MATH DAY 2011 at FAU

Competition B-Teams

NOTE:

1. Enter the name of your team on the answer sheet. **Only one answer sheet per team should be handed in.** Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.

2. **Starred Problems** Twenty of the problems are multiple choice. For the other five problems (identified with a star beside their number) the answer is **in every case** a positive integer which you enter directly beside the problem number on the answer sheet. Make sure you write clearly.

3. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."

4. The notation AB is used to indicate the line through the points A, B, or the segment from A to B; $|AB|$ denotes the length of the segment AB.

5. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.

6. If n is a non-negative integer, then $n!$ stands for the product of all positive integers in the range $1 - n$ if $n \geq 1$, with 0! defined to be 1. That is:

   $0! = 1, 1! = 1, 2! = 2, 3! = 2{\cdot}3 = 6, 4! = 2{\cdot}3{\cdot}4 = 24, 5! = 2{\cdot}3{\cdot}4{\cdot}5 = 120, etc.$

7. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

THE QUESTIONS

1. If $4^x - 4^{x-1} = 24$, then $(2x)^x$ is equal to:

   (A) $5\sqrt{5}$ **(B)** 125 **(C)** $25\sqrt{5}$ **(D)** 25 **(E)** NA

2. When the census taker asked Mary Smith how many children she had, she replied "fewer than 10." Pressed for a more definite answer, she said "if you meet two of my children at random, the probability that both will be brown eyed is $1/2$". Assuming Mrs. Smith has at least two children, how many brown eyed children does Mrs. Smith have?

   (A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** NA

3.* $\lfloor x \rfloor$ is the greatest integer that is $\leq x$. Thus $\lfloor 3 \rfloor = 3$, $\lfloor \sqrt{2} \rfloor = 1$, $\lfloor 1/10 \rfloor = 0$. Compute

$$\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{99} \rfloor + \lfloor \sqrt{100} \rfloor.$$

   Write the answer directly on the answer sheet.

4.* Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. For each nonempty subset $B \subset A$, we form the product of its elements. Find the sum of all the possible products. The product of a subset having a single element is that element. Thus the product of the subset consisting of $\{3\}$ alone is 3. Write your answer directly on the answer sheet. It is a relatively large number; all digits must be correctly displayed.

5.* How many integers x are there such that $1 \leq x \leq 100$, and $x^3 + 4x + 2$ is divisible by 7? Write your answer directly onto the answer sheet. Count carefully!, a miss is as bad as a mile.

6. If it is known that $\log_3 a + \log_3 b \geq 6$, what is the smallest value that $a + b$ can assume?

(A) $3\sqrt{6}$ **(B)** 6 **(C)** $36\sqrt{2}$ **(D)** 54 **(E)** NA

7. The sum of the **squares** of all the real roots of the equation $25^{4x^2-3x} = 5^{6x-1}$ equals:

(A) 1 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

8. Which of the following statements about the equation $2 \cdot 9^x + 3 \cdot 9^{-x} = 7$, is **false**. There is only one false one; circle the corresponding letter on the answer sheet. A solution of the equation is a value x for which the equation is true.

(A) The equation has at least one rational solution (a solution that is a rational number; ie. of the form a/b, a, b integers).

(B) The equation has no integer solutions (solutions that are integers).

(C) The equation has two positive solutions.

(D) The sum of all the real solutions of the equation is < 1.

(E) At least one of the preceding statements is false.

9.* Determine $|A| + |B| + |C|$ (sum of absolute values) where A, B, C are such that $A(\sin(y))^5 + B(\sin(y))^3 + C\sin(y) = \sin(5y)$ for all real numbers y. Write your answer directly on the answer sheet.

10. If $x^2 + 3x + 5$ is a factor of $x^4 + ax^2 + b$, then $a + b$ equals

(A) 0 **(B)** 17 **(C)** 23 **(D)** 26 **(E)** 31

11.* The four roots x_1, x_2, x_3, x_4 of the equation

$$x^4 - 45x^3 + 630x^2 - 3240x + m = 0$$

are in geometric progression. What is the sum of m plus the roots; that is, what is $m + x_1 + x_2 + x_3 + x_4$? Write your answer on the answer sheet. A hint that can be used to shorten the solving time: The roots, and m are integers.

