MATH DAY 2018 at FAU

Competition A–Individual

THE QUESTIONS WITH SOLUTIONS

1. The sum of a 4-digit number and its four digits is 2018. What is the sum of digits of this number?

(A) 7 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 13

Solution. Suppose the number is $x = 1000a + 100b + 10c + d$ so that

$$1001a + 101b + 11c + 2d = 2018.$$

Clearly $1 \leq a \leq 2$. Assuming $a = 1$ we must have $b = 9$ and $c = 9$; otherwise

$$1001a + 101b + 11c + 2d \leq 1001 + 909 + 88 + 18 = 2016 < 2018.$$

However, with $a = 1, b = c = 9$ we get

$$1001a + 101b + 11c + 2d = 2009 + 2d,$$

which cannot equal 2018 since it would require $2d = 9$. Thus $a = 2$. Then it is not too hard to see that one must have $b = 0$ to avoid overshooting. If $c \geq 2$ we also will be overshooting so $c = 0$ or 1. But $c = 1$ implies

$$1001a + 101b + 11c = 2013$$

requiring $2d = 5$; impossible. Thus $c = 0$ and we see that we must have $2d = 16$, thus $d = 8$ and the number is 2008. The correct solution is **C.**

2. Find $m + n$, where m, n are two positive integers such that $2018 = m^2 + n^2$.

(A) 56 **(B)** 64 **(C)** 76 **(D)** 80 **(E)** 86

Solution. Because the square of even numbers is divisible by four and 2018 is not divisible by 4, m and n must be odd. The last digits of m and n must thus be in the list $1, 5, 9$ and the only way in which $m^2 + n^2$ can have a last digit of 8 is if both m^2 and n^2 end in 9, implying that the last digits of m and n are either 3 or 7. Now $47^2 = 2209 > 2018$ so that limits considerably the possibilities for m and n. Working downward by trial and error we see that

$$2018 - 43^2 = 169 = 13^2$$

so that $m = 43, n = 13$ is a choice with $m + n = 56$. The correct solution is **A.**

3. In a class of 20 students on grading a test, the teacher found that if she graded everybody's test but Marvin's, the average came out to 75 for the class. After grading Marvin's test, the average dropped to 74. What was Marvin's grade on this test.

(A) 74 **(B)** 70 **(C)** 63 **(D)** 55 **(E)** 50

Solution. If N is the sum of the scores obtained by all students except Marvin then $N = 19 \cdot 75 = 1425$. Let x be Marvin's score. The average score of all 20 students is

$$74 = \frac{N + x}{20} = \frac{1425 + x}{20}$$

from which

$$x = 74 \cdot 20 - 1425 = 55.$$

The correct solution is **D.**

4. The famous trio of Abelard, Belinda and Consuelo got a number of vinyl records from one of their performances. Consuelo got 4 times as many records as Belinda, Belinda got 2 and a half times; that is, 2.5 times, as many records as Abelard. Consuelo decided to give a number of her records to Abelard and Belinda so that all ended by having the same number of records. What fraction of her records did Consuelo give Belinda?

$$\textbf{(A)}\ \frac{1}{20} \qquad \textbf{(B)}\ \frac{1}{10} \qquad \textbf{(C)}\ \frac{1}{5} \qquad \textbf{(D)}\ \frac{1}{4} \qquad \textbf{(E)}\ \frac{1}{2}$$

Solution. Lets call the performers A, B, and C. We may assume that at first A has 2 records, B has 5 and C has 20. Say C gives a records to A and b records to B. Then we should have

$$2 + a = 5 + b = 20 - a - b.$$

From this we see that $a = 7, b = 4$. The fraction of C's records going to B is $4/20 = 1/5$. The correct solution is **C.**

5. Find the sum of the first 2018 terms of the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots.$$

$$\textbf{(A)}\ 16246 \qquad \textbf{(B)}\ 32288 \qquad \textbf{(C)}\ 48436 \qquad \textbf{(D)}\ 64576 \qquad \textbf{(E)}\ 85472$$

Solution. It helps to know that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad \text{and} \quad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let n be the last integer that will appear n times in the first 2018 terms of the sequence. Then

