MATH DAY 2010 at FAU

Competition B-Teams

NOTE:

1. Enter the name of your team on the answer sheet. **Only one answer sheet per team should be handed in.** Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.
2. **Starred Problems** Twenty of the problems are multiple choice. For the other five problems (identified with a star beside their number) the answer is **in every case** a positive integer which you enter directly beside the problem number on the answer sheet. Make sure you write clearly.
3. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."
4. In all questions, i stands for the imaginary unit; $i^2 = -1$.
5. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.
6. If n is a non-negative integer, then $n!$ stands for the product of all positive integers in the range $1 - n$ if $n \geq 1$, with $0!$ defined to be 1. That is:

   $$0! = 1, 1! = 1, 2! = 2, 3! = 2{\cdot}3 = 6, 4! = 2{\cdot}3{\cdot}4 = 24, 5! = 2{\cdot}3{\cdot}4{\cdot}5 = 120, etc.$$

7. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

THE QUESTIONS

1. The percentage of salt in n ounces of a salt-water solution is n%. How many ounces of salt must be added to make a solution that is $2n$% salt?

   (A) $\dfrac{n}{100+n}$ **(B)** $\dfrac{2n}{100+n}$ **(C)** $\dfrac{2n}{100-2n}$ **(D)** $\dfrac{n^2}{100+2n}$ **(E)** $\dfrac{n^2}{100-2n}$ **(F)** NA

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2. There are $5! = 120$ five-digit numbers that can be formed by permuting $1, 2, 3, 4, 5$. The sum of all these numbers is:

   (A) $3,999,960$ **(B)** $2,876,540$ **(C)** $4,969,960$ **(D)** $5,600,610$ **(E)** $6,975,640$ **(F)** NA

3. 62 people are staying in a series of rooms labeled from 1 to 62, one person per room. For reasons unknown, at some point all the people are told to leave their rooms. People in rooms labeled 1 to 31 are told to go to the room that has twice the number of their original room. People in the higher numbered rooms are then told to occupy the remaining empty rooms (the odd numbered ones) keeping the order in which they were before; that is, the person who was in room 32 goes to room 1, the one who was in room 33 goes to room 3, etc.; the one who was in room 62 goes to room 61. Suppose this operation gets repeated several times; how many times must it be repeated for everybody to be back in the rooms they occupied originally? For example, if instead of 62 rooms and 62 people we'd only have had 4 rooms and 4 people we can label P_1, P_2, P_3, P_4, writing them out in the order of the rooms they are occupying, here is how things work out:

Original order	:	P_1	P_2	P_3	P_4
After one operation	:	P_3	P_1	P_4	P_2
After two operations	:	P_4	P_3	P_2	P_1
After three operations	:	P_2	P_4	P_1	P_3
After four operations	:	P_1	P_2	P_3	P_4

After 4 operations, people are back in their original rooms. Thinking modulo 63 can help. The question is: After how many operations of room changing will the 62 people be back in their original rooms?

(A) They'll never be back in original order **(B)** 6
(C) 62 **(D)** $62!$ **(E)** NA

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4. When $(x+y+z)^{2010}$ is expanded and like terms are collected, how many terms will there be? For example $(x+y)^4$ has 5 terms after expansion and simplification, while $(x+y+z)^2$ has 6 terms.

(A) 2012 **(B)** $44,100$ **(C)** $2,021,055$ **(D)** $2,023,066$ **(E)** NA

5.* Determine the largest real number solving the equation

$$\log_3(x+2) - \log_9(x+8) = \frac{1}{2}.$$

Enter your answer, which should be a positive integer, directly in the provided space in the answer sheet. **Please write clearly.**

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6. How many pairs of numbers (a, b) are there such that $a < b$ and $a^b = b^a$?

(A) Exactly one such pair **(B)** Exactly two such pairs **(C)** There is **no** such pair
(D) There is an infinite number of such pairs **(E)** NA

7. The function $\arctan x$ is defined for all real x by $y = \arctan x$ if and only if $|y| < \pi/2$ and $\tan y = x$. Which of the following equals $\cos(\arctan x)$ for all x?

(A) $x + \sqrt{1+x^2}$ **(B)** $\sqrt{1+x^2}$ **(C)** $\dfrac{1}{\sqrt{1+x^2}}$ **(D)** $\dfrac{x}{\sqrt{1+x^2}}$ **(E)** NA

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8. The following product

$$\tan 1^\circ \tan 2^\circ \tan 3^\circ \cdots \tan 87^\circ \tan 88^\circ \tan 89^\circ$$

is equal to

(A) 1 **(B)** $\sqrt{3}$ **(C)** $\dfrac{1}{\sqrt{2}}$ **(D)** $\dfrac{\pi}{4}$ **(E)** NA

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9. Suppose that $(x+iy)^2 = 16 + 30i$, where x, y are real numbers and $x > 0$. Then $x + y$ equals

(A) $4 + \sqrt{30}$ **(B)** 4 **(C)** $\sqrt{30}$ **(D)** 8 **(E)** NA

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10. Suppose $x^2 + xy + x = 14$ and $y^2 + xy + y = 28$. Which of the following is a possible value of $x + y$?

