MATH DAY 2012 at FAU

Competition A–Individual

NOTE:

1. Enter your name on the answer sheet. Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.
2. **Starred Problems** Twenty of the problems are multiple choice. For the other five problems (identified with a star beside their number) the answer is **in every case** a positive integer which you enter directly beside the problem number on the answer sheet. Make sure you write clearly.
3. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."
4. The notation AB is used to indicate the line through the points A, B, or the segment from A to B; $|AB|$ denotes the length of the segment AB.
5. $\log_b a$ denotes the logarithm in base b of a;
6. As a symbol, i denotes the imaginary unit; $i^2 = -1$. $\log_b a = c$ if and only if $b^c = a$.
7. If n is a non-negative integer, then $n!$ stands for the product of all positive integers in the range $1 - n$ if $n \geq 1$, with $0!$ defined to be 1. That is:

   $0! = 1, 1! = 1, 2! = 2, 3! = 2{\cdot}3 = 6, 4! = 2{\cdot}3{\cdot}4 = 24, 5! = 2{\cdot}3{\cdot}4{\cdot}5 = 120, etc.$
8. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.
9. The problems are not ordered by degree of difficulty. Problem n could be harder than problem $n + 1$.

THE QUESTIONS

1. How many positive integers, including 1 and 2012, are there which exactly divide 2012?

   (A) 6 **(B)** 18 **(C)** 24 **(D)** 36 **(E)** NA

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2. What is the largest integer dividing $n^2(n^2 - 1)$ for all positive integers n?

   (A) 3 **(B)** 6 **(C)** 12 **(D)** 18 **(E)** 24 **(F)** NA

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3. The arithmetic mean (average) of a set of 100 numbers is 80. If two numbers, namely 51 and 60, are discarded, the mean of the remaining set of numbers is:

   (A) 79.5 **(B)** 79 **(C)** 80 **(D)** 80.5 **(E)** 80.75 **(F)** NA

4. By adding a constant to each of 11, 15, 20 a geometric progression results. The common ratio is:

(A) $\frac{3}{5}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{5}{4}$ **(D)** $\frac{5}{3}$ **(E)** 5 **(F)** NA

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5. If the sum of the first $3n$ positive integers is 150 more than the sum of the first n positive integers, then the sum of the first $2n$ positive integers is

(A) 21 **(B)** 78 **(C)** 171 **(D)** 300 **(E)** NA

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6. The sum of two numbers is 10; their product is 15. The sum of their reciprocals is:

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{3}$ **(D)** 2 **(E)** 3 **(F)** NA

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7. If

$$(2+i)^7-6(2+i)^6+15(2+i)^5-20(2+i)^4+15(2+i)^3-6(2+i)^2+(2+i) = a+bi$$

(here $i = \sqrt{-1}$; $i^2 = -1$), where a, b are real, then $a + b$ equals

(A) -8 **(B)** 0 **(C)** 8 **(D)** 64 **(E)** 128 **(F)** NA

8.* If $b > 1$ is a real number and

$$(\log_4 b)^2 - 18(\log_b 4)^2 = 7,$$

what is b? Write your answer directly onto the answer sheet.

9.* If $(3x^2 + 5x + 2)^6$ is expanded, what is the coefficient of x^{10}? That is, if

$$(3x^2 + 5x + 2)^6 = a_{12}x^{12} + a_{11}x^{11} + \cdots + a_1x + a_0,$$

what is a_{10}. Write your answer directly onto the answer sheet.

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10. The equation

$$|x - 3| + |x + 1| = 4$$

has exactly

(A) 1 real root **(B)** 2 real roots **(C)** 3 real roots **(D)** 4 real roots **(E)** 5 real roots **(F)** NA

11.* The equation $x^4 - 18x^3 + 59x^2 + mx + n = 0$ has four real roots that form an arithmetic progression. What is m? Write your answer on the answer sheet.

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12. The symbol $|x|$ means x if x is not negative and $-x$ if x is not positive. The number of solutions of the equation

$$|x|^2 + |x| - 12 = 0$$

is:

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4 **(F)** NA

13.* The sum of all possible products of 2 different integers from the set $\{1, 2, 3, 4\}$ equals :

$$1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 35.$$

Determine the sum of all possible products of 2 different integers from the set $\{1, 2, \ldots, 20\}$. Write the answer in the space provided on the answer sheet. As an aid, formulas you may or may not need, recall that

$$\begin{aligned} 1 + 2 + \cdots + n &= \frac{n(n+1)}{2}, \\ 1^2 + 2^2 + 3^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

14.* For a positive integer n, let $d_5(n)$ be the sum of its base 5 digits. For example, $d_5(4) = 4$, $d_5(5) = 1$, $d_5(7) = 3$, $d_5(349) = d_5\left((12344)_5\right) = 14$. If you know that $n!$ has exactly 2012 trailing zeroes, and $d_5(n) = 13$, determine n. Write out your answer directly on the answer sheet.