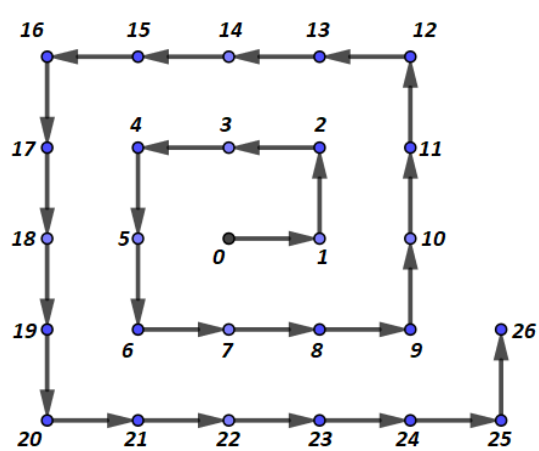
$$1 + 2 + \cdots + n \leq 2018 < 1 + 2 + \cdots + (n+1); \quad \text{i.e.,} \quad \frac{n(+1)}{2} \leq 2018 < \frac{(n=1)(n+2)}{2}.$$

By trial and error or otherwise, one sees that $n = 63$ and since $63 \cdot 64/2 = 2016$, there are two additional terms of 64 to be counted. The sum is

$$1^2 + 2^2 + 3^2 + \cdots + 63^2 + 2 \cdot 64 = \frac{63 \cdot (63+1) \cdot (2 \cdot 63 + 1)}{6} + 2 \cdot 64 = 85472.$$

The correct solution is **85472.**

6. The points with integer coordinates in the xy-plane are labelled by integers 0, 1, 2, 3, 4, ... counter-clockwise, beginning with the origin $(0, 0)$, then $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 1)$, ...,



What are the coordinates (x, y) of the point labelled 2018? Write x and y in the appropriate spot on the answer sheet.

Solution. Here are some obvious labels: for every positive integer n,
(i) the point (n, n) has label $(2n-1)(2n)$,
(ii) the point $(-n, n)$ has label $(2n)^2$,
(iii) the point $(-n, -n)$ has label $2n(2n+1)$,
(iv) the point $(n+1, -n)$ has label $(2n+1)^2$.With $n = 22$, we have the labels 1980 and 2025 for the points $(-22, -22)$ and $(23, -22)$. Therefore, 2018 is the label of $(16, -22)$. The correct solution is $x = 16, y = -22$**.**

7. Let n be the number of distinct real solutions of the equation

$$x^2 - |x-3| - 3 = 0$$

and let $x_1, \ldots, x_n$ be the roots. Find $n + x_1 + \cdots + x_n$. (If there is only one root, the sum to find is $1 + x_1$; if there are two roots, it is $2 + x_1 + x_2$; if three, $3 + x_1 + x_2 + x_3$; etc.)

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution. Case 1. $x \geq 3$: $x^2 - (x-3) - 3 = 0$; $x^2 - x = 0$; $x = 0,\ 1$.
None of these satisfies $x \geq 3$.
Case 2: $x < 3$: $x^2 + (x-3) - 3 = 0$; $x^2 + x - 6 = 0$;$(x-2)(x+3) = 0$; $x = -3,\ 2$. Both roots are admissible and $n = 2$; $2 + (-3) + 2 = 1$. The correct solution is **B.**

8. Two students miscopied the quadratic equation $x^2 + bx + c = 0$ that their teacher wrote on the board. Aaron copied b correctly but c incorrectly; his equation has roots 4 and 5. Betty copied c correctly but b incorrectly; her equation has roots 2 and 4. What are the roots of the original equation?

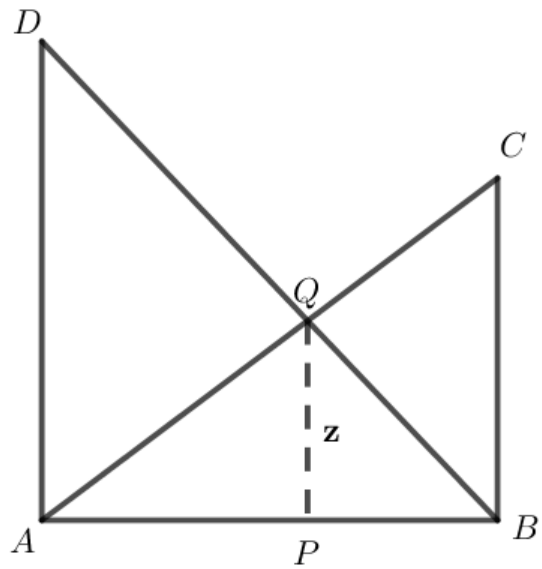
(A) 1 and 8 **(B)** 2 and 5 **(C)** 3 and 4 **(D)** 4 and 6 **(E)** 5 and 7

