MATH DAY 2010 at FAU

Competition B-Teams

NOTE:

1. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."
2. In all questions, i stands for the imaginary unit; $i^2 = -1$.
3. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.
4. If n is a non-negative integer, then $n!$ stands for the product of all positive integers in the range $1 - n$ if $n \geq 1$, with 0! defined to be 1. That is:

   $0! = 1, 1! = 1, 2! = 2, 3! = 2{\cdot}3 = 6, 4! = 2{\cdot}3{\cdot}4 = 24, 5! = 2{\cdot}3{\cdot}4{\cdot}5 = 120, etc.$

5. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

THE ANSWERS

1. The percentage of salt in n ounces of a salt-water solution is $n\%$. How many ounces of salt must be added to make a solution that is $2n\%$ salt?

   (A) $\dfrac{n}{100+n}$ **(B)** $\dfrac{2n}{100+n}$ **(C)** $\dfrac{2n}{100-2n}$ **(D)** $\dfrac{n^2}{100+2n}$ **(E)** $\dfrac{n^2}{100-2n}$ **(F)** NA

   Solution. If we add x ounces of salt there will be $\dfrac{n}{100}n + x = \dfrac{n^2+100x}{1}00$ ounces of salt in the solution; the weight of the solution has increased to $n + x$ ounces. We want

   $$\frac{\frac{n^2+100x}{100}}{n+x} = \frac{2n}{100}; \quad \text{that is} \quad \frac{n^2+100x}{100(n+x)} = \frac{2n}{100}.$$

   Solving $x = n^2/(100 - 2n)$.

   The correct solution is **E.** ■

2. There are $5! = 120$ five-digit numbers that can be formed by permuting $1, 2, 3, 4, 5$. The sum of all these numbers is:

   (A) $3,999,960$ **(B)** $2,876,540$ **(C)** $4,969,960$ **(D)** $5,600,610$ **(E)** $6,975,640$ **(F)** NA

   Solution. Each one of these numbers is of the form $10^4a_1 + 10^3a_2 + 10^2a_3 + 10a_4 + a_5$, where $a_1, \ldots, a_5$ can take any value in the range $1, 2, 3, 4, 5$. Each one of the allowed digits will appear in each one of the positions a_1, a_2, a_3, a_4, a_5 surrounded (or followed if it is the first position, preceded if it is the last) by all

possible combinations of the other four digits; in other words, $4! = 24$ times. When we add up all the numbers, each power of ten will multiply

$$24 \times (1 + 2 + 3 + 4 + 5) = 360$$

so that the total sum is

$$360(10^4 + 10^3 + 10^2 + 10 + 1) = 360 \times 11,111 = 3,999,960.$$

The correct solution is **A.** ■

3. 62 people are staying in a series of rooms labeled from 1 to 62, one person per room. For reasons unknown, at some point all the people are told to leave their rooms. People in rooms labeled 1 to 31 are told to go to the room that has twice the number of their original room. People in the higher numbered rooms are then told to occupy the remaining empty rooms (the odd numbered ones) keeping the order in which they were before; that is, the person who was in room 32 goes to room 1, the one who was in room 33 goes to room 3, etc.; the one who was in room 62 goes to room 61. Suppose this operation gets repeated several times; how many times must it be repeated for everybody to be back in the rooms they occupied originally? For example, if instead of 62 rooms and 62 people we'd only have had 4 rooms and 4 people we can label P_1, P_2, P_3, P_4, writing them out in the order of the rooms they are occupying, here is how things work out:

Original order	:	P_1	P_2	P_3	P_4
After one operation	:	P_3	P_1	P_4	P_2
After two operations	:	P_4	P_3	P_2	P_1
After three operations	:	P_2	P_4	P_1	P_3
After four operations	:	P_1	P_2	P_3	P_4

After 4 operations, people are back in their original rooms. Thinking modulo 63 can help. The question is: After how many operations of room changing will the 62 people be back in their original rooms?

