 2 trailing zeros.)

Solution. From part a), $k = 41$ so $n = 410$. The number of trailing zeros in $410! = 410 \cdot 409 \cdot 408 \cdots 3 \cdot 2 \cdot 1$ equals the highest power of 10 that divides 410!; that is, the number ℓ such that 410! is a multiple of 10^ℓ but not of $10^{\ell+1}$. Now $10 = 2 \times 5$ and there are many more factors of 2 appearing in 410! than factors of 5, so the number ℓ we are looking for is also the number such that 5^ℓ but not $5^{\ell+1}$ divides 410!. We now can count as follows: Every multiple of 5 contributes one zero; multiples of 25 contribute an additional zero, multiples of 125 one additional 0. So we have:

$$\text{\# of multiples of 5:} \qquad \lfloor \frac{410}{5} \rfloor = 82$$

$$\text{\# of multiples of 25:} \qquad \lfloor \frac{410}{25} 5 \rfloor = 16$$

$$\text{\# of multiples of 5:} \qquad \lfloor \frac{410}{125} \rfloor = 3$$

Adding up, the answer is $\boxed{101.}$

Question 4 − c

Let k be the answer from part b) and let $s = 2\lfloor \sqrt{k} \rfloor$; twice the largest integer $\leq \sqrt{k}$. If A, B are angles in the first quadrant with $\cos A = \dfrac{s-8}{s-7}$, $\tan B = \dfrac{s-5}{s-12}$, evaluate $\sin(A+B)$. Your answer should be of the form m/n with m, n integers with no common divisor but 1; $n > 0$. Enter $m+n$ on the answer sheet and pass the sheet to your next teammate.

Solution. From part b), $k = 101$ so $s = 20$; $\cos A = \dfrac{12}{13}$, $\tan B = \dfrac{15}{8}$. From this we obtain easily that $\sin A = \dfrac{5}{13}$, $\sin B = \dfrac{15}{17}$, $\cos B = \dfrac{8}{17}$; using that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ we see that $\sin(A+B) = \dfrac{220}{221}$. Then $m+n = \boxed{441}$.

Question 4 − d

Let k be the solution to part c) Let $p = k/9$. The equation

$$px^3 - bx^2 + 138x - 40 = 0$$

has three real roots in arithmetic progression. Determine b and solve the equation. If d is the difference of the progression, r is the largest root, then your answer should be $b + d + r$ written in the form m/n, where m and n are integers with no common divisor other than 1, $n > 0$.

The *difference* of an arithmetic progression $x_1, x_2, \ldots, x_m$ is the number d such that $x_j - x_{j-1} = d$ for $j = 2, \ldots, m$.

Solution. From c), $k = 441$ so that $p = 49$. It is probably best to write the roots in the form $a - d, a, a + d$. Then, by Viète's relations,

$$\begin{aligned} 3a &= (a-d) + a + (a+d) = \frac{b}{49}, \\ 3a^2 - d^2 &= a(a-d) + a(a+d)(a-d)(a+d) = \frac{138}{49}, \\ a^3 - a\delta^2 &= a(a-d)(a+d) = \frac{4}{0}49. \end{aligned}$$

From the second and third equation we can solve to get $98a^3 - 138a + 40 = 0$. One solution is easy to guess, namely $a = 1$. If we are pressed for time we might just go with it. Then $b = \frac{3}{49}$, $d^2 = 3 - \frac{138}{49} = \frac{9}{49}$ so that $d = \frac{3}{7}$ and the roots are $\frac{4}{7}, 1, \frac{10}{7}$ and it all works out. We don't need to investigate any further. The answer is

$$\frac{3}{49} + \frac{3}{7} + \frac{10}{7} = \boxed{\frac{94}{49}.}$$

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2 Question 5 – a

Find the sum of all integers k, $1 \le k \le 8$, for which there exists an integer $n \ge k$ such that $\sqrt{n-k} + \sqrt{n+k}$ is rational.

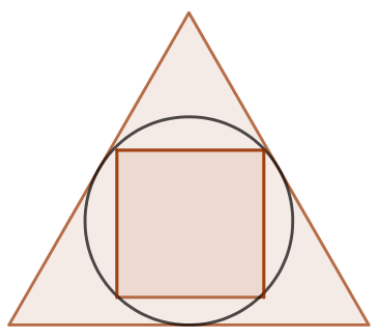
Solution. Suppose $\sqrt{n-k} + \sqrt{n+k} = r$; r a rational number. Squaring, and doing some rearranging, $\sqrt{n^2 - k^2} = \dfrac{r^2 - 2n}{2}$. The square root of an integer is rational if and only if the integer is a perfect square; thus, there has to exist an integer m such that $n^2 - k^2 = m^2$. Thus either $n = k$ (and $m = 0$) or there is m such that k, m, n is a pythagorean triple. The possibilities with $m \neq 0$ are

k	m	n	$\sqrt{n-k} + \sqrt{n+k}$
3	4	5	$\sqrt{2} + \sqrt{8}$
4	3	5	$\sqrt{1} + \sqrt{9} = 4$
6	8	10	$\sqrt{4} + \sqrt{16} = 6$
8	6	10	$\sqrt{2} + \sqrt{18}$
8	15	17	$\sqrt{9} + \sqrt{25} = 8$
7	14	25	$\sqrt{18} + \sqrt{32}$