Solution. Aaron's equation: $0 = (x-4)(x-5) = x^2 - 9x + 20$; so $b = -9$.
Betty's equation: $0 = (x-2)(x-4) = x^2 - 6x + 8$; so $c = 8$.The correct equation is $x^2 - 9x + 8 = 0$; that is, $(x-1)(x-8) = 0$. The correct solution is **A.**

9. The positive integers that can be expressed as the sum of 21 consecutive (not necessarily positive) integers are listed in increasing order. Determine the 21-st integer in the list. Write your answer directly on the answer sheet.

Solution. The sum of the 21 consecutive integers beginning with k is $\frac{21}{2}(k + k + 20) = 21(k + 10)$. This is positive if $k \geq -9$. Therefore, the 21-st sum in the list corresponds to $k = -9 + 20 = 11$, and is $21 \cdot 21 = 441$. The correct solution is **441.**

10. Two walls at A and B are 40 feet apart. A ladder 50 feet long has its lower end at the base of the wall at A and its upper end at C, against the Wall at B. Another ladder 58 feet long has its lower end at the base of the wall at B and its upper end D against the wall at A. How high above the ground is the point Q where the ladders intersect?



(A) 16.5 **(B)** 17 **(C)** 17.5 **(D)** 18 **(E)** 18.5

Solution. Let $z = |PQ|$ be the height to be found. By the theorem of Pythagoras,

$$|BC| = \sqrt{50^2 - 40^2} = 30 \quad \text{and} \quad |AD| = \sqrt{58^2 - 40^2} = 42.$$

By similarity of triangles

$$\frac{z}{|BC|} = \frac{|AP|}{|AB|}, \quad \text{thus} \quad z = \frac{3}{4}|AP|.$$

Similarly,

$$\frac{z}{|AD|} = \frac{|PB|}{|AB|}, \quad \text{thus} \quad z = \frac{21}{20}|AP|.$$

It follows that

$$\frac{4}{3}z + \frac{20}{21}z = |AP| + |PB| = 40$$

from which $z = 17.5$. The correct solution is **C.**

11. For how many integers x, $1 \le x \le 100$ does it hold that $x^3 + 2x^2 + 2x + 8$ is divisible by 7. As a help, $x = 2$ is one such integer. The expression is also divisible by 7 for $x = -1$, but $x = -1$ is not in the range that is being considered. Write your answer directly onto the answer sheet.

Solution. Let $f(x) = x^3 + 2x^2 + 2x + 8$. Then $28 = f(2) = 2^3 + 2 \cdot 2^2 + 2 \cdot 2 + 8$ so that

$$f(x) - 28 = x^3 - 2^3 + 2(x^2 - 2^2) + 2(x - 2) = (x - 2)(x^2 + 4x + 10).$$

Let $g(x) = x^2 + 4x + 10$. The information that the expression was divisible by -1 now comes in handy; $g(-1) = 7$; we can write $7 = g(-1) = (-1)^2 + 4 \cdot (-1) + 10$ and

$$g(x) - 7 = x^2 - 1 + 4(x + 1) = (x + 1)(x + 3).$$

Thus

$$f(x) = 28 + 7(x-2) + (x-2)(x+1)(x+3)$$

and since 7 is prime, $f(x)$ will be divisible by 7 if and only if one of $x-2, x+1, x+3$ is divisible by 7. That means that x has to have one of the following forms for some integer k in the specified range:

$$\begin{aligned} x &= 7k+2, & 0 \le k \le 14, \\ x &= 7k-1, & 1 \le k \le 14, \\ x &= 7k-3, & 1 \le k \le 14. \end{aligned}$$

In particular, x has to be congruent to one of 2, 6, or 4 modulo 7, and no number can be congruent to two of these, so that we have a total of $15+14+14$ values of x in the range 0–100 for which $f(x)$ is divisible by 7. The correct solution is **43.**

12. The lengths of the sides of a triangle are three consecutive integers. If the largest angle is twice the smallest one, determine the side lengths.

