MATH DAY 2010 at FAU

Competition A–Individual

NOTE:

1. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."
2. In all questions, i stands for the imaginary unit; $i^2 = -1$.
3. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.
4. If n is a non-negative integer, then $n!$ stands for the product of all positive integers in the range $1 - n$ if $n \geq 1$, with $0!$ defined to be 1. That is:

   $0! = 1, 1! = 1, 2! = 2, 3! = 2{\cdot}3 = 6, 4! = 2{\cdot}3{\cdot}4 = 24, 5! = 2{\cdot}3{\cdot}4{\cdot}5 = 120, etc.$

5. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

THE ANSWERS

1. The least number of students that must be in a classroom to ensure that there are at least 10 boys or at least 10 girls is

   (A) 10 **(B)** 11 **(C)** 19 **(D)** 20 **(E)** NA

   Solution. With 19 students, if there are less than 10 students of one sex, there have to be at least 10 of the other. If there are less than 10 girls and less than 10 boys, then there are at most $9 + 9 = 18$ students.

   The correct solution is **C.** ■

2.* If greeting cards cost \$2.50 for a box of 12, \$1.25 for a packet of three, or 50 cents each, determine the greatest number of cards that can be purchased for \$14.75.

   Solution. We can buy 5 boxes, one packet and 2 cards, for a total of $5 \times 12 + 3 + 2 = 65$ cards.

   The correct solution is **65.** ■

3. A store cuts the price of an article by 25%. To restore its price to its original value, the store must increase the price by:

   (A) $24\frac{2}{3}\%$ **(B)** 25% **(C)** 30% **(D)** $33\frac{1}{3}\%$ **(E)** NA

   Solution. Let us say that the original price is P. After the cut, the price is $P - .25P = .75P$. If we now increase this price by a percentage x, the price becomes $(1 + \frac{x}{100})0.75P$; equating to P and solving for x we get $x = 100[(1/.75) - 1] = 33.333\ldots$.

   The correct solution is **D.** ■

4. Adam ran up a hill at a speed of 3 miles per hour, and then ran down at a speed of 6 miles per hour. His average speed for the round trip was (in miles per hour)

(A) 4 **(B)** 4.5 **(C)** 5 **(D)** 5.5 **(E)** NA

Solution. Let us call L the distance (in miles) to the top of the hill. The time it took Adam to reach the top was $L/3$ hours. It then only took him $L/6$ hours to get back to the bottom of the hill, so the total time for the round trip was $(L/3)+(L/6)$. The total distance for the round trip being $2L$, his average speed was

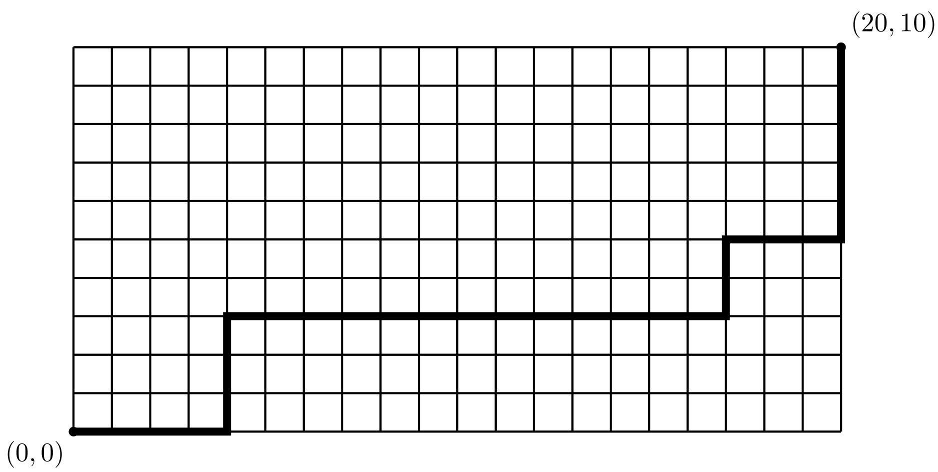
$$\frac{2L}{\frac{L}{3}+\frac{L}{6}} = \frac{2\times 3\times 6\times \cancel{L}}{(3+6)\cancel{L}} = \frac{36}{9} = 4.$$

(Incidentally, given two positive numbers a, b, the number $c = 2ab/(a+b)$ which has the property that its reciprocal $1/c$ is the mean between the reciprocals $1/a, 1/b$ of a, b, is called the *harmonic mean* between a and b. The average velocity in this case is the harmonic mean of the two velocities.)

