MATH DAY 2013 at FAU

Competition A–Individual

NOTE:

1. Enter your name on the answer sheet. Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.
2. **Starred Problems** Twenty of the problems are multiple choice. For the other five problems (identified with a star beside their number) the answer is **in every case** a positive integer which you enter directly beside the problem number on the answer sheet. Make sure you write clearly.
3. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."
4. The notation AB is used to indicate the line through the points A, B, or the segment from A to B; $|AB|$ denotes the length of the segment AB.
5. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.
6. As a symbol, i denotes the imaginary unit; $i^2 = -1$.
7. If n is a non-negative integer, then $n!$ stands for the product of all positive integers in the range $1 - n$ if $n \geq 1$, with $0!$ defined to be 1. That is:
$$0! = 1, 1! = 1, 2! = 2, 3! = 2 \cdot 3 = 6, 4! = 2 \cdot 3 \cdot 4 = 24, 5! = 2 \cdot 3 \cdot 4 \cdot 5 = 120, etc.$$
8. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.
9. The problems are not ordered by degree of difficulty. Problem n could be harder than problem $n + 1$.

THE QUESTIONS

1. Of 4 tests, each out of 100 points, my average is 87. What is the lowest possible score I can have on any of these tests?

   (A) 0 **(B)** 48 **(C)** 80 **(D)** 86 **(E)** NA

2. A lattice point in the plane is a point both of whose coordinates are integers. How many lattice points (including the endpoints) are there on the line segment joining the points $(2, 0)$ and $(16, 203)$?

   (A) 15 **(B)** 14 **(C)** 9 **(D)** 8 **(E)** NA

3. A certain function f satisfies $f(x) + 2f(6 - x) = x$ for all real numbers x. Find $f(1)$.

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

4. Given $y > 0$, $x > y$, and $z \neq 0$. Which of the following inequalities is **not** always correct?

   (A) $x - z > y - z$ **(B)** $xz > yz$ **(C)** $\frac{x}{z^2} > \frac{y}{z^2}$ **(D)** $xz^2 > yz^2$ **(E)** NA

5. Barton and Gabrielle are brother and sister. Barton has twice as many sisters as brothers. Gabrielle has twice as many brothers as sisters. The number of girls in the family is

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

6. What condition on the real number a is necessary and sufficient so that the equation

$$|x-1|+|x-2|=a$$

has exactly two real roots?

(A) $a>1$ **(B)** $a=1$ **(C)** $1\le a$ **(D)** $1<a$ **(E)** NA

7. Five students are to sit from left to right for a photo. One of them is the tallest. How many seating arrangements are possible if the tallest student cannot sit at the leftmost or the rightmost?

(A) 36 **(B)** 72 **(C)** 96 **(D)** 120 **(E)** NA

8. Consider the following sequence

$$-,+,+,-,+,+,\cdots$$

where every minus sign is followed by two plus signs. How many minus signs are there in the first 2014 terms?

(A) 670 **(B)** 671 **(C)** 672 **(D)** 673 **(E)** NA

9.* Find the smallest positive number that has a remainder of 3 when divided by 4, of 4 when divided by 5, and of 1 when divided by 7.

Write your answer directly onto the answer sheet.

10. With i denoting the imaginary unit $\sqrt{-1}$, let a, b be real numbers such that

$$(a+bi)^2=16+30i.$$

If $a>0$, then a equals

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

11. With i denoting the imaginary unit $\sqrt{-1}$, assume a, b are real numbers such that

$$(a+bi)^5=41+38i.$$

Find a^2+b^2.

(A) 5 **(B)** 79 **(C)** $\sqrt[5]{79}$ **(D)** $\sqrt[5]{158}$ **(E)** NA

12. The roots of the equation $2x^3-9x^2-11x+c=0$ are in arithmetic progression. Determine c

(A) 30 **(B)** 25 **(C)** 20 **(D)** 15 **(E)** 10

13.* How many numbers of the form $1^n+2^n+3^n+4^n$ are divisible by 5, if n is an integer, $100\le n\le 200$?

Write your answer directly onto the answer sheet.

14. What are the last three digits of 3^{2013}?

(A) 123 **(B)** 323 **(C)** 523 **(D)** 723 **(E)** 923

15. Which of the following numbers is closest to the number of digits of 2^{125}? Write the answer directly onto the answer sheet.