(A) They'll never be back in original order **(B)** 6
(C) 62 **(D)** 62! **(E)** NA

Solution. The key here is to see that after each operation the person who was in room n ends in room $2n$ modulo 63. Thus after k operations, the person in room n will be in room $2^k n$ modulo 63. People will be back in their original rooms when 2^k equals 1, modulo 63. The first time this happens is for $k = 6$; then $2^6 = 64$ and 64 is 1 modulo 63.

The correct solution is **B.** ■

4. When $(x + y + z)^{2010}$ is expanded and like terms are collected, how many terms will there be? For example $(x + y)^4$ has 5 terms after expansion and simplification, while $(x + y + z)^2$ has 6 terms.

(A) 2012 **(B)** $44,100$ **(C)** $2,021,055$ **(D)** $2,023,066$ **(E)** NA

Solution. When expanded, the expression will be a homogeneous polynomial of degree 2010 in which all terms are of the form $cx^\ell y^m z^n$, where c is a constant

and ℓ, m, n are non-negative integers such that $\ell + m + n = 2010$. Moreover all such powers appear. So the total number of terms equals the number of such powers. To count them, we notice that $\ell + m$ can be any number in the range $0-2010$, but then n is uniquely determined by $n = 2010-\ell-m$. So the question becomes how many choices of ℓ, m are there such that $\ell + m = k \leq 2010$? For any k, $0 \leq k \leq 2010$ we have exactly $k + 1$ choices, namely ℓ any number in the range $0 - k$ and $m = k - \ell$. The total number of terms is thus

$$(0+1)+(1+1)+\cdots+(2010+1) = 1+2+\cdots+2011 = \frac{2011 \cdot 2012}{2} = 2023066.$$

The correct solution is **D.** ■

5. Determine the largest real number solving the equation

$$\log_3(x+2) - \log_9(x+8) = \frac{1}{2}.$$

Solution. For any positive number y, $\log_3 y = \log_9 y^2$ because $9^{\log_3 y} = 3^{2\log_3 y} = 3^{\log_3 y^2}$. The equation can be written in the form

$$\log_9(x+2)^2 = \log_9(x+8) + \frac{1}{2}$$

and removing the logarithms we get

$$(x+2)^2 = 9^{1/2}(x+8); \text{ expanding, } x^2 + x - 20 = 0.$$

The solutions of this quadratic equation are -5 and 4, but the only solution that makes sense for the original equation is $x = 4$. (If $x = -5$, then $x + 2 = -3$ and $\log_3(x + 2)$ is undefined). Thus 4 is not only the largest real solution, it is the only real solution.

The correct solution is **4.** ■.

6. How many pairs of numbers (a, b) are there such that $a < b$ and $a^b = b^a$?

(A) Exactly one such pair **(B)** Exactly two such pairs **(C)** There is **no** such pair
(D) There is an infinite number of such pairs **(E)** NA

Solution. There is at least one pair, namely $(2, 4)$: $2^4 = 16 = 4^2$. To decide if there is another such pair notice that $a^b = b^a$ is the same as $a^{1/a} = b^{1/b}$. Consider now the graph of the function $y = x^{1/x}$ as it goes from $x = 2$ to $x = 4$. If it goes up after $x = 2$, it has to come down again and an infinity of horizontal lines will intersect it at least two times giving and infinity of pairs (x_1, x_2), $x_1 < x_2$ such that $x_1^{1/x_1} = x_2^{1/x_2}$; i.e., $x_1^{x_2} = x_2^{x_1}$. The same is true if the graph goes down. In the unlikely event that the graph stays flat, every pair (x_1, x_2) with $2 < x_1 < x_2 < 4$ would have the property.

The correct solution is **D.** ■

(**Note** What actually happens between 2 and 4 is that after 2 the graph of $y = x^{1/x}$ increases steadily until $x = e$; then it decreases steadily. For every value of a, $2 \leq a < e$, there is b, $e < b \leq 4$ such that $a^b = b^a$.)