The correct solution is **A.** ∎

5. The picture below shows grid lines in the plane, all grid lines intersect at points with integer coordinates. The lower left corner of the picture is the origin, the point at $(0,0)$, the point in the upper right corner has coordinates $(20,10)$. We will call a path from the origin to the point $(20,10)$ *proper* if it stays on the grid lines and has a total length of 30. This means that a proper path can only go up or to the right, never down or to the left. One such path is shown with thick lines.

How many proper paths are there from $(0,0)$ to $(20,10)$?



(A) $\frac{30!}{10!20!}$ **(B)** $\frac{30!}{10!}$ **(C)** $\frac{30!}{20!}$ **(D)** $\frac{10!30!}{20!}$ **(E)** $30!$ **(F)** NA

Solution. The trick here is to realize that every path consists of twenty moves of one unit to the right and ten moves of one unit up, in some order. If we write r for a move of one unit to to the right, u for a move of one unit up, every path can be seen as string of length 30 of r's and u's, with exactly 20 r's and 10 u's. For example the path shown in the picture can be coded as

$$rrrruuurrrrrrrrrrrrrruurrruuuuu$$

There are u's precisely in positions 5, 6, 7, 21, 22, 26, 27, 28, 29, 30. It is now easy to see that we get a path precisely for every subset of ten elements of the set $\{1,\ldots,30\}$; we assign to each number in that set the letter u, to each number in the complement of this set the letter r. The number of subsets of 10 elements of a set of 30 elements is

$$\binom{30}{10} = \frac{30!}{10!20!}.$$

The correct solution is **A.** $\blacksquare$

6. The number $10! = 3,628,800$ ends in two zeros. In how many zeros does $500!$ end.

Solution. We can factor $500! = 2^e 5^f a$, where e, f are positive integers and a is an odd number not divisible by 5. It is sort of obvious that $e > f$ so that there will be exactly f trailing 0's. (Every 5 can be multiplied by a 2 to produce a factor of 10.) There are 100 multiples of 5 in the range 1-500. Of these 20 are also multiples of 25; each one of these adds another factor of 5. Of these 20, 4 are also multiples of 125, adding one more 5. There are thus $100+20+4 = 124$ trailing zeros.

The correct solution is **124.** $\blacksquare$

7. For a certain integer n, the numbers $5n+16$ and $8n+29$ have a common factor larger than one. That common factor is:

(A) 11 **(B)** 13 **(C)** 17 **(D)** 19 **(E)** NA

Solution. The common factor has to divide also $5(8n+29)-8(5n+16) = 17$. The factor must be 17.

Another way of coming up with the answer is by trial and error, writing out values of the two expressions. Beginning with $n = 1, 2,$, etc., for $n = 7$, one gets the first case of a common factor; $5 \cdot 7 + 16 = 51 = 3 \cdot 17$ and $8 \cdot 7 + 29 = 85 = 5 \cdot 17$.