(A) 25 **(B)** 30 **(C)** 35 **(D)** 40 **(E)** 45

16.* Abe is a compulsive hoarder of Lincoln pennies. He has more than 10,000 pennies filling three boxes. One fifth (1/5) of his pennies are stashed in one of these boxes. A second box is divided into a number of compartments, each compartment contains one nineteenth (1/19) of his collection. Finally, the third box contains 2013 pennies. How many pennies does he have in all?

Write your answer directly onto the answer sheet.

17. Find the remainder of dividing $x^{101} + 3x^{74} + 4x^{26} + 7x^2$ by $x^5 + x^2$.

(A) $x^4 - 3x^2$ **(B)** $17x^3 + x^2$ **(C)** $9x^3 - 5x^2$ **(D)** $13x^2$ **(E)** NA

18.* Find the sum of all distinct three digit numbers containing only the digits 1, 2, 3, and 4, each no more than once.

Write your answer directly onto the answer sheet.

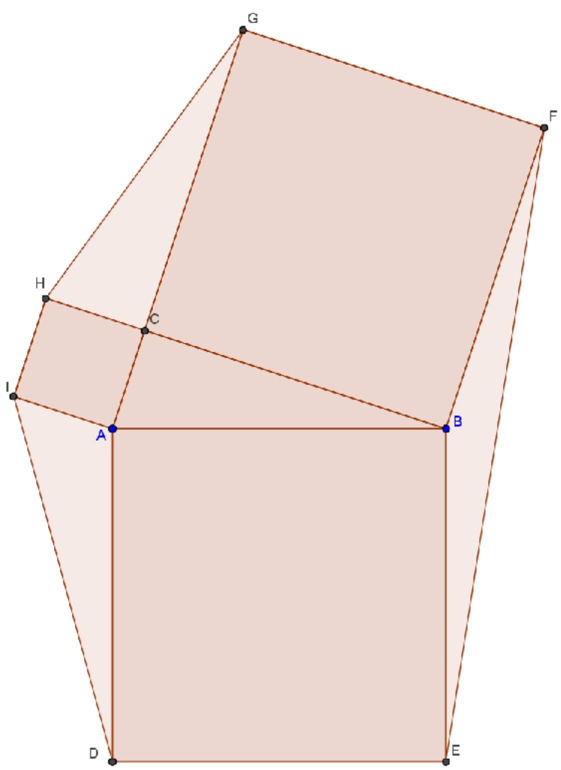
19. How many roots does the equation $\log_{10} x = \sin x$ have? (In $\sin x$, x is measured in radians.)

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NA

20. A triangle has sides of lengths $a, 3a/2$, and $2a$, where a is a positive real number. If the area is A and the square of the area is $A^2 = 15$, then a equals

(A) 4/3 **(B)** 3/4 **(C)** 2/3 **(D)** 3/2 **(E)** NA

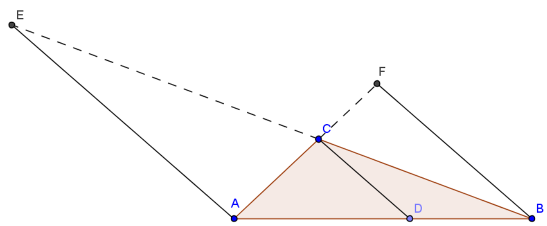
21.* An irregular hexagon $DEFGHI$ is drawn as follows: We start with a right triangle ABC, draw the squares on the legs and hypotenuse, and then join in sequence the free vertices of the squares.



If the hypotenuse of triangle ABC has **length** 13 ($|AB| = 13$) and the triangle ABC has **area** 25, find the area of the hexagon $DEFGHI$.

Write your answer directly onto the answer sheet.

22. A line segment is drawn from a point D on side AB of triangle ABC to the opposite vertex C. Lines parallel to DC are drawn through A and B; the extension of side BC intersects the first parallel at E, the extension of AC intersects the second parallel at F.