7. The function $\arctan x$ is defined for all real x by $y = \arctan x$ if and only if $|y| < \pi/2$ and $\tan y = x$. Which of the following equals $\cos(\arctan x)$ for all x?

(A) $x + \sqrt{1+x^2}$ **(B)** $\sqrt{1+x^2}$ **(C)** $\dfrac{1}{\sqrt{1+x^2}}$ **(D)** $\dfrac{x}{\sqrt{1+x^2}}$ **(E)** NA

Solution. The easiest way to get the correct answer is setting up a right triangle so that one angle has tangent equal to x; for example by letting the leg opposite to that angle be x units in length and the adjacent leg be one unit in length. Then the hypotenuse has length $\sqrt{1+x^2}$ and we see that the cosine is $1/\sqrt{1+x^2}$. This assumes implicitly that $x > 0$ but since both $\cos(\arctan x)$ and $1/\sqrt{1+x^2}$ are even functions, the equality will also hold for negative values of x.

The correct solution is **C.** ■

8. The following product

$$\tan 1^\circ \tan 2^\circ \tan 3^\circ \cdots \tan 87^\circ \tan 88^\circ \tan 89^\circ$$

is equal to

(A) 1 **(B)** $\sqrt{3}$ **(C)** $\dfrac{1}{\sqrt{2}}$ **(D)** $\dfrac{\pi}{4}$ **(E)** NA

Solution. The key to this problem is to realize that if $a^\circ + b^\circ = 90^\circ$, then $\tan a^\circ = 1/\tan b^\circ$; that is $\tan a^\circ \tan b^\circ = 1$. In the given product we can group all the factors but one into pairs of the form $\tan a^\circ \tan b^\circ$ with $a^\circ + b^\circ = 90^\circ$; each one of these pairs multiplies to 1. The factor left over is $\tan 45^\circ = 1/\sqrt{2}$.

9. Suppose that $(x+iy)^2 = 16 + 30i$, where x, y are real numbers and $x > 0$. Then $x + y$ equals

(A) $4 + \sqrt{30}$ **(B)** 4 **(C)** $\sqrt{30}$ **(D)** 8 **(E)** NA

Solution. If $(x+iy)^2 = 16+30i$, then $x^2 - y^2 = 16$, $2xy = 30$. Solving the second equation for y, substituting into the first, we get $x^4 - 16x^2 - 225 = 0$. Solving this quadratic equation in x^2,

$$x^2 = \frac{16 \pm \sqrt{16^2 + 4 \times 225}}{2} = 8 \pm 17.$$

Discarding the (meaningless) negative root, we get $x^2 = 25$, thus $x = 5$ (since $x > 0$), thus $y = 15/x = 3$ and $x + y = 8$.

The correct solution is **D.** ■

10. Suppose $x^2 + xy + x = 14$ and $y^2 + xy + y = 28$. Which of the following is a possible value of $x + y$?

(A) -7 **(B)** -6 **(C)** 0 **(D)** 7 **(E)** NA

Solution. Adding the two equations we get $x^2 + 2xy + y^2 + x + y = 42$; that is, $(x+y)^2 + (x+y) = 42$. Solving this quadratic equation we see that $x + y = 6$ or -7.

The correct solution is **A.** ■

11. When $x^{100} + 3x^{99} - 1$ is divided by $x^2 + 2x - 3$, the remainder is:

(A) $x + 2$ **(B)** $x + 1$ **(C)** $2x + 1$ **(D)** $x - 1$ **(E)** $3x - 2$ **(F)** NA

Solution. We have $x^2 + 2x - 3 = (x - 1)(x + 3)$. The remainder must be a polynomial of degree at most 1; that is an expression of the form $ax + b$; if $p(x)$ is the quotient then

$$x^{100} + 3x^{99} - 1 = p(x)(x - 1)(x + 3) + (ax + b).$$

Setting $x = 1$ we get $a + b = 3$. Setting $x = -3$ we get $-3a + b = -1$. Solving these equation we get $a = 1$, $b = 2$.