We see that we get rational numbers (integers) for $k = 4, 6, 8$. If $k = n$ we need to have $\sqrt{2k}$ rational and in the range $2 \le k \le 8$ this only happens if $k = 2$ or 8. So the complete list is $k = 2, 4, 6, 8$ and the answer is $\boxed{20.}$

Question 5 – b

Let k be the solution of part a) and let $\ell = k/3$. A circle is inscribed in an equilateral triangle of side ℓ and a square is inscribed in the circle.

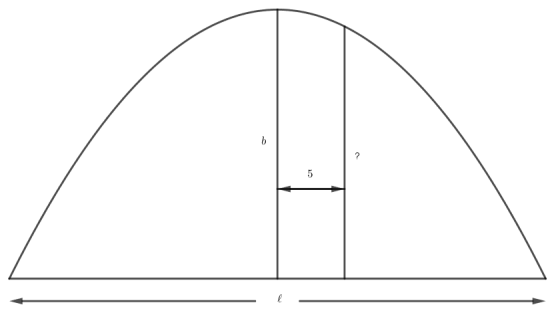


Find the area of the square. Enter the answer in the appropriate place on the answer sheet and pass the sheet to your next teammate.

Solution. From a), $k = 20$ so that $\ell = 20/3$. The altitude of an equilateral triangle of this side is $h = 10/\sqrt{3}$. The inscribed circle has its center at a distance from the base equal to $\frac{1}{3}h$, thus its radius (which equals this distance) is $10/(3\sqrt{3})$. A square inscribed in a circle of radius r has sides of length $\sqrt{2}\,r$ so, in this case, $\sqrt{1}0\sqrt{2}/(3\sqrt{3})$. Squaring we get the area equal to $\boxed{\dfrac{200}{27}.}$

Question 5 – c

Let k be the solution to part c) and let $b = k^2$, $\ell = 30 + 5k$. A parabolic arch has a height of b and a span of ℓ. Find the height at a point at distance 5 from the center, Enter the answer in the appropriate place on the answer sheet and pass the sheet to your next teammate.

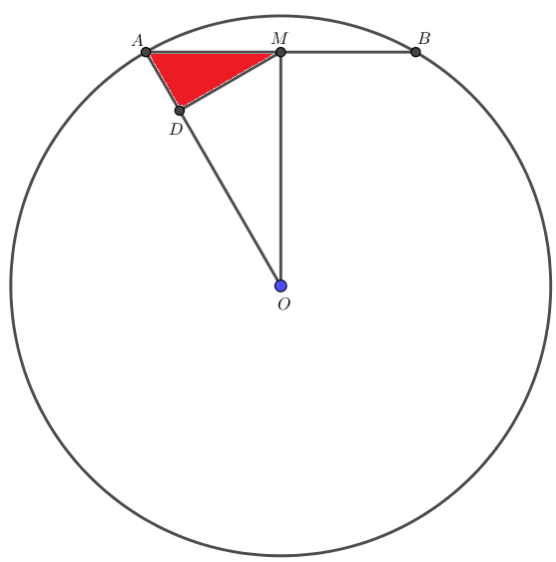


Remember that pictures are not necessarily to scale.
Solution. The correct solution involves some horrendous numbers. With $k = 200/27$, one gets that the equation of the parabola is $y = b - ax^2$, where $b = 40,000/729$, $a = 4b/\ell^2$ $(\ell 1810/27)$ so $a = 1600/32761$ and finally

$$y(5) = \boxed{\frac{1,281,280,000}{23,882,769}}.$$

Question 5 – d

Let $r = k/5$ where k is the solution to part d) In a circle with center O and radius r a chord, AB also of length r is drawn. A perpendicular segment from O meets AB at M. From M a perpendicular segment meets OA at$D$. Find the area of the triangle MDA. Your answer should be in the form $\dfrac{a\sqrt{b}}{c}$ where b is a, b, c are positive integers, a, c have no factor in common (other than 1) and b is square free.



Solution. It is best to work out a formula; use the given value of r only at the very end. Because AM is half of $AO = r$, the triangle AOM is half an equilateral triangle; the same holds for AMD. Then $|AD| = r/4$, $MD = r\sqrt{3}/4$ so that $[MDA] = \dfrac{r^2\sqrt{3}}{3}2$. Using the value of $r = k/5$ we get

$$[MDA] = \boxed{\frac{2,052,098,048,000,000\sqrt{3}}{570,386,655,107,361}}.$$