(A) $3, 4, 5$ **(B)** $4, 5, 6$ **(C)** $5, 6, 7$ **(D)** $6, 7, 8$ **(E)** $7, 8, 9$

Solution. Let ABC be a triangle with sides $BC = x-1$, $CA = x$, and $AB = x+1$. Thus, $A < B < C$. If $C = 2A$, $\sin C = \sin 2A = 2\sin A \cos A$, and

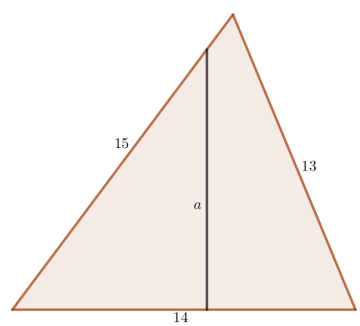
$$\frac{\sin C}{\sin A} = 2\cos A.$$

This means

$$\frac{x+1}{x-1} = 2 \cdot \frac{x^2 + (x+1)^2 - (x-1)^2}{2x(x+1)} = \frac{x^2+4x}{x(x+1)} = \frac{x+4}{x+1}.$$

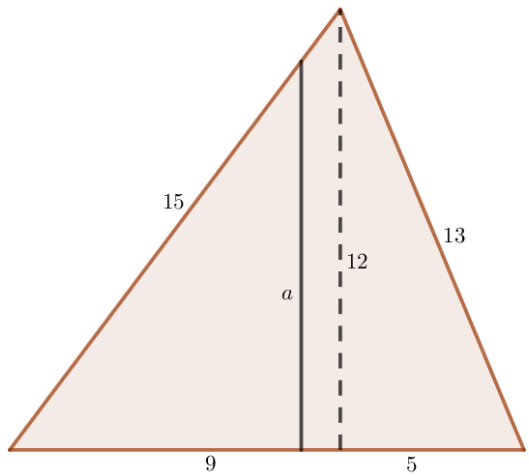
Therefore, $(x+1)^2 = (x-1)(x+4) \implies 2x+1 = 3x-4 \implies x = 5$. The correct solution is **B.**

13. A triangle has sides 13, 14, and 15. A line perpendicular to the side of length 14 divides the interior of the triangle into two regions of equal area. Find the length a of the segment of the perpendicular that lies within the triangle.



(A) $4\sqrt{7}$ **(B)** $7\sqrt{3}$ **(C)** $\dfrac{13\sqrt{7}}{3}$ **(D)** $\dfrac{5\sqrt{13}}{2}$ **(E)** NA

Solution. We can find the area of the triangle using Heron's formula; we get that the area is 84. This makes the altitude shown as a dashed line in the picture below have length equal to $2 \cdot 84/14 = 12$. Alternatively a triangle with sides 13, 14, 15 is made up of two right triangles, of sides 12, 9, 15, and 12, 5, 13, joined along the sides of length 12.

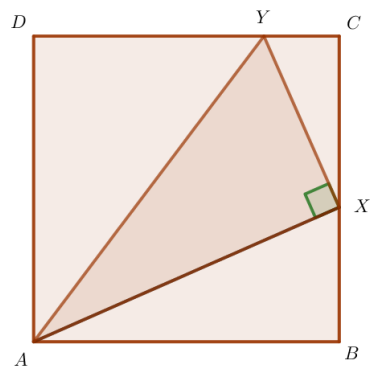


A segment perpendicular to the side of length 14 bisecting the area must be in the interior of the $12-9-15$ triangle. If its length is a, then by similarity of triangles