The correct solution is **A.** ■

12. Suppose that $a + b = 3$ and $a^2 + b^2 = 7$. Then $a^4 + b^4$ is equal to:

(A) 45 **(B)** 47 **(C)** 49 **(D)** 51 **(E)** 81 **(F)** NA

Solution. Squaring $a^2 + b^2 = 7$ tells us that $a^4 + 2a^2b^2 + b^4 = 49$. To determine $2a^2b^2$, we square $a + b = 3$ and get

$$a^2+b^2+2ab = 9; \quad \text{thus } ab = \frac{1}{2}(9-a^2-b^2) = \frac{1}{2}(977) = 1; \quad \text{squaring } a^2b^2 = 1, 2a^2b^2 = 2.$$

Thus $a^4 + b^4 = 49 - 2 = 47$.

The correct solution is **B.** ■

13. Let $(x_1, x_2, x_3, x_4, x_5)$ solve the system

$$\begin{array}{rcrcrcrcrcr} 3x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 1 \\ x_1 & + & 3x_2 & + & x_3 & + & x_4 & + & x_5 & = & 4 \\ x_1 & + & x_2 & + & 3x_3 & + & x_4 & + & x_5 & = & 6 \\ x_1 & + & x_2 & + & x_3 & + & 3x_4 & + & x_5 & = & 3 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & + & 3x_5 & = & 0 \end{array}$$

Determine the product $x_1x_2x_3x_4x_5$.

(A) 0 **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** NA

Solution. One can solve this linear system by Gaussian elimination, but the fastest way is to observe that if we add all equations, the left hand sides add up to $7(x_1 + x_2 + x_3 + x_4 + x_5)$, the right hand sides to 14, thus $x_1 + x_2 + x_3 + x_4 + x_5 = 2$. Subtracting this last equation successively from all the equations of the systems gives us $2x_1 = -1$, $2x_2 = 2$, $2x_3 = 4$, $2x_4 = 1$, $2x_5 = -2$. Thus

$$x_1 = -1/2,\ x_2 = 1,\ x_3 = 2,\ x_4 = 1/2,\ x_5 = -1 \quad \text{and } x_1x_2x_3x_4x_5 = 1/2.$$

The correct solution is **B.** ■

14. The equation $x^4 - 19x^3 + bx^2 + 473x - 2010 = 0$ has four distinct real roots. The product of two of these roots is -30 (minus 30). Determine the value of b.

Solution. Suppose the roots are x_1, x_2, x_3, x_4, where $x_1x_2 = -30$. By Viète's equations, we have

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 19, \\ x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 &= b, \\ x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 &= -473, \\ x_1x_2x_3x_4 &= -2010. \end{aligned}$$

From the last equation, we get $x_3x_4 = 2010/(-30) = 67$. The third equation can now be rewritten in the form

$$-473 = x_1x_2(x_3+x_4)+(x_1+x_2)x_3x_4; \quad \text{that is, } 67(x_1+x_2)-30(x_3+x_4) = -473.$$

Combining with the first equation we get the following system of equations in the unknowns $x_1 + x_2, x_3 + x_4$:

$$\begin{aligned} (x_1 + x_2) + (x_3 + x_4) &= 19, \\ 67(x_1 + x_2) - 30(x_3 + x_4) &= -473. \end{aligned}$$

Solving, we get $x_1 + x_2 = 1$, $x_3 + x_4 = 18$. Finally, we notice that the second on of Viète's equations can be rewritten in the form

$$b = (x_1 + x_2))x_3 + x_4) + x_1x_2 + x_3x_4 = 1 \cdot 18 - 30 + 67 = 55.$$

The correct solution is **55.** ■

15. If k is the **smallest** number with the following property: If we write a list of k different names there must be two names having the same first letters and the same last letters (such as Jane and Julie, or Charles and Curtis), then

(A) $k > 676$ **(B)** $k > 700$ **(C)** $k < 672$ **(D)** $k > 678$ **(E)** NA

(Only the usual letters from A to Z are allowed; there is no distinction between capitals and lower case).

