MATH DAY 2011 at FAU

Competition A–Individual

NOTE:

1. Enter your name on the answer sheet. Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.
2. **Starred Problems** Twenty of the problems are multiple choice. For the other five problems (identified with a star beside their number) the answer is **in every case** a positive integer which you enter directly beside the problem number on the answer sheet. Make sure you write clearly.
3. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."
4. The notation AB is used to indicate the line through the points A, B, or the segment from A to B; $|AB|$ denotes the length of the segment AB.
5. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.
6. If n is a non-negative integer, then $n!$ stands for the product of all positive integers in the range $1 - n$ if $n \geq 1$, with 0! defined to be 1. That is:

   $0! = 1, 1! = 1, 2! = 2, 3! = 2{\cdot}3 = 6, 4! = 2{\cdot}3{\cdot}4 = 24, 5! = 2{\cdot}3{\cdot}4{\cdot}5 = 120, etc.$
7. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.
8. The problems are not ordered by degree of difficulty. Problem n could be harder than problem $n + 1$.

THE QUESTIONS

1. Because the company is in trouble, at the end of 2008 Jason's boss reduces Jason's salary of \$40,000 by 10%. The company does better in 2010, so the boss raises Jason's salary by 10%. The result is:

   (A) Jason is back at his old salary. **(B)** Jason is making \$400 less than he used to.
   (C) Jason is making \$400 more than he used to. **(D)** Jason is making \$4000 less than he used to.
   (E) Jason is making \$4000 more than he used to.

2. What is the unit digit of 2^{2011}?

   (A) 1 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

3. A cube measuring 100 units on each side is painted only on the outside and cut into unit cubes. The number of unit cubes with paint on exactly two sides is:

   (A) 1000 **(B)** 1125 **(C)** 1176 **(D)** 980 **(E)** NA

4. A bus travels up a hill at an average speed of 50 miles per hour. At what average speed would it have to travel down the hill to average 60 miles per hour for the entire trip?

(A) 68 **(B)** 70 **(C)** 72 **(D)** 73 **(E)** NA

5.* At a meeting of 100 people, every person shakes hands with every other person exactly once. How many handshakes are there in total? Write your answer directly on the answer sheet.

6. Five counterfeit coins are mixed with nine authentic coins. If two coins are drawn at random, what is the probability that one is good and one is counterfeit?

(A) $\frac{1}{2}$ **(B)** $\frac{45}{91}$ **(C)** $\frac{1}{14}$ **(D)** $\frac{4}{45}$ **(E)** NA

7. Samantha wants to run a tournament for 20 players. If each player competes as an individual and matches consist of two competitors, how many matches must Samantha allow for, assuming that competitors stay in the competition until they have lost three times and the tournament is to produce a single champion? Assume there are no ties.

(A) 55 **(B)** 57 **(C)** 58 **(D)** 59 **(E)** 60

8. The remainder of dividing $3^{2011}+1$ by 2011 is

(A) 0 **(B)** 4 **(C)** 192 **(D)** 2001 **(E)** 2009

9.* The sum of the third and fourth terms in a sequence of consecutive integers is 47. Find the sum of the first five terms of the sequence. Write your answer directly on the answer sheet.

10.* The sum

$$\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{99\cdot 100}$$

equals a rational number a/b where a, b are positive integers and the greatest common divisor of a and b is 1. Write the value of $a+b$ on the answer sheet.

11.* The sum of a 4-digit number and its four digits is 2011. What is the number? Write your answer directly on the answer sheet.

12. What is the greatest common divisor of $2^{15}+3^{15}$ and $2^{25}+3^{25}$?

(A) 5 **(B)** 11 **(C)** 55 **(D)** 275 **(E)** NA

13. Find the number of positive divisors of 3456. Be sure to count the numbers 1 and 3456 among the divisors.

(A) 18 **(B)** 27 **(C)** 32 **(D)** 40 **(E)** NA

14. What is the maximum value that can be assumed by $3\sin x - 4\cos x$ as x ranges over the real numbers?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

15. If $\log_3 x = y$ then $\log_9 x =$

(A) $\frac{y}{2}$ **(B)** $2y$ **(C)** $3y$ **(D)** $y+3$ **(E)** NA

16. Find the **sum** of all roots of the equation

$$\frac{1}{\sin x} + \frac{\sqrt{3}}{\cos x} = 4$$

in the interval $0 < x < \frac{\pi}{2}$. (Angles are measured in radians for this problem).