$$\left(\frac{a}{12}\right)^2 = \frac{42}{54} \implies a = 4\sqrt{7}.$$

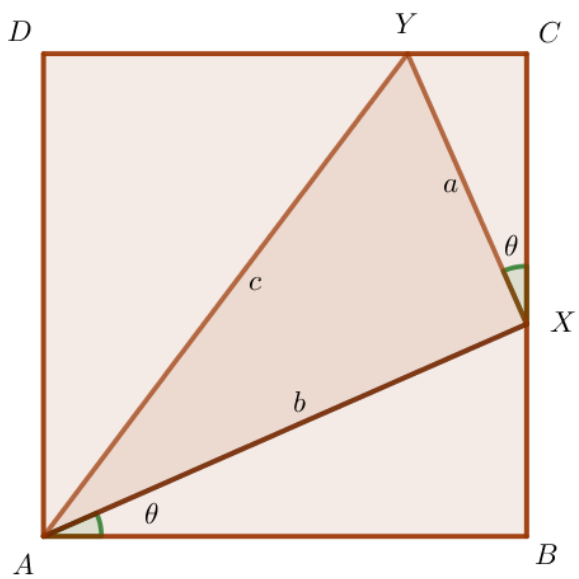
The correct solution is **A.**

14. X and Y are points on the sides BC and CD of a square $ABCD$ such that triangle AXY is a right triangle with $AX = 4$, $XY = 3$ and $AY = 5$, Find the length of each side of the square.



(A) $\frac{14}{\sqrt{15}}$ **(B)** $\frac{16}{\sqrt{17}}$ **(C)** $\frac{18}{\sqrt{23}}$ **(D)** $\frac{19}{3\sqrt{5}}$ **(E)** $\frac{20}{\sqrt{19}}$

Solution. Let $a < b < c$ be the sides of the right triangle and let $\theta = \angle XAB = \angle YXB$.



Then

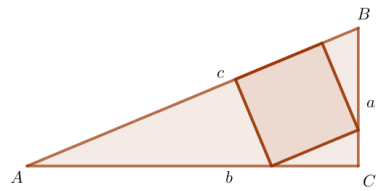
$$b\cos\theta = b\sin\theta + a\cos\theta \implies b\sin\theta = (b-a)\cos\theta.$$

From this, $\tan\theta = \frac{b-a}{b}$, $\cos\theta = \frac{b}{\sqrt{b^2+(b-a)^2}}$.

The length of a side of the square is $b\cos\theta = \dfrac{b^2}{\sqrt{b^2+(b-a)^2}}$.

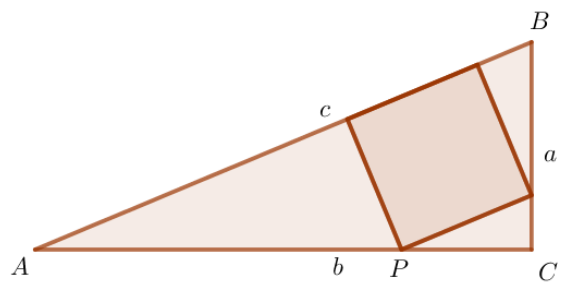
With $a = 3$ and $b = 4$, we have the length of each side of the square $= \frac{16}{\sqrt{17}}$.The correct solution is **B.**

15. A square is inscribed in the right triangle ABC of sides $a = 5, b = 12, c = 13$. The square has sides of length m/n where m, n are positive integers with no common divisor other than 1. Find $m - n$.



(A) 30 **(B)** 128 **(C)** 347 **(D)** 523 **(E)** NA

Solution. Let x be the common length of the sides of the square, let P be the vertex of the square on side AC and let $z = |AP|$.



Notice that the three triangles that form the complement of the square in the larger triangle are all similar to the large triangle. By similarity of triangles

$$\frac{z}{x} = \frac{c}{a}$$
$$\frac{b-z}{x} = \frac{b}{c},$$

thus

$$z = cx/a, \quad b - z = bx/c,$$

adding and solving for x we get

$$x = \frac{abc}{ab + c^2} = \frac{780}{229}.$$

Thus $m - n = 780 - 229 = 551$, which differs from the values in A, B, C, D. The correct solution is **E.**

16.* The equality

$$\left(\sqrt{3} - \sqrt{2}\right)^5 = \sqrt{a} - \sqrt{a-1}$$

holds for an integer a. Find a. Write your answer directly on the answer sheet.

Solution. Assuming the equality is true (and we are told it is), we also have

$$\frac{1}{\left(\sqrt{3} - \sqrt{2}\right)^5} = \frac{1}{\sqrt{a} - \sqrt{a-1}};$$

thus also

$$\frac{\left(\sqrt{3} + \sqrt{2}\right)^5}{\left(\sqrt{3} - \sqrt{2}\right)^5 \left(\sqrt{3} + \sqrt{2}\right)^5} = \frac{\sqrt{a} + \sqrt{a-1}}{\left(\sqrt{a} - \sqrt{a-1}\right)\left(\sqrt{a} + \sqrt{a-1}\right)};$$