The correct solution is **C.** $\blacksquare$

8. In how many ways can one buy 44-cent and 90-cent stamps with exactly 50 dollars?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NA

Solution. Suppose we buy x 44-cent stamps and y 90-cent stamps. Then $44x + 90y = 5000$. Equations of the form $ax + by = c$ have integer solutions if and only the greatest common divisor g of a, b divides c and then all solutions are of the form $x = x_0 + (b/g)k$, $y = y_0 - (a/g)k$, where (x_0, y_0) is a particular solution and $k = 0, \pm 1, \pm 2, \ldots$. In our case $a = 44$, $b = 90$, so $g = 2$, which divides 5000. There are several ways of finding a particular solution (using the Euclidean algorithm, for example), but guessing and/or trial and error can be as good as any. For example, we might notice that two times 44 is 88, that is $44(-2) + 90(1) = 2$, from which $44(-5000) + 90(2500) = 5000$. Thus one solution is $(x_0, y_0) = (-5000, 2500)$. Of course, this solution would mean buying a large negative number of stamps, so it does not apply. But we now know that **all** solutions are of the form $x = -5000 + 45k, y = 2500 - 22k$, k an integer. To get x to be non-negative we need to have $k \geq 5000/45$; that is $k \geq 112$; to get $y \geq 0$ we need $k \leq 2500/22$; that is, $k \leq 113$. For $k = 112$ we get $(x, y) = 40, 36)$; for $k = 113$ we get $(x, y) = (85, 14)$. In other words there are two ways of using up the 50 dollars. We can buy either forty 44-cent stamps and thirty six 90-cent stamps, or we can buy eighty five 44-cent stamps and fourteen 90-cent stamps.

The correct solution is **B.** ■

9. Determine the two last digits of 3^{2010}.

(A) 03 **(B)** 21 **(C)** 27 **(D)** 49 **(E)** 81 **(F)** NA

Solution. We work modulo 100. One approach is to try to determine a pattern among the last two digits as we keep multiplying by three. If we do this, retaining only the last two digits we get the following sequence:

$$03, 09, 27, 81, 43, 29, 87, 61, 83, 49, 47, 41, 23, 69, 07, 21, 63, 89, 67, 01$$

and from now on everything repeats. We have a cycle of length 20. The closest multiple of 20 to 2010 is 2000, so that now, working mod 100 and using $\equiv$ to mean equivalence mod 100 (same last two digits):

$$3^{2010} = 3^{2000}3^{10} \equiv 1 \times 49 = 49$$

(49 being the tenth entry in our list).

One can shorten this process significantly using Euler's extension of Fermat's little theorem; namely that if n is a positive integer and a is an integer relatively prime with n, then $a^{\phi(n)} \equiv 1$ (mod n), where

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_r}\right)$$

is the so called Euler ϕ function of n, and $p_1, \ldots, p_r$ are all the distinct prime factors of n. Then $\phi(100) = 100(1 - \frac{1}{2})(1 - \frac{1}{5}) = 40$ and we would have started with $3^{40} \equiv 1$ (mod 100).

The correct solution is **D.** ■

10. If $\log_a x = 28$ and $\log_b x = 14$, then $\log_b a$ equals

(A) $\dfrac{1}{2}$ **(B)** 2 **(C)** 42 **(D)** 24 **(E)** Can't be determined **(F)** NA

Solution. $\log_a x = 28, \log_b x = 14$ means that $x = a^{28} = b^{14}$; solving $a = b^{14/28} = b^{1/2}$. Taking logarithm in base b, $\log_b a = 1/2$.

The correct solution is **A.** ■

11. Let $f(x)$ be a function such that $f(x) + 2f(-x) = \sin x$ for every real number x. What is the value of $f\left(\frac{\pi}{2}\right)$?

(A) -1 **(B)** $-\dfrac{1}{2}$ **(C)** $\dfrac{1}{2}$ **(D)** 1 **(E)** NA

Solution. Replacing x by $-x$ in the equation for f we get $f(-x) + 2f(x) = -\sin x$. Together with the original equation $f(x) + 2f(-x) = \sin x$ we get a system of two equations and we can solve for $f(x)$ to get $f(x) = -\sin x$. Thus $f(\pi/2) = -\sin(\pi/2) = -1$.

The correct solution is **A.** ■

12. Determine the coefficient of x^5 if the expression $\left((1+x^2)^3-2x\right)^4$ is expanded and written in standard polynomial form.

