MATH DAY 2018 at FAU

Competition A–Individual

NOTE:

1. Enter your name on the answer sheet. Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.

2. **Starred Problems** Twelve of the problems are multiple choice. Of the other eight problems, five (the very last five problems) are starred; a correct answer to a starred problem is worth twice the correct answer to an unstarred problem. For the non multiple choice problem (starred or not) the answer is **in every case** one or more integers and, except where mentioned, positive integers, which you enter directly beside the problem number on the answer sheet.

   Integers **MUST** be entered in standard base 10 notation. For example, if the answer to a problem is 25, entering 5^2 as your answer will be considered wrong. **Make sure you write clearly.**

3. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."

4. The notation AB is used to indicate the line through the points A, B, or the segment from A to B; $|AB|$ denotes the length of the segment AB.

5. If ABC is a triangle, then $[ABC]$ denotes the area of the triangle ABC; if $ABCD$ is a quadrilateral, then $[ABCD]$ denotes the area of $ABCD$.

6. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.

7. As a symbol, i denotes the imaginary unit; $i^2 = -1$.

8. If n is a non-negative integer, then $n!$ stands for the product of all positive integers in the range 1 to n if $n \geq 1$, with $0!$ defined to be 1. That is:

$$0! = 1,\ 1! = 1,\ 2! = 2,\ 3! = 2 \cdot 3 = 6,\ 4! = 2 \cdot 3 \cdot 4 = 24,\ 5! = 2 \cdot 3 \cdot 4 \cdot 5 = 120,\ etc.$$

9. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

10. **The problems are not necessarily ordered by degree of difficulty. Problem n could be harder than problem $n+1$.**

THE QUESTIONS

1. The sum of a 4-digit number and its four digits is 2018. What is the sum of digits of this number?

   (A) 7 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 13

2. Find $m + n$, where m, n are two positive integers such that $2018 = m^2 + n^2$.

   (A) 56 **(B)** 64 **(C)** 76 **(D)** 80 **(E)** 86

3. In a class of 20 students on grading a test, the teacher found that if she graded everybody's test but Marvin's, the average came out to 75 for the class. After grading Marvin's test, the average dropped to 74. What was Marvin's grade on this test.

   (A) 74 **(B)** 70 **(C)** 63 **(D)** 55 **(E)** 50

4. The famous trio of Abelard, Belinda and Consuelo got a number of vinyl records from one of their performances. Consuelo got 4 times as many records as Belinda, Belinda got 2 and a half times; that is, 2.5 times, as many records as Abelard. Consuelo decided to give a number of her records to Abelard and Belinda so that all ended by having the same number of records. What fraction of her records did Consuelo give Belinda?

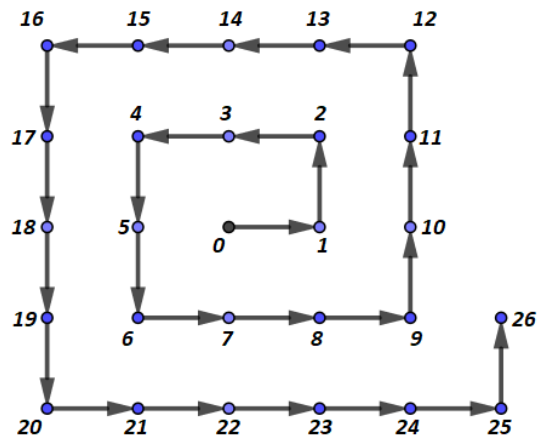
$$\textbf{(A)}\ \frac{1}{20} \qquad \textbf{(B)}\ \frac{1}{10} \qquad \textbf{(C)}\ \frac{1}{5} \qquad \textbf{(D)}\ \frac{1}{4} \qquad \textbf{(E)}\ \frac{1}{2}$$

5. Find the sum of the first 2018 terms of the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots.$$

$$\textbf{(A)}\ 16246 \qquad \textbf{(B)}\ 32288 \qquad \textbf{(C)}\ 48436 \qquad \textbf{(D)}\ 64576 \qquad \textbf{(E)}\ 85472$$

6. The points with integer coordinates in the xy-plane are labelled by integers 0, 1, 2, 3, 4, ... counter-clockwise, beginning with the origin $(0, 0)$, then $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 1)$, ...,



What are the coordinates (x, y) of the point labelled 2018? Write x and y in the appropriate spot on the answer sheet.