that is,

$$\left(\sqrt{3} + \sqrt{2}\right)^5 = \sqrt{a} + \sqrt{a-1}.$$

From this,

$$\begin{aligned}
2\sqrt{a} &= \left(\sqrt{3}-\sqrt{2}\right)^5 + \left(\sqrt{3}+\sqrt{2}\right)^5 = \sum_{k=0}^{5}\binom{5}{k}(\sqrt{3})^{5-k}(-\sqrt{2})^k + \sum_{k=0}^{5}\binom{5}{k}(\sqrt{3})^{5-k}(\sqrt{2})^k \\
&= 2\left(\binom{5}{0}(\sqrt{3})^5 + \binom{5}{2}(\sqrt{3})^3(\sqrt{2})^2 + \binom{5}{4}(\sqrt{3})(\sqrt{2})^4\right) \\
&= 2(1\cdot 9 + 10\cdot 3\cdot 2 + 5\cdot 4)\sqrt{3} = 2\cdot 89\sqrt{3}.
\end{aligned}$$

Squaring and dividing by 4, $a = 89^2 \cdot 3 = 23763$. The correct solution is **23763.**

17.* Find the lattice point on the curve

$$4x^2 - 7xy + 3y^2 - 6x + 5y = 0$$

in the interior of the first quadrant. A lattice point in the interior of the first quadrant is a point whose coordinates are positive integers. If the point has coordinates (m, n) write the values of m, n directly on the answer sheet.

Solution. Rewrite the equation as

$$4x^2 - (7y+6)x + (3y+5)y = 0.$$

Solving, we have

$$x = \frac{1}{8}\left(7y + 6 \pm \sqrt{y^2 + 4y + 36}\right).$$

For x to be an integer, $y^2 + 4y + 36 = z^2$ for an integer z.
From this, $z^2 - (y+2)^2 = 32$,
$(z - y - 2)(z + y + 2) = 2^5$.
Since $z + y + 2 > 0$, $z - y - 2 > 0$ must be positive.
It follows that each of $z - y - 2$ and $z + y + 2$ is a power of 2.
Note that $z - y - 2$ must be even since $(z - y - 2) + (z + y + 2) = 2z$ is even.
The only possibilities are $(z - y - 2,\ z + y + 2) = (2,\ 2^4)$ or $(2^2, 2^3)$.
From these, $(z, y) = (9, 5)$ or $(6,\ 0)$.
Only the first is admissible, and

$$x = \frac{1}{8}\left(7\cdot 5 + 6 \pm \sqrt{5^2 + 4\cdot 5 + 36}\right) = \frac{1}{8}(41 \pm 9).$$

This is an integer $x = 4$ when we choose the minus sign.
The correct solution is $(4, 5)$**.**
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18.* A *snake* of length n is a permutation $x_1, x_2, \cdots, x_n$ of the integers $1, 2, \ldots, n$ such that

$$x_1 < x_2, x_2 > x_3, x_3 < x_4, \ldots.$$

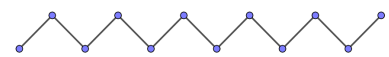
The only snake of length 1 is 1; the only snake of length 2 is 1, 2. The following are all the snakes of length 3:

$$\begin{gathered} 1, 3, 2 \\ 2, 3, 1. \end{gathered}$$

The following are all the snakes of length 4:

$$\begin{gathered} 1, 3, 2, 4 \\ 1, 4, 2, 3 \\ 2, 3, 1, 4 \\ 2, 4, 1, 3 \\ 3, 4, 1, 2 \end{gathered}$$

Let s_n be the number of different snakes of length n. Thus $s_1 = s_2 = 1$, $s_3 = 2$, $s_4 = 5$. Find s_7, the number of all different snakes of length 7. Write your answer directly onto the answer sheet.

Solution. A good picture of a snake is that the permutation looks like  . It is not too difficult to see that the numbers s_n obey the following recurrence formula

$$s_{n+1} = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2k-1} s_{2k-1} s_{n-2k+1}$$

where $s_0 = 1$. Using this formula one gets that $s_5 = 16, s_6 = 61, s_7 = 272$. The correct solution is **272.**

19.* Find the solution x, y, z of

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ x^2 & + & y^2 & + & z^2 & = & 62 \\ x^3 & + & y^3 & + & z^3 & = & 308 \end{array}$$

such that $x < y < z$. Write the values of x, y, z in the appropriate place on the answer sheet.