12. How many pairs (m, n) of **positive** integers satisfy the equation

$$n^4 + 2n^3 + 16n^2 + 30n + 15 = m^2 \quad ?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

13. Suppose that 1 and 2 are roots of the equation $x^3 + ax^2 + bx + c = 0$ and that $a + b = -15$. Find a.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

14. The coefficient of x^{17} in the expansion of $(x^4 + x^3 + 3)^6$ equals

(A) 6 **(B)** 24 **(C)** 48 **(D)** 120 **(E)** 180

15. For how many **positive** integers a does the equation $x^5 + ax^2 - 9 = 0$ have integer solutions?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** infinity

16. The equation

$$3x^5 - 6x^4 - 18x^3 + 30x^2 - x - 5 = 0$$

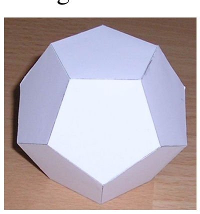
has 5 distinct real roots x_1, x_2, x_3, x_4, x_5. What is $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$?

(A) 15 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 30

17. The hypotenuse of a right triangle is 5 centimeters long, and the radius of the inscribed circle is 2 centimeters. The perimeter of the triangle in centimeters is

(A) $10cm$ **(B)** $12cm$ **(C)** $14cm$ **(D)** $16cm$ **(E)** $18cm$

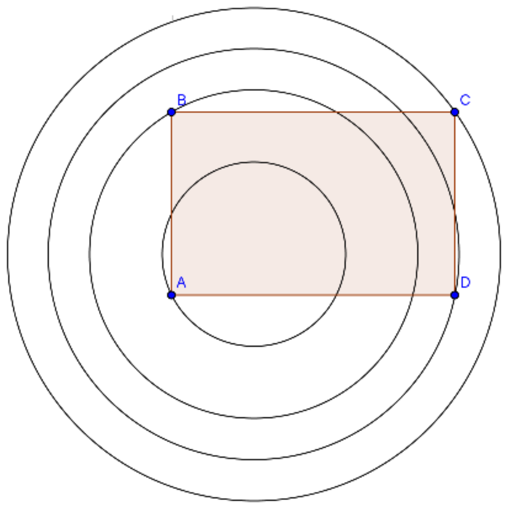
18. George constructs a regular dodecahedron (twelve faces all regular pentagons).



In how many ways can he paint three faces red and the other faces green, if two paintings are to be considered the same if one can be rotated into the other?

(A) 5 **(B)** 12 **(C)** 30 **(D)** 72 **(E)** 100

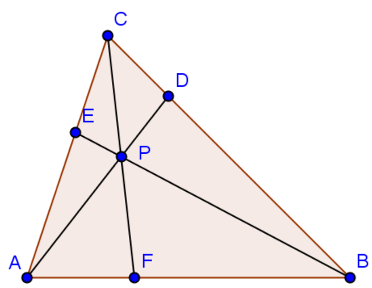
19. The **rectangle** $ABCD$ has one vertex on one each of four concentric circles, as in the picture.



If the radii of the three larger circles are 6, 5, and 4, then the radius of the smallest circle is

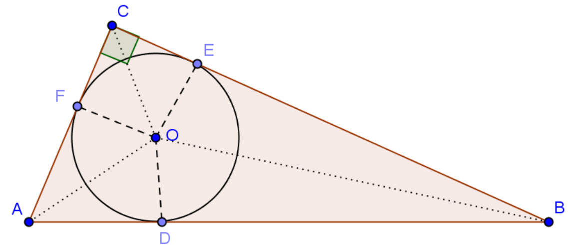
(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $\sqrt{5}$ **(E)** $\sqrt{6}$

20. In the triangle ABC a line segment is drawn from vertex C to a point F on side AB. A line segment is then drawn from vertex A to a point D on side BC, intersecting the segment CF at its **midpoint** P. Finally,a line segment from vertex C to the point E on AC also goes through P. If $|AP| = 2|PD|$ and $|BP| = 3$, determine $|PE|$.



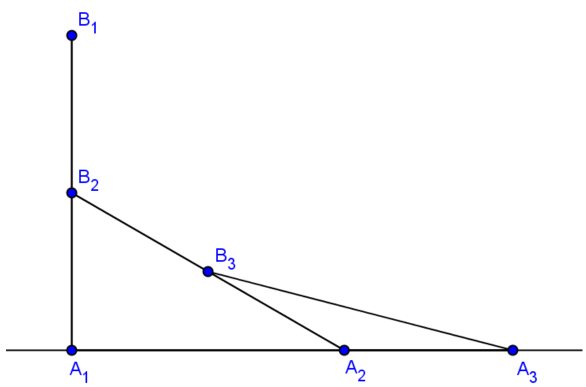
(A) 0.5 **(B)** 0.6 **(C)** 0.75 **(D)** 0.9 **(E)** 1

21. The inscribed circle of a right triangle ABC is tangent to the hypotenuse AB at D. If $|AD| = x$ and $|DB| = y$, find the area of the triangle in terms of x and y.