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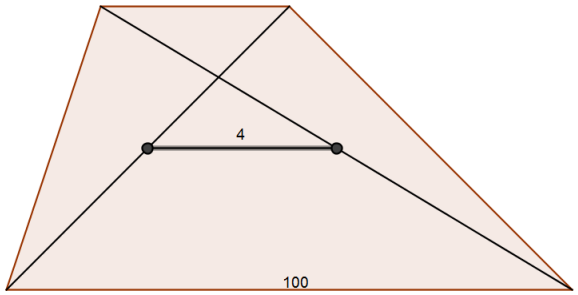
15. The points of intersection of $xy = 12$ and $x^2 + y^2 = 25$ are joined in succession. The resulting figure is (give the most precise answer):

(A) a straight line **(B)** an equilateral triangle **(C)** a rhombus
(D) a rectangle **(E)** a square

16. The line segment joining the midpoints of the diagonals of a trapezoid has length 4.



If the longer base is 100, then the shorter base is:

(A) 94 **(B)** 92 **(C)** 91 **(D)** 90 **(E)** 89 **(F)** NA

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17. A quadrilateral has sides of 4, 5, 7, and 10. What is the largest possible area?

(A) 9 **(B)** 16 **(C)** 18 **(D)** 24 **(E)** 36 **(F)** NA

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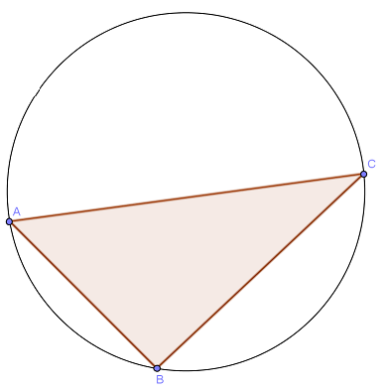
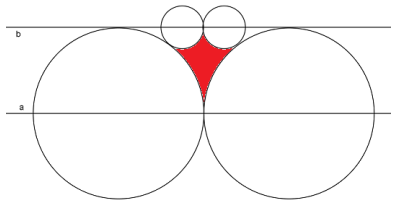
18. Triangle ABC is inscribed in a circle of radius $r = 25$. If $|AB| = 14$, $|BC| = 25$ and Points B, C are on opposite sides of the diameter through B, determine the length of the side AC.



(A) $24 + 7\sqrt{3}$ **(B)** $25 + 7\sqrt{2}$ **(C)** $24 + 7\sqrt{2}$ **(D)** $25 + 7\sqrt{3}$ **(E)** NA

19. The picture shows two large circles with centers on a line a and two smaller circles with centers on a line b parallel to a. The two large circles are tangent to each other and to one of the smaller circles; the two small circles are tangent to each other and each one is tangent to one of the larger circles.



Compute the area of the shaded region bounded by all four circles if the radius of the larger circles is $R = 6$ and that of the smaller circles is $r = 2$.

(A) $16\sqrt{3}+\frac{26\pi}{3}$ **(B)** $20\sqrt{3}+\frac{7\pi}{3}$ **(C)** $28\sqrt{3}-\frac{5\pi}{3}$ **(D)** $32\sqrt{3}-\frac{44\pi}{3}$ **(E)** NA

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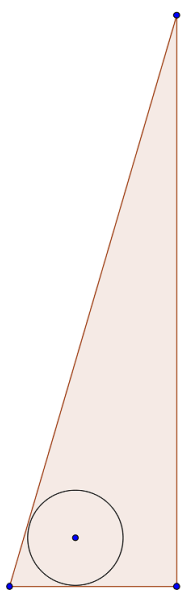
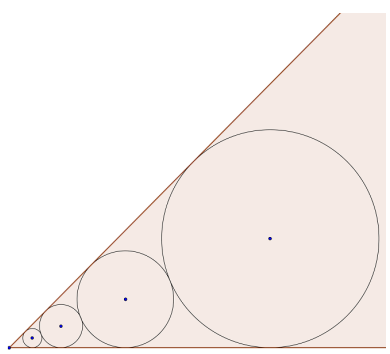
20. A circle of radius $R_1 = 4$ is drawn inside a right triangle of sides 14, 48, 50 so that it is tangent to the leg of length 14 and to the hypotenuse. Then a second circle of radius R_2 is drawn closer to the vertex determined by the hypotenuse and the leg of length 14, tangent to the first circle and to the shorter leg and to the hypotenuse. And so forth, ad infinitum. Once a circle is drawn, the next circle is tangent and to the left of the one just drawn, tangent also to the horizontal leg and the hypotenuse. The first picture shows the first circle. The next picture (definitely not to scale!) is a magnified view showing the first four circles.