(A) $\frac{4\pi}{9}$ **(B)** $\frac{7\pi}{6}$ **(C)** $\frac{5\pi}{8}$ **(D)** $\frac{2\pi}{5}$ **(E)** NA

17. A function is defined by $f(x) = Ax + B$ where A, B are real constants, $A > 0$. If $f(1) = 0$ and $f(f(0)) = -20$, then A equals

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NA

18. The minimum value taken by the expression $x^2 + x + 1$ for real values of x is

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{3}{4}$ **(D)** 1 **(E)** NA

19. The remainder of dividing the polynomial $x^{81} + 3x^{49} + 7x^{25} + 3x^9 + x$ by $x^3 - x$ is

(A) $15x$ **(B)** $12x^2 - 3$ **(C)** $x + 1$ **(D)** 12 **(E)** NA

20. In a rectangle we increase the shorter side by 3 and it becomes a square having an area twice of that of the original rectangle. What is the area of the original rectangle?

(A) 9 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 36

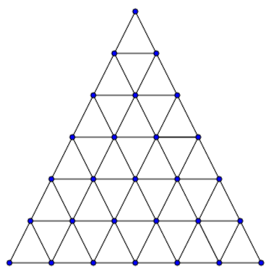
21. A triangle of sides measuring 6, 8, and 10 centimeters is inscribed in a circle. What is the area of the circle in square centimeters?

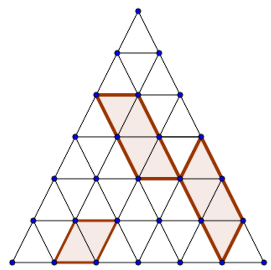
(A) 50π **(B)** 48π **(C)** 25π **(D)** 24π **(E)** 20π

22. The interior of a rectangular container is 1 meter wide, 2 meters long, and a little bit less than a meter deep. It is filled with water to a depth of $\frac{1}{2}$ meter. A cube of gold is placed flat in the tub, and water rises to exactly the top of the cube without overflowing. Find the length of the side of the cube, rounded off to two digits after the decimal point.

(A) 0.60 **(B)** 0.62 **(C)** 0.67 **(D)** 0.75 **(E)** 0.81

23. Each side of an equilateral triangle is partitioned into 6 equal parts and then the triangle is partitioned by lines parallel to the sides of the triangle using the partitioning of the sides as a guide. The result is the configuration on the left below. **How many parallelograms are there in this configuration?**. The picture on the right is there only for your convenience; it shows three of these parallelograms; there are many more. I suggest that rather than counting directly, you try to find a general formula.



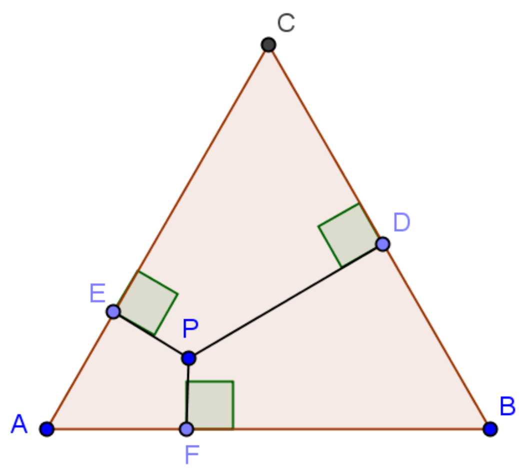


(A) 78 **(B)** 132 **(C)** 210 **(D)** 325 **(E)** 516

24. As you may recall, the sum of the interior angles of a triangle is $180°$. How many sides must a polygon have if the sum of its interior angles is $2520°$?

(A) 12 **(B)** 14 **(C)** 16 **(D)** 28 **(E)** 42

25.* From a point P inside the **equilateral** triangle ABC **perpendicular** line segments have been drawn to the three sides, intersecting these sides at D, E, F.



If the sides of the triangle have length 5 ($|AB| = |BC| = |CA| = 5$), determine the sum $|PD|+|PE|+|PF|$. Your answer should have the form $\dfrac{a\sqrt{b}}{c}$ where a, c are positive integers with no factor in common other than 1, and b is a square-free positive integer (no square other than 1 divides b). Write the value of $a + b + c$ directly on the answer sheet.