MATH DAY 2017 at FAU

Competition A–Individual

NOTE:

1. Enter your name on the answer sheet. Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.

2. **Starred Problems** Fifteen of the problems are multiple choice. For the other five problems (identified with a star beside their number) the answer is **in every case** one or more integers and, except where mentioned, positive integers, which you enter directly beside the problem number on the answer sheet. **Be aware that these starred problems are worth twice as much as the unstarred ones.**

   Integers **MUST** be entered in standard base 10 notation. For example, if the answer to a problem is 25, entering 5^2 as your answer will be considered wrong. **Make sure you write clearly.**

3. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."

4. The notation AB is used to indicate the line through the points A, B, or the segment from A to B; $|AB|$ denotes the length of the segment AB.

5. If ABC is a triangle, then $[ABC]$ denotes the area of the triangle ABC; if $ABCD$ is a quadrilateral, then $[ABCD]$ denotes the area of $ABCD$.

6. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.

7. As a symbol, i denotes the imaginary unit; $i^2 = -1$.

8. If n is a non-negative integer, then $n!$ stands for the product of all positive integers in the range 1 to n if $n \geq 1$, with $0!$ defined to be 1. That is:

$$0! = 1,\ 1! = 1,\ 2! = 2,\ 3! = 2 \cdot 3 = 6,\ 4! = 2 \cdot 3 \cdot 4 = 24,\ 5! = 2 \cdot 3 \cdot 4 \cdot 5 = 120,\ \textit{etc.}$$

9. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

10. **The problems are not ordered by degree of difficulty. Problem n could be harder than problem $n + 1$.**

THE QUESTIONS

1. How many positive integers divide 25×27?

   (A) $2 + 3$ **(B)** 2×3 **(C)** $3 + 4$ **(D)** 3×4 **(E)** NA

2. How many positive integers x larger than 100 but less than 1000 satisfy: The remainder of x when divided by 3 is 2, when divided by 5 is 4, and when divided by 7 is 6?

   (A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** NA

3. The sum of a 4-digit number and its four digits is 2017. The sum of the last two digits of the smallest such number is

   (A) 3 **(B)** 7 **(C)** 13 **(D)** 17 **(E)** NA

4.* Find the sum of all positive integers dividing 1000. Write your answer directly on the answer sheet.

5. Let a and b be positive integers such that $a^2 + b^2 = 2017$. The value of $a + b$ is:

(A) 53 **(B)** 65 **(C)** 79 **(D)** 121 **(E)** No such integers exist

6. $f(x)$ is a quadratic polynomial with $f(0) = 1$, $f(1) = 2$, and $f(2) = 4$. What is $f(3)$?

(A) 3 **(B)** 4 **(C)** 7 **(D)** 8 **(E)** NA

7. The length of a rectangle is increased by 25 percent. By what percentage should the width be decreased to maintain the same area of the rectangle?

(A) 30 **(B)** 27 **(C)** 25 **(D)** 20 **(E)** NA

8. What is the units digit of 2017^{2017}?

(A) 1 **(B)** 3 **(C)** 7 **(D)** 9 **(E)** NA

9. A wineglass is half full of wine and a second wineglass, twice the size of the first one, is one third full of wine. Then they are both filled up with water and the contents mixed. What is the proportion of wine in the final mixture?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{5}{17}$ **(D)** $\frac{7}{18}$ **(E)** NA

10.* The equation

$$x^5 - 11x^4 + 24x^3 + 55x^2 - 224x + 175 = 0$$

has five distinct real roots x_1, x_2, x_3, x_4, x_5. Find: $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$. Write your answer directly on the answer sheet.

11. How many positive integers have the property that they are both a square and equal to a prime number plus one?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** An infinite number

12. The sum of ALL the roots of the equation $x^{4\log_2 x} = 8x^4$ equals:

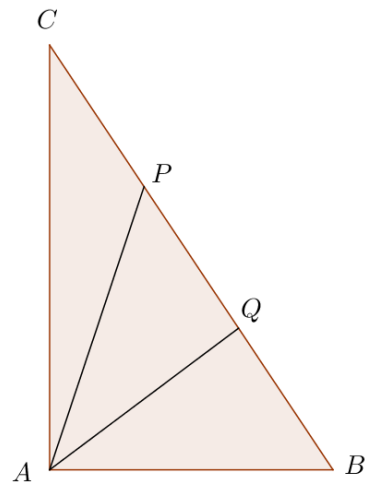
(A) 1 **(B)** $\sqrt{2}$ **(C)** $\frac{3}{2}\sqrt{2}$ **(D)** 2 **(E)** $\frac{5}{2}\sqrt{2}$

13.* Assume $0^\circ \leq \theta \leq 90^\circ$ and

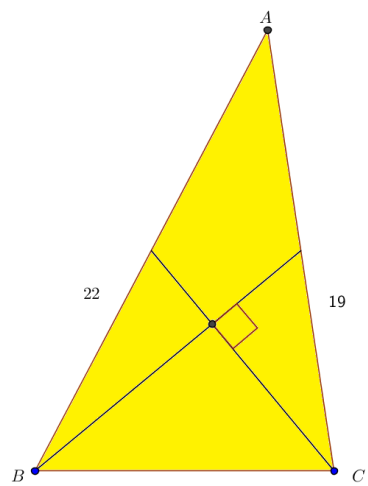
$$\tan\theta = \frac{1 - \tan 25^\circ}{1 - \tan 20^\circ}.$$

Determine θ (in degrees). Write your answer directly on the answer sheet.