(A) 0 **(B)** -187 **(C)** -300 **(D)** -384 **(E)** NA

Solution.

$\left((1+x^2)^3-2x\right)^4=(1+x^2)^{12}-8x(1+x^2)^9+24x^2(1+x^2)^6-32x^3(1+x^2)^3+16x^4.$

Of these terms, only $-8x(1+x^2)^9$ and $-32x^3(1+x^2)^3$ will contribute a fifth power of x. If we remember our binomial expansions one sees that the total contribution is

$$-8\binom{9}{2}-32\binom{3}{1}=-384.$$

The correct solution is **D.** ■

13. Suppose it is known that one root of the equation $3x^2+ax+b=0$, where a, b are real numbers, is $2+3i$. The value of b is

(A) Undetermined **(B)** -3 **(C)** 13 **(D)** 26 **(E)** 39 **(F)** NA

Solution. Because all coefficients are real, if a complex number solves the equation, so does its conjugate. It follows that the roots of the equation are $2+3i, 2-3i$. Since the leading coefficient is 1, we have $b/3=(2+3i)(2-3i)=13$.

The correct solution is **E.** ■

14. The remainder of dividing $x^{33}-4x^{31}+32x-7$ by $x-2$ is

(A) 0 **(B)** 2^{31} **(C)** -7 **(D)** 57 **(E)** 64 **(F)** NA

Solution. The remainder of dividing a polynomial by a monomial of the form $x-a$ equals the value of the polynomial at a. In our case the remainder is $2^{33}-4\cdot 2^{31}+32\cdot 2-7=57$.

The correct solution is **D.** ■

15. Tom has 13 numbers. He adds up all possible products of the numbers and finds a sum S. Now, he adds a 14-th number to the list, adds up all possible products of the numbers, and finds a sum T. What is the 14-th number?

(A) $\dfrac{T}{S}$ **(B)** $\dfrac{1+T}{1+S}$ **(C)** $\dfrac{T-S}{1+S}$ **(D)** $\dfrac{T+S}{S-1}$ **(E)** NA

Note. Among "all possible products" we have to include products of a single factor; that is, the 13 numbers themselves so that if the numbers are $a_1, \ldots a_{13}$, then

$$S=a_1+\cdots+a_{13}+a_1a_2+\cdots+a_{12}a_{13}+a_1a_2a_3+\cdots+a_1a_2\cdots a_{13}.$$

Solution. Suppose the fourteenth number is x. Then the possible products consist of all the previous possible products (which add up to S), plus x, plus every previous product multiplied by x (which adds up to Sx). Thus $T=S+x+Sx$ so that $x=(T-S)/(1+S)$

The correct solution is **C.** ■

16. Determine the largest number n such that $1 \leq n \leq 1000$ having an odd number of positive divisors.

(A) 789 **(B)** 841 **(C)** 900 **(D)** 999 **(E)** NA

Solution. Divisors come in pairs; if a divides n so does n/a. The only case in which one has an odd number of divisors is when there is a divisor a such that $a = n/a$; that is, when n is a perfect square. We thus have to determine the largest perfect square ≤ 1000, which is easily calculated as being 961.

The correct solution is **E.** $\blacksquare$

17.* How many positive integers divide **at least one** of the following two numbers: $3^{10} \cdot 35^{20}$, $6^{15} \cdot 5^{10}$?

(Just to make sure, the first number is three to the power ten times thirty five to the power twenty; the second number is six to the fifteen times five to the ten.)

Solution. The number we are looking for is the total number of divisors of $3^{10}35^{20}$, plus the total number of divisors of $6^{15}5^{10}$, minus the number of common divisors.