Solution. There are $26 \times 26 = 676$ possible first and last letter combinations.

The correct solution is **A.** ■

16. Determine the smallest positive integer with the property that the last digit of its decimal representation is 9, and such that if we erase this last digit of 9 and place it in front of the number, the resulting number equals four times the original number.

Solution. Let x be the number we are looking for. Then x is of the form $x = 10a + 9$, for some integer a. erasing the 9 and putting it in front changes the number to $9 \cdot 10^d + a$, where d is the number of digits of a. We want

$$9 \cdot 10^d + a = 4x = 40a + 36, \text{ solving, } 39a = 9 \cdot 10^d - 36.$$

We need to find a value of d so that $9 \cdot 10^d - 36$ is a multiple of 39. It is probably best to go by trial and error; at least for a while. We see that

$$\begin{aligned} (90 - 36)/39 &= 54/39 = 1.38\ldots, \\ (900 - 36)/39 &= 864/39 = 22.15\ldots, \\ (9000 - 36)/39 &= 8964/39 = 229.846\ldots, \\ (90000 - 36)/39 &= 89964/39 = 2306.769\ldots, \\ (900000 - 36)/39 &= 899964/39 = 23076 \quad \text{exactly} \end{aligned}$$

The smallest possible number is the one with $a = 23076$.

The correct solution is 230769 **or** $230,769$**.** ■

17. A *lattice point* in the plane is a point of coordinates (x, y) such that both x and y are integers. We assume of course that we have set up a system of cartesian coordinates. For example, the origin $(0, 0)$ is a lattice point. So are the points of coordinates $(1, 1)$, $(-2, 3)$, etc.

    A line through the origin forming an angle of 60 degrees with the x-axis will go through

    (A) Exactly one lattice point **(B)** Exactly two lattice points
    (C) A finite number ≥ 3 of lattice points **(D)** An infinite number of lattice points **(E)** NA

    Solution. Since the tangent of 60 degrees equals $\sqrt{3}$, the equation of a line through the origin forming an angle of 60 degrees with the x-axis is $y = \sqrt{3}\,x$. If the line goes through a point of coordinates (x, y), then $\sqrt{3} = y/x$, assuming $x \neq 0$. It is known that $\sqrt{3}$ is irrational, thus cannot be the quotient of two integers. It follows that the line goes through a single lattice point, namely $(0, 0)$.

    The correct solution is **A.** ■

18.* Of all points of the plane of coordinates (x, y) satisfying $x^2 + y^2 = 2x - 4y + 4$, the distance to the origin of the one that is closest to the origin equals $a - \sqrt{b}$, where a, b are positive integers. Determine $a + b$.

    Solution. The equation satisfied by the points (x, y) can be rewritten in the form $(x - 1)^2 + (y + 2)^2 = 9$, which is the equation of a circle of radius 3, center at $(1, -2)$. Suppose P is the point on this circle closest to the origin. The line segment joining it to the origin will then be perpendicular to the tangent at P, but that means that this line segment will be part of a radius of the circle. Since the radius is 3 and the distance of the center of the circle to the origin is $\sqrt{1^2 + 2^2} = \sqrt{5}$, the distance of the point P to the origin is $3 - \sqrt{5}$.