Hint: Think of x, y, z as zeroes of a polynomial.

Solution. Suppose x, y, z are the zeroes of $P(t) = t^3 + at^2 + bt + c$. By Viète's relations

$$\begin{aligned} a &= -(x + y + z) = -2 \\ b &= xy + yz + zx = \frac{1}{2}\left((x + y + z)^2 - (x^2 + y^2 + z^2)\right) = -29 \end{aligned}$$

One can also express xyz in terms of $x^k + y^k + z^k$ with $k = 1, 2, 3$, but it is easier to notice that

$$\begin{aligned} x^3 - 2x^2 - 29x + c &= 0 \\ y^3 - 2y^2 - 29y + c &= 0 \\ z^3 - 2z^2 - 29z + c &= 0 \end{aligned}$$

so that adding termwise we get

$$(x^3 + y^3 + z^3) - 2(x^2 + y^2 + z^2) - 29(x + y + z) + 3c = 0,$$

from which we get

$$c = -\frac{1}{3}\left(308 - 2 \cdot 62 - 29 \cdot 2\right) = -42$$

so that the polynomial is $P(t) = t^3 - 2t^2 - 29t - 42$. The problem suggests that x, y, z must be integers; being roots of a monic polynomial with constant coefficient -42 implies they must be divisors of -42. Trying all possible divisors by magnitude we see that $P(\pm 1) \neq 0$ but $P(-2) = 0$. Dividing by $t + 2$ we get $P(t) = (t + 2)(t^2 - 4t - 21)$ and see that the other two roots are -3 and 7. The correct solution is $\mathbf{x = -3,\ y = -2,\ z = 7.}$

20.* Call a number *appropriate* if its square is a 6 digit number such that the number represented by its last 3 digits is the number following the number represented by its first 3 digits. Thus x is appropriate if $x^2 = abcdef$ with $def = abc + 1$. Find the sum of all appropriate numbers. If there are none your answer should be 0. Write your answer directly onto the answer sheet.

Solution. We must have $x^2 = 1000y + (y+1)$ for some 3 digit number y, thus $x^2 = 1001y + 1$. Since $1001 = 7 \cdot 11 \cdot 13$ it means that $x^2 \equiv 1 \bmod 7$, 11, and 13. Since these are prime numbers it follows that $x \equiv \pm 1 \bmod 7$, 11, 13. This gives 8 possibilities for x. Notice that for x^2 to have 6 digits we must have $316 \leq x \leq 999$. We can discard at once $x \equiv 1$ for all three moduli; then $x = 1 + 1001k > 999$.
Consider next $x \equiv 1 \pmod 7$, $x \equiv 1 \pmod{11}$, $x \equiv -1 \pmod{13}$. Solving the system of congruences, $x \equiv 155 \pmod{1001}$; too large or too small.
The case $x \equiv 1 \pmod 7$, $x \equiv -1 \pmod{11}$, $x \equiv 1 \pmod{13}$ results in $x \equiv 274 \pmod{1001}$ also too small or too large.
Case $x \equiv 1 \pmod 7$, $x \equiv -1 \pmod{11}$, $x \equiv -1 \pmod{13}$. The solution is $x \equiv 428 \pmod{1001}$ and $428^2 = 183184$ fits the bill.
Case $x \equiv -1 \pmod 7$, $x \equiv 1 \pmod{11}$, $x \equiv 1 \pmod{13}$. The solution is $x \equiv 573 \pmod{1001}$ and $573^2 = 328329$.
Case $x \equiv -1 \pmod 7$, $x \equiv 1 \pmod{11}$, $x \equiv -1 \pmod{13}$. The solution is $x \equiv 727 \pmod{1001}$ and $727^2 = 528529$.
Case $x \equiv -1 \pmod 7$, $x \equiv -1 \pmod{11}$, $x \equiv 1 \pmod{13}$. The solution is $x \equiv 846 \pmod{1001}$ and $846^2 = 715716$.
Finally, if $x \equiv -1 \pmod 7$, $x \equiv -1 \pmod{11}$, $x \equiv -1 \pmod{13}$ we get $x \equiv 1000 \pmod{1001}$, once again out of range.
Now $428 + 573 + 727 + 846 = 2574$. The correct solution is **2574.**