The total number of positive divisors of a number whose prime factorization is $p_1^{e_1} \cdots p_n^{e_n}$, where $p_1, \ldots, p_n$ are distinct primes and $e_1, \ldots, e_n$ are positive integral powers, is $(e_1 + 1) \cdots (e_n + 1)$. Thus $3^{10}35^{20} = 3^{10}5^{20}7^{20}$ has a total of $11 \times 21 \times 21 = 4851$ divisors, $6^{15}5^{10} = 2^{15}3^{15}5^{10}$ has $16 \times 16 \times 11 = 2816$ divisors. A positive divisor of both numbers must have the form 3^e5^f, where $0 \leq e \leq 10$, $0 \leq f \leq 11$, a total of $11 \times 11 = 121$ numbers. Now $4851 + 2816 - 121 = 7546$.

The correct solution is **7491.** $\blacksquare$

18.* Determine the smallest positive number with exactly 56 divisors.

Solution.

If the prime factorization of a number n is of the form

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

where $p_1, \ldots, p_r$ are distinct primes and $e_1, \ldots, e_r$ are positive integers, then the divisors of n are all (and no more than) the numbers of the form

$$p_1^{f_1} p_2^{f_2} \cdots p_r^{f_r}$$

where $0 \leq f_k \leq e_k$ for $k = 1, \ldots, r$; a total of $(e_1 + 1)(e_2 + 1) \cdots (e_r + 1)$ numbers. If a number is to have 56 divisors, then we must have $(e_1 + 1)(e_2 + 1) \cdots (e_r + 1) = 56$. The possible factorizations of 56 are (not ignoring the case $r = 1$).

$$\begin{aligned} 56 &= 56, \\ 56 &= 7 \cdot 8, \\ &= 2 \cdot 4 \cdot 7, \\ &= 2 \cdot 2 \cdot 2 \cdot 7, \end{aligned}$$

We try to get the smallest number corresponding to each one of these. Concerning the first factorization ($r = 1$), we have a single prime involved with a power of $e_1 = 55$; the smallest number with 56 divisors and a single prime factor is 2^{55}, a pretty large number. Chances are we won't need to compute its exact value. The factorization $56 = 7 \cdot 8$ involves two prime factors; we'd get $n = p_1^6 p_2^7$; obviously the smallest such number is $3^6 2^7 = 93,312$. The factorization $56 = 2 \cdot 4 \cdot 7$ corresponds to numbers of the form $p_1 p_2^3 p_3^6$; the smallest such number is $5 \cdot 3^3 \cdot 2^6 = 8640$. Finally, the factorization $56 = 2 \cdot 2 \cdot 2 \cdot 7$ has as smallest number the number $7 \cdot 5 \cdot 3 \cdot 2^6 = 6720$.

The correct solution is **6720.** ■

19. The equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

has the roots $1, 2, 3, 4$, and 5. Determine c.

(A) 0 **(B)** -121 **(C)** -200 **(D)** -220 **(E)** -225 **(F)** NA

Solution. By Viète's relations,

$$c = -(1\cdot2\cdot3+1\cdot2\cdot4+1\cdot2\cdot5+1\cdot3\cdot4+1\cdot3\cdot5+1\cdot4\cdot5+2\cdot3\cdot4+2\cdot3\cdot5+2\cdot4\cdot5+3\cdot4\cdot5) = -225$$

The correct solution is **E.** ■

20. Suppose it is known that an equation of the form

$$x^n + a_{n-1}x^{n-1} + \cdots a_1 x + 10 = 0,$$

where all coefficients $a_{n-1}, a_{n-2}, \ldots, a_1$ are integers, has four **distinct positive integer** roots m_1, m_2, m_3 and m_4. Determine $m_1 + m_2 + m_3 + m_4$.

(A) 10 **(B)** 18 **(C)** 20 **(D)** 30 **(E)** 36 **(F)** NA

Solution. If $m > 0$ is an integer root of the equation then

$$m^n + a_{n-1}m^{n-1} + \cdots a_1 m + 10 = 0, \text{ thus } m\left(m^{n-1} + a_{n-1}m^{n-2} + \cdots a_1\right) = -10$$

and it follows that m divides 10. Now 10 has exactly four positive divisors, namely 1, 2, 5 and 10 so if the equation has four distinct positive integral roots, they must be $1, 2, 5, 10$, which add up to 18.