    Of course, if one knows calculus, one can also use calculus. But it seems to me that such an approach is considerably harder. A **correct** calculus approach yields the distance in the form

    $$\sqrt{14 - 6\sqrt{5}}$$

    and one now has to figure out how to write this in the form $a - \sqrt{b}$. This is not hard, but it entails further calculations. Suppose $\sqrt{14 - 6\sqrt{5}} = a - \sqrt{b}$. Both sides being positive, the equality holds if and only if there is equality between the squares of both sides; that is, if and only if $\left(\sqrt{14 - 6\sqrt{5}}\right)^2 = (a - \sqrt{b})^2$. Squaring, the equality holds if and only if

    $$14 - 6\sqrt{5} = a^2 + b - 2a\sqrt{b}, \quad \text{hence } a^2 + b - 14 = 2a\sqrt{b} - 6\sqrt{5}.$$

    The right hand side of this last equality is either 0 or irrational; the left hand side is an integer (so rational). Thus $a^2 + b - 14 = 0$, $2a\sqrt{b} = 6\sqrt{5}$. Solving this one gets as possible solutions $a^2 = 5$ and $b = 9$, or $a^2 = 9$, $b = 5$. Since a is a positive integer, the one and only solution is $a = 3, b = 5$, giving $a + b = 8$. Except if there is a shortcut I failed to see, the calculus solution is way harder then the geometric one.

The correct solution is **8.** ■

19. Equation $x^2 + 4y^2 - 6x + 5 = 0$ represents an ellipse. Determine the distance between its foci.

(A) 4 **(B)** $\sqrt{3}$ **(C)** $2\sqrt{3}$ **(D)** $\sqrt{5}$ **(E)** $2\sqrt{5}$ **(F)** NA

Solution. Completing squares and dividing by 4, we can write the equation in the form

$$\frac{(x-3)^2}{4} + y^2 = 1.$$

This is the equation of an ellipse of half axes $a = 2, b = 1$; the distance between the foci is $2\sqrt{a^2 - b^2} = 2\sqrt{3}$.

The correct solution is **C.** ■

20. Two parallel chords in a circle have lengths 6 and 8. The distance between them is 1. Then the diameter of the circle is:

(A) 9 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** NA

Solution. If a chord of length 2ℓ is at distance h from the center of a circle of radius r, the theorem of Pythagoras tells us that $\ell^2 + h^2 = r^2$. In our case if the longer chord ($\ell = 4$) is at distance h from the center, the shorter one ($\ell = 3$) is at distance $h + 1$, so that we have the equations

$$\begin{aligned} 16 + h^2 &= r^2 \\ 9 + (h+1)^2 &= r^2. \end{aligned}$$

Subtracting the second equation from the first, we get $7 - 2h - 1 = 0$; i.e., $h = 3$. Using this in the first equation, we get $r = 5$. The diameter is thus 10.

The correct solution is **B.** ■

21.* A hexagon having 4 sides of length 15 and the remaining two sides of length 41, is inscribed in a circle. Determine the radius of the circle.

Solution. Let r be the radius of the circle. As per picture, the hexagon can be dissected into six isosceles triangles; all of them have equal sides equal to r, four of them have a third side of length 15, the other two of length 41. Let θ be

the angle between the equal sides for the first three triangles, ψ for the last two. Given an isosceles triangle ABC with $|AC| = |BC|$, a bit of trigonometry (law of sines) shows that

$$|AC| = \frac{|AB|}{2\sin(\angle ACB/2)};$$

; in our case leading to

$$\frac{15}{2\sin(\theta/2)} = r = \frac{41}{2\sin(\psi/2)}.$$

Now $4\theta + 2\psi = 2\pi$ so that $\sin(\psi/2) = \cos\theta$; solving

$$\frac{15}{2\sin(\theta/2)} = \frac{41}{2\cos\theta} = \frac{41}{2(1 - 2\sin^2(\theta/2))},$$

and discarding the negative root, we get $\sin(\theta/2) = 3/10$. Returning to $\frac{15}{2\sin(\theta/2)} = r$, we get $r = 25$

The correct solution is **25.** ■

22. If the circle in the picture below has radius 1 and the angle θ equals 30 degrees, compute the area of the shaded region.