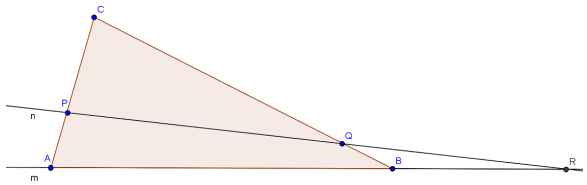




Calculate $R_1 + R_2 + \cdots$, the sum of all the radii.

(A) $\frac{16}{3}$ **(B)** 5 **(C)** $\frac{14}{3}$ **(D)** $\frac{9}{2}$ **(E)** NA

21. Triangle ABC has its side AB on a line m. Line n intersects sides AC and BC in points P and Q, and line m in R.



If $|AB| = 9$, $|AC| = 5$, $|BC| = 7$, $|AP| = 2$, $|BQ| = 1$, find $|BR|$.

(A) 3 **(B)** 3.5 **(C)** 4 **(D)** 4.5 **(E)** NA

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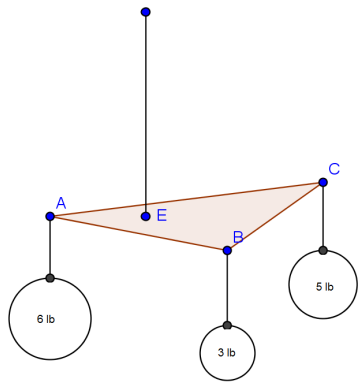
22. A wooden board shaped like an equilateral triangle of sides of length 4 ft. hangs from the ceiling of a room, suspended from point E.



Masses weighing 6 lb, 3 lb, and 5 lb are hung from the vertices labeled A, B, C, respectively. The board is perfectly balanced (all of its vertices are at the same distance from the ground). Determine $|AE|$

(A) $\sqrt{3}$ **(B)** 2 **(C)** $2\sqrt{3}$ **(D)** $3\sqrt{2}$ **(E)** NA

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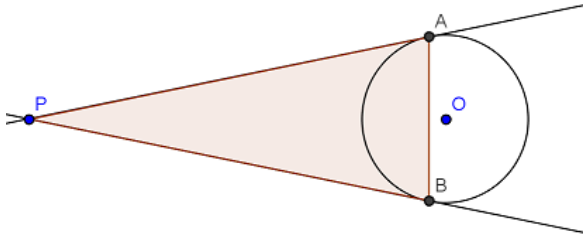
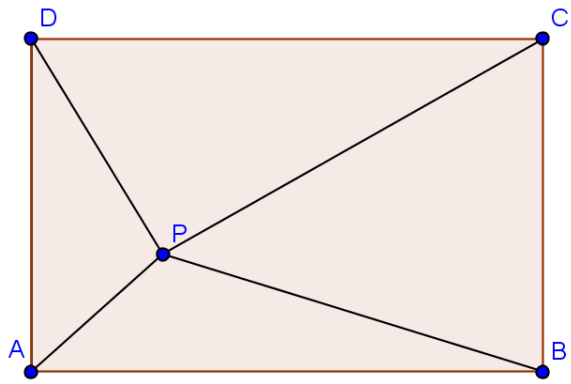
23. The two tangents to a circle of radius $r = 3$ are drawn from a point P at a distance $|OP| = 5$ from the center of the circle. The tangents touch the circle at A and B.



Determine the area of the triangle APB.

(A) $\frac{129}{25}$ **(B)** $\frac{219}{25}$ **(C)** $\frac{291}{25}$ **(D)** $\frac{912}{25}$ **(E)** $\frac{192}{25}$ **(F)** NA

24. A point P is selected inside the rectangle $ABCD$.



If $|PA| = 2$, $|PC| = 5$, and $|PD| = 3$, then $|PB|$ equals

(A) 4 **(B)** $2\sqrt{5}$ **(C)** $5\sqrt{2}$ **(D)** $3\sqrt{3}$ **(E)** NA

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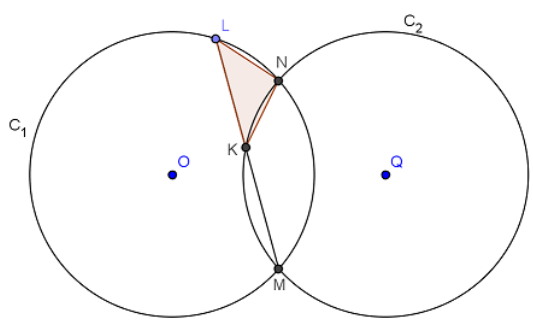
25. Two circles C_1, C_2 of equal radius r have their centers at a distance equal to $(3/2)r$ apart. The circles intersect at M, N. A line is drawn from M, it intersects again the circle C_2 at K and then C_1 at L.



If $|KL| = 6$, what is the area of triangle KLN?

(A) $2\sqrt{11}$ **(B)** $3\sqrt{7}$ **(C)** $4\sqrt{5}$ **(D)** $5\sqrt{3}$ **(E)** NA