(A) -7 **(B)** -6 **(C)** 0 **(D)** 7 **(E)** NA

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11. When $x^{100} + 3x^{99} - 1$ is divided by $x^2 + 2x - 3$, the remainder is:

(A) $x + 2$ **(B)** $x + 1$ **(C)** $2x + 1$ **(D)** $x - 1$ **(E)** $3x - 2$ **(F)** NA

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12. Suppose that $a + b = 3$ and $a^2 + b^2 = 7$. Then $a^4 + b^4$ is equal to:

(A) 45 **(B)** 47 **(C)** 49 **(D)** 51 **(E)** 81 **(F)** NA

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13. Let $(x_1, x_2, x_3, x_4, x_5)$ solve the system

$$\begin{array}{ccccccccccc}
3x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 1 \\
x_1 & + & 3x_2 & + & x_3 & + & x_4 & + & x_5 & = & 4 \\
x_1 & + & x_2 & + & 3x_3 & + & x_4 & + & x_5 & = & 6 \\
x_1 & + & x_2 & + & x_3 & + & 3x_4 & + & x_5 & = & 3 \\
x_1 & + & x_2 & + & x_3 & + & x_4 & + & 3x_5 & = & 0
\end{array}$$

Determine the product $x_1 x_2 x_3 x_4 x_5$.

(A) 0 **(B)** $\dfrac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** NA

14.* The equation $x^4 - 19x^3 + bx^2 + 473x - 2010 = 0$ has four distinct real roots. The product of two of these roots is -30 (minus 30). Determine the value of b.

Enter your answer, which should be a positive integer, directly in the provided space in the answer sheet. **Please write clearly.**

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15. If k is the **smallest** number with the following property: If we write a list of k different names there must be two names having the same first letters and the same last letters (such as Jane and Julie, or Charles and Curtis), then

(A) $k > 676$ **(B)** $k > 700$ **(C)** $k < 672$ **(D)** $k > 678$ **(E)** NA

(Only the usual letters from A to Z are allowed; there is no distinction between capitals and lower case).

16.* Determine the smallest positive integer with the property that the last digit of its decimal representation is 9, and such that if we erase this last digit of 9 and place it in front of the number, the resulting number equals four times the original number.

Enter your answer, which should be a positive integer, directly in the provided space in the answer sheet. **Please write clearly.**

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17. A *lattice point* in the plane is a point of coordinates (x, y) such that both x and y are integers. We assume of course that we have set up a system of cartesian coordinates. For example, the origin $(0, 0)$ is a lattice point. So are the points of coordinates $(1, 1)$, $(-2, 3)$, etc.

A line through the origin forming an angle of 60 degrees with the x-axis will go through

(A) Exactly one lattice point **(B)** Exactly two lattice points
(C) A finite number ≥ 3 of lattice points **(D)** An infinite number of lattice points **(E)** NA

18.* Of all points of the plane of coordinates (x, y) satisfying $x^2 + y^2 = 2x - 4y + 4$, the distance to the origin of the one that is closest to the origin equals $a - \sqrt{b}$, where a, b are positive integers. Determine $a + b$.

Enter your answer, which should be a positive integer, directly in the provided space in the answer sheet. **Please write clearly.**

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19. Equation $x^2 + 4y^2 - 6x + 5 = 0$ represents an ellipse. Determine the distance between its foci.

(A) 4 **(B)** $\sqrt{3}$ **(C)** $2\sqrt{3}$ **(D)** $\sqrt{5}$ **(E)** $2\sqrt{5}$ **(F)** NA

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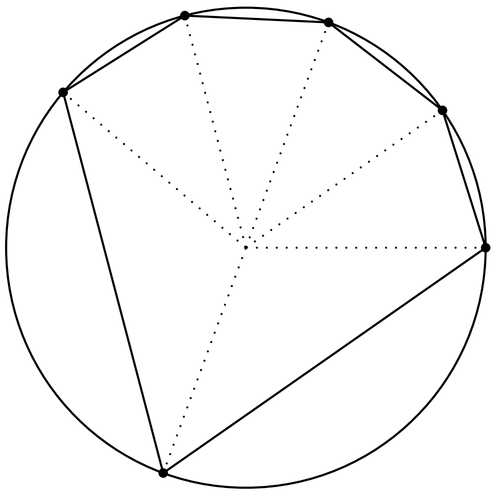
20. Two parallel chords in a circle have lengths 6 and 8. The distance between them is 1. Then the diameter of the circle is:

(A) 9 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** NA

21.* A hexagon having 4 sides of length 15 and the remaining two sides of length 41, is inscribed in a circle. Determine the radius of the circle.