7. Let n be the number of distinct real solutions of the equation

$$x^2 - |x - 3| - 3 = 0$$

and let $x_1, \ldots, x_n$ be the roots. Find $n + x_1 + \cdots + x_n$. (If there is only one root, the sum to find is $1 + x_1$; if there are two roots, it is $2 + x_1 + x_2$; if three, $3 + x_1 + x_2 + x_3$; etc.)

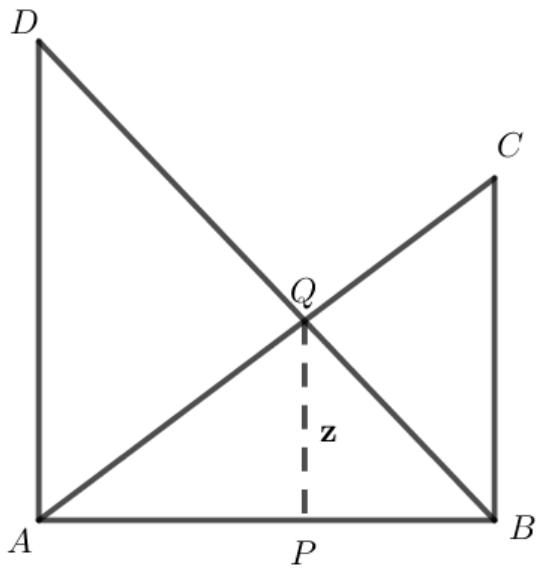
$$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 4$$

8. Two students miscopied the quadratic equation $x^2 + bx + c = 0$ that their teacher wrote on the board. Aaron copied b correctly but c incorrectly; his equation has roots 4 and 5. Betty copied c correctly but b incorrectly; her equation has roots 2 and 4. What are the roots of the original equation?

$$\textbf{(A)}\ 1 \text{ and } 8 \qquad \textbf{(B)}\ 2 \text{ and } 5 \qquad \textbf{(C)}\ 3 \text{ and } 4 \qquad \textbf{(D)}\ 4 \text{ and } 6 \qquad \textbf{(E)}\ 5 \text{ and } 7$$

9. The positive integers that can be expressed as the sum of 21 consecutive (not necessarily positive) integers are listed in increasing order. Determine the 21-st integer in the list. Write your answer directly on the answer sheet.

10. Two walls at A and B are 40 feet apart. A ladder 50 feet long has its lower end at the base of the wall at A and its upper end at C, against the Wall at B. Another ladder 58 feet long has its lower end at the base of the wall at B and its upper end D against the wall at A. How high above the ground is the point Q where the ladders intersect?



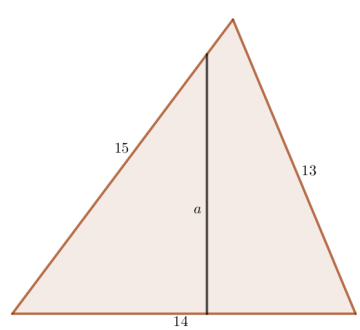
(A) 16.5 **(B)** 17 **(C)** 17.5 **(D)** 18 **(E)** 18.5

11. For how many integers x, $1 \leq x \leq 100$ does it hold that $x^3 + 2x^2 + 2x + 8$ is divisible by 7. As a help, $x = 2$ is one such integer. The expression is also divisible by 7 for $x = -1$, but $x = -1$ is not in the range that is being considered. Write your answer directly onto the answer sheet.

12. The lengths of the sides of a triangle are three consecutive integers. If the largest angle is twice the smallest one, determine the side lengths.