The correct solution is **B.** ■

21. Let x, y, z be real numbers such that $x + y + z = 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Then

(A) They are all different, and none of them is 1. **(B)** All three of them **must** coincide.
(C) One of them **must** be 1. **(D)** Two of them **must** equal 1. **(E)** NA

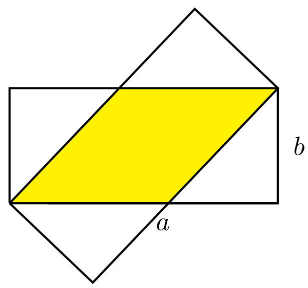
Solution. Another way of writing the equations satisfied by x, y, z is $x + y + z = 1$, $xy + yz + zx = xyz$, which suggests that x, y, z are roots of an equation $X^3 - X^2 + aX - a = 0$. This equation clearly has a root equal to 1. It suggests doing the product $(1-x)(1-y)(1-z)$. One gets

$$(1-x)(1-y)(1-z) = 1 - (x+y+z) + (xy+xz+yz) - xyz = 0.$$

This **proves** that at least one of x, y, z must equal 1; say $x = 1$. Then $y = -z$ and $1/y = -1/z$ (which is the same as $y = -z$), in particular $y \neq z$ (neither y nor z can be 0). At this point we know that the correct answer is either C or D. The key word now is **must**. We can have two of the numbers equal 1, but that is not a necessity. For example, $x = 1, y = 2, z = -2$ is a possible solution.

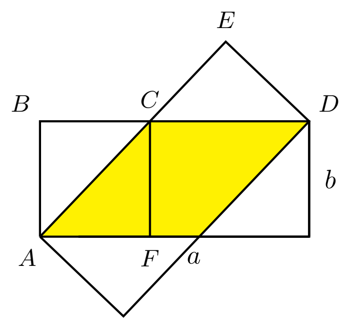
The correct solution is **C.** ■

22. Two congruent rectangles of dimensions $a \times b$ share a common diagonal as shown in the diagram below. Find the area of their overlap.



(A) $\dfrac{a(a^2+b^2)}{b}$ **(B)** $\dfrac{b(a^2+b^2)}{a}$ **(C)** $\dfrac{a(a^2+b^2)}{2b}$ **(D)** $\dfrac{b(a^2+b^2)}{2a}$ **(E)** NA

Solution. Let us add a few labels and lines to the figure for easy reference.



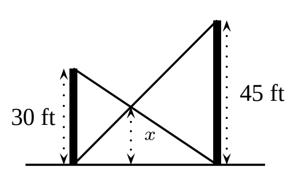
Now $a = |BD|$, $b = |AB|$. It is easy to see that $\triangle ABC$ is congruent to $\triangle DEC$ and the area of the shaded region is $|CD| \times |CF| = |CD| \times b$. Noticing that $|CE| = a - |CD|$ we get by Pythagoras

$|CD|^2 = b^2 + |CE|^2 = b^2 + (a - |CD|)^2$; solving for $|CD|$, $|CD| = \dfrac{a^2+b^2}{2a}$.

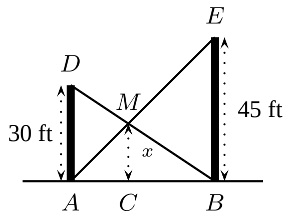
The area is thus $|CD| \times b = \dfrac{b(a^2+b^2)}{2a}$.

The correct solution is **D.** ∎

23.* Two poles, one 30 feet tall, the other one 45 feet tall, are set up near each other. From the top of each pole a wire is stretched to the base of the other pole. How high above the ground do the two wires meet (intersect)? (Determine x in the picture below.)



Solution. Let the poles be at points A, B and let C be the point on the ground above which the two wires meet, as in the picture below. Let M be the point where the wires meet, so $x = |CM|$. Let also D, E be as in the picture.