Enter your answer, which should be a positive integer, directly in the provided space in the answer sheet. **Please write clearly.**

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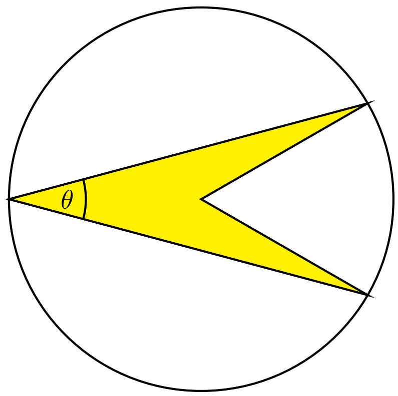
22. If the circle in the picture below has radius 1 and the angle θ equals 30 degrees, compute the area of the shaded region.

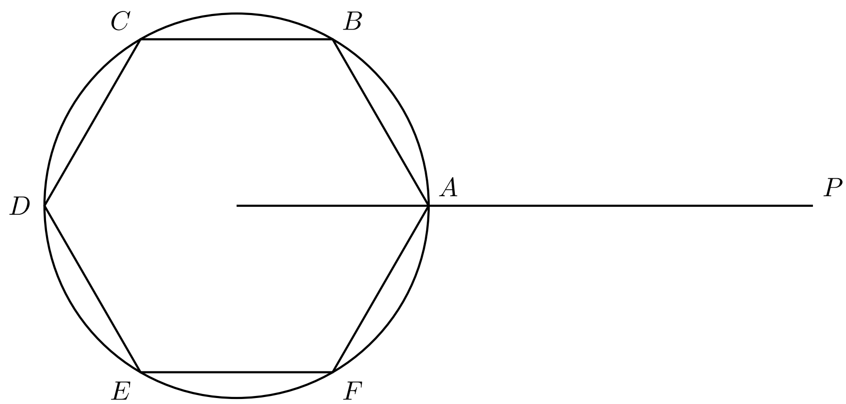


(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{1}{2}\sqrt{3}$ **(D)** $\frac{1}{4}\sqrt{3}$ **(E)** NA

23. The point P is at distance 3 from the center of a circle of radius 1. The regular hexagon of vertices A, B, C, D, E, F is inscribed in the circle in such a way the A is on the line from the center of the circle to P. Determine the product

$$|PA| \cdot |PB| \cdot |PC| \cdot |PD| \cdot |PE| \cdot |PF|$$

of the distances from P to the vertices of the hexagon.



(A) 728 **(B)** 729 **(C)** $420\sqrt{3}$ **(D)** $421\sqrt{3}$ **(E)** NA

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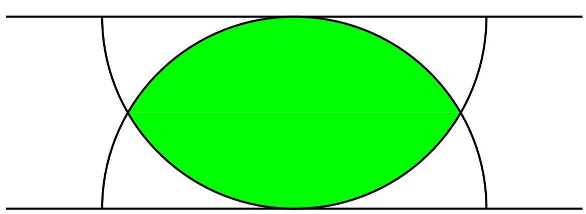
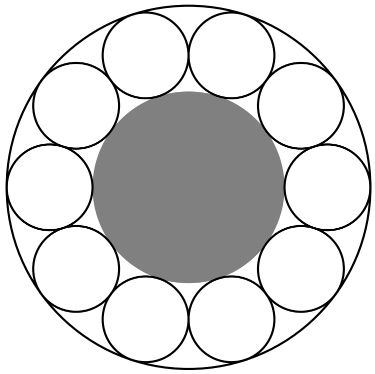
24. Determine the area common to two half-circles that have their diameters on two parallel lines 1 unit apart, and their centers on a line perpendicular to the parallel lines, as in the figure below.



(A) $2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$ **(B)** $2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)$ **(C)** $\frac{\sqrt{3}\,\pi}{4}$ **(D)** $\frac{3\sqrt{2}\,\pi}{4}$ **(E)** NA

25. In the picture, exactly ten circles fit between the inner (shaded) circle and the outer circle, so that each of the ten little circles are tangent to the inner circle, to the outer circle, no two intersect, and each one is tangent to two others (one on each side).

If the radius of the inner circle is 1, determine which of the following is closest to the radius of the outer circle



(A) 1.8940 **(B)** 1.8944 **(C)** 1.8948 **(D)** 1.8952 **(E)** 1.8956