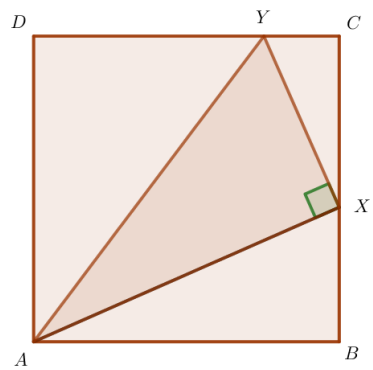
(A) $3, 4, 5$ **(B)** $4, 5, 6$ **(C)** $5, 6, 7$ **(D)** $6, 7, 8$ **(E)** $7, 8, 9$

13. A triangle has sides 13, 14, and 15. A line perpendicular to the side of length 14 divides the interior of the triangle into two regions of equal area. Find the length a of the segment of the perpendicular that lies within the triangle.



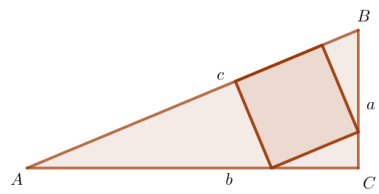
(A) $4\sqrt{7}$ **(B)** $7\sqrt{3}$ **(C)** $\frac{13\sqrt{7}}{3}$ **(D)** $\frac{5\sqrt{13}}{2}$ **(E)** NA

14. X and Y are points on the sides BC and CD of a square $ABCD$ such that triangle AXY is a right triangle with $AX = 4$, $XY = 3$ and $AY = 5$, Find the length of each side of the square.



(A) $\frac{14}{\sqrt{15}}$ **(B)** $\frac{16}{\sqrt{17}}$ **(C)** $\frac{18}{\sqrt{23}}$ **(D)** $\frac{19}{3\sqrt{5}}$ **(E)** $\frac{20}{\sqrt{19}}$

15. A square is inscribed in the right triangle ABC of sides $a = 5, b = 12, c = 13$. The square has sides of length m/n where m, n are positive integers with no common divisor other than 1. Find $m - n$.



(A) 30 **(B)** 128 **(C)** 347 **(D)** 523 **(E)** NA

16.* The equality

$$\left(\sqrt{3} - \sqrt{2}\right)^5 = \sqrt{a} - \sqrt{a-1}$$

holds for an integer a. Find a. Write your answer directly on the answer sheet.

17.* Find the lattice point on the curve

$$4x^2 - 7xy + 3y^2 - 6x + 5y = 0$$

in the interior of the first quadrant. A lattice point in the interior of the first quadrant is a point whose coordinates are positive integers. If the point has coordinates (m, n) write the values of m, n directly on the answer sheet.

18.* A *snake* of length n is a permutation $x_1, x_2, \cdots, x_n$ of the integers $1, 2, \ldots, n$ such that

$$x_1 < x_2, x_2 > x_3, x_3 < x_4, \ldots.$$

The only snake of length 1 is 1; the only snake of length 2 is 1, 2. The following are all the snakes of length 3:

$$1, 3, 2$$
$$2, 3, 1.$$

The following are all the snakes of length 4:

$$1, 3, 2, 4$$
$$1, 4, 2, 3$$
$$2, 3, 1, 4$$
$$2, 4, 1, 3$$
$$3, 4, 1, 2$$

Let s_n be the number of different snakes of length n. Thus $s_1 = s_2 = 1$, $s_3 = 2$, $s_4 = 5$. Find s_7, the number of all different snakes of length 7. Write your answer directly onto the answer sheet.

19.* Find the solution x, y, z of

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ x^2 & + & y^2 & + & z^2 & = & 62 \\ x^3 & + & y^3 & + & z^3 & = & 308 \end{array}$$

such that $x < y < z$. Write the values of x, y, z in the appropriate place on the answer sheet.

Hint: Think of x, y, z as zeroes of a polynomial.

20.* Call a number *appropriate* if its square is a 6 digit number such that the number represented by its last 3 digits is the number following the number represented by its first 3 digits. Thus x is appropriate if $x^2 = abcdef$ with $def = abc + 1$. Find the sum of all appropriate numbers. If there are none your answer should be 0. Write your answer directly onto the answer sheet.