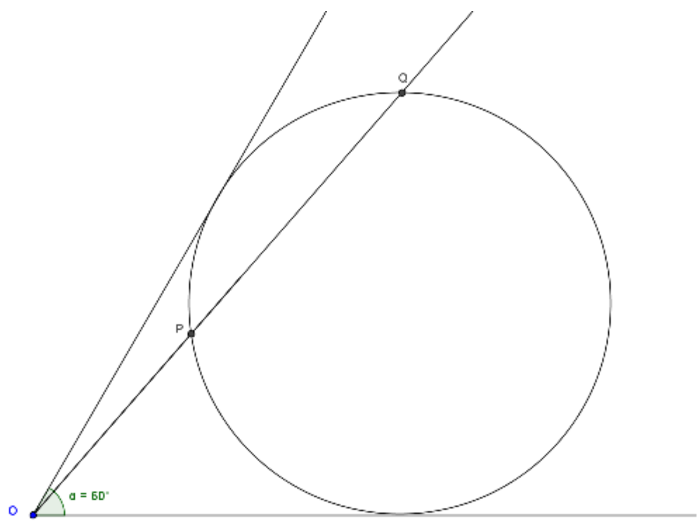


If $|AE| = 6$ and $|BF| = 4$, what is $|DC|$?

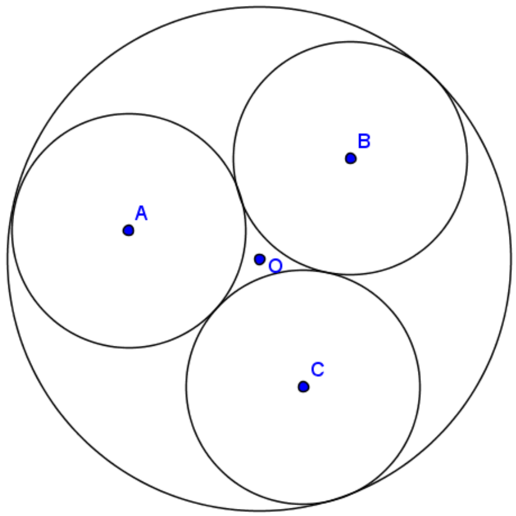
(A) 2 **(B)** 2.2 **(C)** 2.4 **(D)** 2.6 **(E)** 2.8

23. A circle of radius r is between and tangent to two lines intersecting at O at an angle of 60°. A third line drawn from O intersects the circle at points P and Q. Determine $|OP|$, the length of the segment OP, if $|PQ| = r/2$.

(A) r **(B)** $3r/2$ **(C)** $2r$ **(D)** $5r/2$ **(E)** NA



24. Three circles of equal radius r are drawn inside a large circle of radius R so that they all are tangent to each other, and to the large circle.



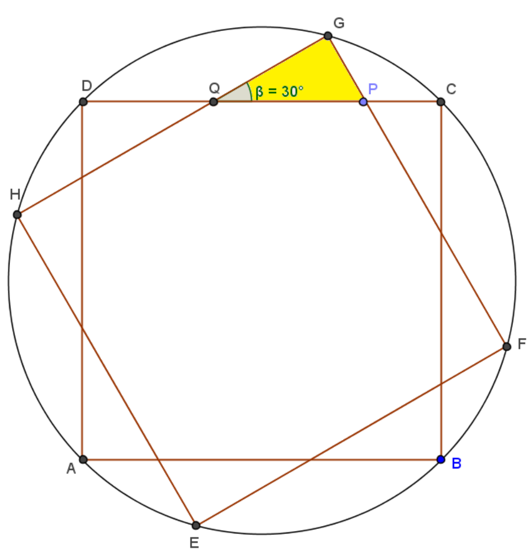
The relation between r and R can be expressed in the form

$$r = \frac{\sqrt{a}}{b + \sqrt{a}} R$$

where a, b are prime numbers. What is $a^2 + b$?

(A) 7 **(B)** 11 **(C)** 14 **(D)** 28 **(E)** NA

25. Squares $ABCD$ and $EFGH$ are inscribed in a circle in such a way that one side of square $EFGH$ intersects a side of $ABCD$ forming an angle of $30°$, as shown.



If the inscribed squares have sides of length ℓ (so, for example, $|AB| = \ell$), determine the area of the triangle QPG.

(A) $\frac{3}{64}\ell^2$ **(B)** $\frac{3\sqrt{2}-3}{12}\ell^2$ **(C)** $\frac{1}{12}\ell^2$ **(D)** $\frac{2\sqrt{2}-2}{16}\ell^2$ **(E)** $\frac{2\sqrt{3}-3}{12}\ell^2$