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{1}{2}\sqrt{3}$ **(D)** $\frac{1}{4}\sqrt{3}$ **(E)** NA

Solution. Consider half the shaded figure

It is an isosceles triangle with the two equal sides of length 1, base angles of 15 degrees. The area is easily worked out to

$$A = \frac{1}{2}\sin(\theta) = \frac{1}{2}\sin 30^\circ = \frac{1}{4}.$$

Multiplying by 2 we get the area of the figure.

The correct solution is **B.** ■

23. The point P is at distance 3 from the center of a circle of radius 1. The regular hexagon of vertices A, B, C, D, E, F is inscribed in the circle in such a way the A is on the line from the center of the circle to P. Determine the product

$$|PA| \cdot |PB| \cdot |PC| \cdot |PD| \cdot |PE| \cdot |PF|$$

of the distances from P to the vertices of the hexagon.

(A) 728 **(B)** 729 **(C)** $420\sqrt{3}$ **(D)** $421\sqrt{3}$ **(E)** NA

Solution. By the law of cosines, if Q is a point on the circle, then (denoting the center of the circle by O)

$$|PQ|^2 = |OP|^2 + |OQ|^2 - 2|OP| \cdot |OQ| \cos(\angle POQ) = 9 + 1 - 6\cos(\angle POQ) = 10 - 6\cos(\angle POQ).$$

Using this formula we get

$$|PA| \cdot |PB| \cdot |PC| \cdot |PD| \cdot |PE| \cdot |PF| = 4 \cdot 7 \cdot 13 \cdot 16 \cdot 13 \cdot 7 = 728^2$$

The correct solution is **A.** ■

24. Determine the area common to two half-circles that have their diameters on two parallel lines 1 unit apart, and their centers on a line perpendicular to the parallel lines, as in the figure below.

(A) $2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$ **(B)** $2\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)$ **(C)** $\frac{\sqrt{3}\,\pi}{4}$ **(D)** $\frac{3\sqrt{2}\,\pi}{4}$ **(E)** NA

Solution. The area of a segment of a circle of radius r spanning a central angle θ radians is $\frac{1}{2}(\theta-\sin\theta)r^2$.

In our case, θ is the angle between the two equal sides (of length 1) of an isosceles triangle whose height is $1/2$, from which it is easy to see that $\theta = 2\pi/3$ and $\sin\theta = \sqrt{3}/2$ The area of the shaded segment is thus $\frac{1}{2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$, the area we are supposed to find consists of two such segments, thus equals $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$

The correct solution is **A.** ■

25. In the picture, exactly ten circles fit between the inner (shaded) circle and the outer circle, so that each of the ten little circles are tangent to the inner circle, to the outer circle, no two intersect, and each one is tangent to two others (one on each side).

    If the radius of the inner circle is 1, determine which of the following is closest to the radius of the outer circle

(A) 1.8940 **(B)** 1.8944 **(C)** 1.8948 **(D)** 1.8952 **(E)** 1.8956

Solution. Suppose the common radii of the ten little circles is ρ; then the radius of the outer circle is $1 + 2\rho$. If we draw a circle concentric with the large circles and of radius $1 + \rho$, then the centers of the small circles are the vertices of an inscribed regular decagon. The length of the side of a regular decagon inscribed in a circle of radius R is $2R \sin 18°$; in our case $2(1 + \rho) \sin 18°$. Because all the small circles are tangent to their neighbors, the side of the decagon, which equals the distance between the centers of two adjacent small circles, is 2ρ. Thus

$$2(1 + \rho) \sin 18° = 2\rho, \quad \text{solving, } \rho = \frac{\sin 18°}{1 - \sin 18°} \approx 0.4472135954\ldots,$$

thus $1 + 2\rho = 1.894427\ldots$.

The correct solution is **B.** ■