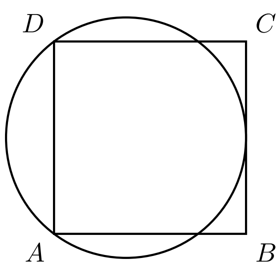
By similarity of triangles $\triangle CBM \sim ABD$, $\dfrac{x}{30} = \dfrac{|CB|}{|AB|}$. By the similarity $\triangle CAM \sim \triangle BAE$, $\dfrac{x}{45} = \dfrac{|AC|}{|AB|}$. Adding:

$$\frac{x}{30} + \frac{x}{45} = \frac{|CB|}{|AB|} + \frac{|AC|}{|AB|} = \frac{|CB| + |AC|}{|AB|} = 1.$$

Solving now for x, $x = (30 \cdot 45)/(30 + 45) = 18$.

The correct solution is **18.** ■

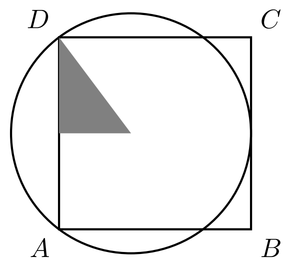
24. A circle is drawn through vertices A and D of a square $ABCD$ in such a way that the circle is tangent to side BC.



If the length of a side of the square is ℓ, the radius of the circle equals

(A) $\dfrac{\ell}{2}$ **(B)** $\dfrac{\ell\sqrt{2}}{2}$ **(C)** $\dfrac{5\ell}{8}$ **(D)** $\dfrac{5\ell\sqrt{2}}{8}$ **(E)** $\dfrac{3\ell}{8}$ **(F)** NA

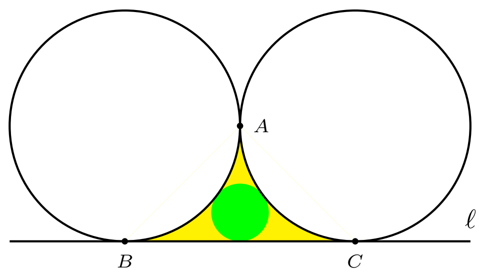
Solution. Consider the shaded triangle in the picture below. It is a right triangle; one leg, the horizontal one, goes from the center of the circle to the side of the square and has length $\ell - r$, the vertical leg has length $\ell/2$. The hypotenuse coincides with the radius of the circle.



By Pythagoras: $r^2 = (r-\ell)^2 + \dfrac{\ell^2}{4}$. Solving for r, $r = 5\ell/8$.

The correct solution is **C.** ■

25. Between two congruent circles that are tangent to each other at A and to a line ℓ at B, C, a third circle is constructed tangent to both circles and also to ℓ. What is the ratio of the area of the small circle to the area of the curvilinear "triangle" ABC?



(A) $\dfrac{1}{2}$ **(B)** $\dfrac{\pi}{8(4-\pi)}$ **(C)** $\dfrac{\pi}{8(4+\pi)}$ **(D)** $\dfrac{1}{3}$ **(E)** NA

Solution. The area of the curvilinear triangle ABC is equal to the area of the rectangle of vertices B, C and the centers of the two congruent (big) circles minus the area of two quarters of these circles; that is, it equals $2r^2 - \dfrac{\pi r^2}{2}$. It is easy to see that the radius ρ of the third (small) circle is $r/4$. For example, if we think of A as being the origin, then the center of the small circle has coordinates $(0, -(r-\rho)$, center of the large circle on the right has coordinates $(r, 0)$. The square of the distance between these two centers is thus $r^2 + (r-\rho)^2)$. On the other hand, because these circles are tangent, the distance is the sum of the radii. Thus $r^2 + (r-\rho)^2) = (r+\rho)^2$ from which $\rho = r/4$ follows. The area of the small circle is thus $\pi r^2/16$ and the answer to the question is

$$\frac{\frac{\pi r^2}{16}}{2r^2 - \frac{\pi r^2}{2}} = \frac{\pi}{8(4-\pi)}.$$

The correct solution is **B.** ■