14.* Triangle ABC is a right triangle. Points P, Q trisect side BC. It is given that $|AP| = 11$, $|AQ| = 8$. Then $|PQ| = \sqrt{a}$ where a is a square free positive integer. What is a? Write your answer directly on the answer sheet. (An integer is square free if does not have any divisors, other than 1, that are squares.)

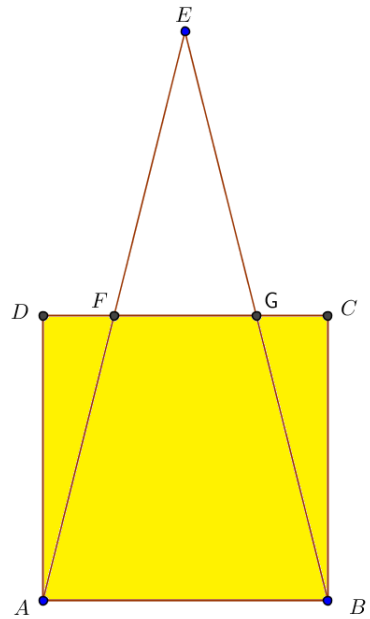


15. ABC is a triangle with $CA = 19$ and $AB = 22$. If the medians on CA and AB are perpendicular, the length of BC is



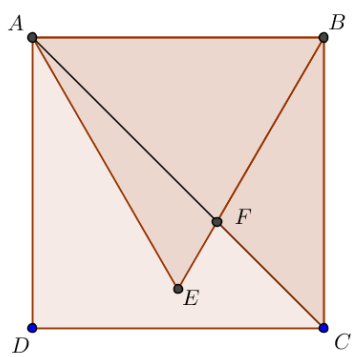
(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

16. A square $ABCD$ and an isosceles triangle ABE have the same base. Their areas are equal; AE and BE intersect CD at F and G. Find the ratio of the areas of triangle EFG and the trapezoid $AFGB$.

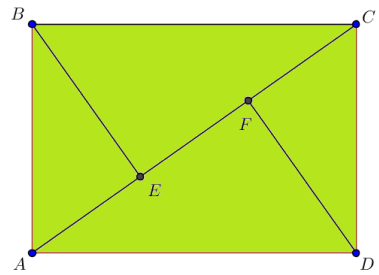


(A) $1:3$ **(B)** $7:20$ **(C)** $2:5$ **(D)** $9:20$ **(E)** $1:2$

17.* A square $ABCD$ and an equilateral triangle ABE have a common base of length 1. The diagonal AC intersects the side BE at F. The area of triangle BFC can be expressed in the form $\dfrac{\sqrt{a}-b}{c}$, where a, b, c are positive integers and a is square free. Write the value of $a+b+c$ directly on the answer sheet.

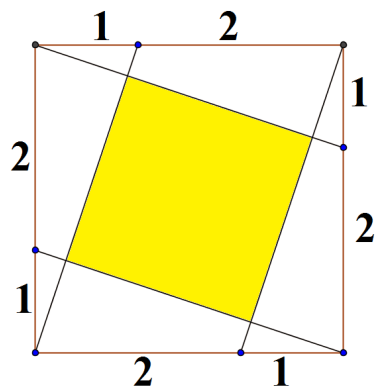


18. $ABCD$ is a rectangle with $|AB| = 1$. E and F are the perpendicular feet of B and D on the diagonal AC such that $|AE| = |EF| = |FC|$. Find the length $|AD|$.



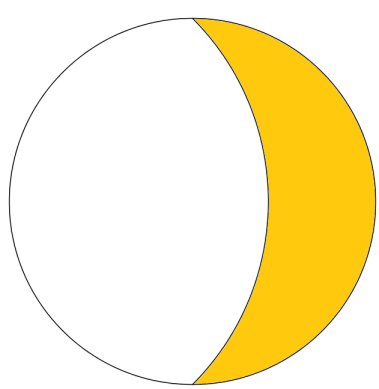
(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** $\frac{4\sqrt{2}}{3}$ **(D)** 2 **(E)** NA

19. In the diagram below, find the ratio of the areas of the smaller and the larger squares.



(A) $1:3$ **(B)** $\sqrt{3}:4$ **(C)** $\sqrt{2}:3$ **(D)** $2:5$ **(E)** $9:20$

20. In the diagram below, the shaded area is known as a *lune*. It has been cut out from the circle by a circular arc with center on the circle and intersecting the circle at diametrically opposite points. Find the ratio of the areas of the unshaded to the shaded regions.



(A) $\pi - 1 : 1$ **(B)** $\pi : 1$ **(C)** $\sqrt{2}\pi - 1 : 1$ **(D)** $\pi - \sqrt{2} : 1$ **(E)** NA