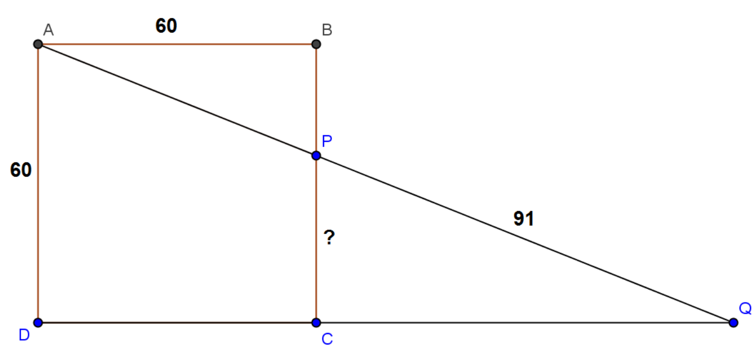
(A) $x^2 + y^2$ **(B)** $(x + y)^2$ **(C)** $\frac{1}{2}(x^2 + y^2)$ **(D)** $\frac{1}{2}(x + y)^2$ **(E)** xy

22. Three equal segments A_1B_1, A_2B_2, A_3B_3 are positioned in such a way that the endpoints B_2, B_3 are the midpoints of A_1B_1, A_2B_2 respectively, while the endpoints A_1, A_2, A_3 are on a line perpendicular to A_1B_1. Find the ratio $\frac{|A_1A_3|}{|A_1A_2|}$ correct up to two places after the decimal point.



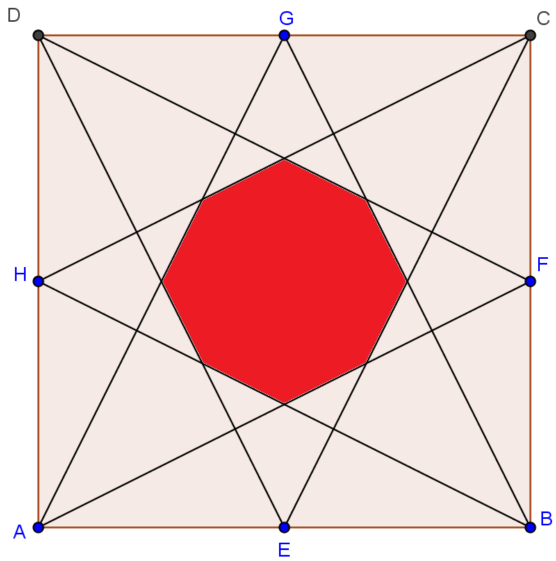
(A) 1.51 **(B)** 1.62 **(C)** 1.73 **(D)** 1.84 **(E)** NA

23. A man has a square field, 60 feet by 60 feet, with other property adjoining the highway. He put up a straight fence in the line of 3 trees, at A, P, Q. If the distance between P and Q is 91 feet, and that from P to C is an exact number of feet, what is this distance?



(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

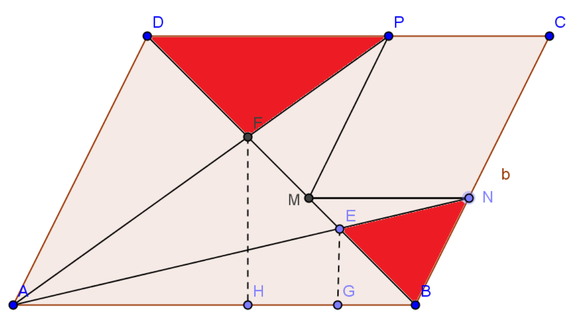
24. Lines are drawn from each vertex of a square to the midpoint of the two non-adjacent sides. An octagon appears in the center of the square.



Find the ratio of the area of the octagon to that of the square.

(A) $\frac{1}{8}$ **(B)** $\frac{1}{7}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{4}$ **(E)** NA

25. A smaller parallelogram $MNCP$ was marked off in the parallelogram ABCD, so that vertex M falls on the diagonal BD. Lines are drawn from vertex A to P and to N. These lines intersect the diagonal BD at F and E, respectively. The line segments EG and FH are perpendicular to the base AB. If $|EG| = 2$ and $|FH| = 4$, and $|AB| = 7$, determine the area of the shaded region; that is determine $[DFP] + [BEN]$. As a hint, the parallelogram $MNCP$ can play an important role in the solution to this problem, especially the point M.



(A) 6 **(B)** 7 **(C)** $3\sqrt{2}$ **(D)** $4\sqrt{2}$ **(E)** NA