MATH DAY 2008 at FAU

Competition B-Teams

NOTE:

1. Enter the name of your team on the answer sheet. **Only one answer sheet per team should be handed in.** Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.
2. **Starred Problems** Twenty of the problems are multiple choice. For the other five problems (identified with a star beside their number) the answer is **in every case** a positive integer which you enter directly beside the problem number on the answer sheet. Make sure you write clearly.
3. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."
4. In all questions, i stands for the imaginary unit; $i^2 = -1$.
5. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.
6. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

THE QUESTIONS

1*. The remainder of the polynomial $x^4 - 3x^3 + ax^2 + bx + 3$ when divided by $x - 1$ is 5. When divided by $x + 1$ the remainder is also 5. **Determine** b. Enter your answer directly on the answer sheet.

2. For $x^2 + 2x + 5$ to be a factor of $x^4 + px^2 + q$, the values of p and q must be, respectively:

(A) $-2, 5$ **(B)** $5, 25$ **(C)** $10, 20$ **(D)** $6, 25$ **(E)** $14, 25$

3. Let $1, x_1, x_2, \cdots, x_{10}$ be the roots of the equation $x^{11} = 1$, where $x_1, x_2, \cdots, x_{10}$ are the ten distinct complex non-real roots. Find $(1 - x_1^2)(1 - x_2^2) \cdots (1 - x_{10}^2)$.

(A) 0 **(B)** 1 **(C)** 10 **(D)** 11 **(E)** NA

4. The minimum value of $x^4 + 4x^3 + 6x^2 + 4x + 3$ is

(A) 3 **(B)** 2 **(C)** 1 **(D)** 0 **(E)** NA

5. The sum of the reciprocals of the roots of the equation $x^2 + px + q = 0$, where $p, q \neq 0$, is:

(A) $-p/q$ **(B)** q/p **(C)** p/q **(D)** $-q/p$ **(E)** pq **(F)** NA

(The reciprocal of a number $x \neq 0$ is the number $1/x$)

6. How many distinct solutions does the equation $x^2 - 8[x] + 7 = 0$ have? Here $[x]$ represents the largest integer not exceeding the real number x, also called the *floor* of x. For example, $[\sqrt{2}] = 1$, $[\pi] = 3$, $[-1.2] = -2$, and $[4] = 4$.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NA

7*. If the sum of the first $3n$ positive integers is 150 more than the sum of the first n positive integers, determine the sum of the first $4n$ positive integers. Enter your answer directly on the answer sheet.

8. Three digit numbers are formed using only odd digits. The sum of all such three digit numbers is:

(A) 19375 **(B)** 34975 **(C)** 6253 **(D)** 34975 **(E)** 69375

9. If you have an unlimited supply of 3-cent and 8-cent postage stamps, what is the largest value (in cents) that you can**not** place on an envelope?

(A) 10 **(B)** 13 **(C)** 17 **(D)** 19 **(E)** NA

10. There is a group of children, in which the oldest is 13 and one of them is 10. The sum of their ages is 50. Also, the ages of the children except the one who is 10 form an arithmetic progression. Find the number of children in the group.

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** NA

11. How many zeros does 100! end in? For example, $10! = 3628800$ ends in two zeros.

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** NA

(If n is a positive integer, then $n! = 1 \cdot 2 \cdots (n-1)n$ is the product of all integers from 1 to n)

12. 873 digits are used to number the pages of a book consecutively from page 1. How many pages are there in the book?

(A) 255 **(B)** 290 **(C)** 320 **(D)** 327 **(E)** NA

13. When simplified the product

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n}\right)$$

becomes:

(A) $\frac{1}{n}$ **(B)** $\frac{2}{n}$ **(C)** $\frac{2(n-1)}{n}$ **(D)** $\frac{2}{n(n+1)}$ **(E)** $\frac{3}{n(n+1)}$

14. If the area of a circle is doubled when its radius r is increased by n, then r equals:

(A) $(\sqrt{2}+1)n$ **(B)** $(\sqrt{2}-1)n$ **(C)** n **(D)** $(2-\sqrt{2})n$ **(E)** NA

15. Let $a > 1$ and suppose that x is a **positive** solution of $6a^x + 6a^{-x} = 13$. Then x equals

(A) $\log_a 3$ **(B)** $\log_a 2$ **(C)** $\log_a 3 - \log_a 2$ **(D)** $\log_a 13$ **(E)** $\log_a 6$ **(F)** NA

16. If a, b are real numbers such that $(a+bi)^2 = 3+4i$ and $a < 0$ then b equals:

(A) 1 **(B)** -1 **(C)** 2 **(D)** -2 **(E)** NA

17. Let x, y be **complex** numbers such that $x \neq 0$, $y \neq 0$, $x+y \neq 0$, and satisfying the equation

$$x^3 + x^2y + xy^2 + y^3 = 0.$$

Determine the value of

$$\left(1+\frac{x}{y}\right)^{2008} + \left(1+\frac{y}{x}\right)^{2008}.$$

(A) 2^{1004} **(B)** 2^{1005} **(C)** 2^{2007} **(D)** 2^{2008} **(E)** $2 \cdot 2^{2008}$ **(F)** NA

18*. What is the smallest that $\log_a b + \log_b a$ can be if $a > 1, b > 1$? Enter your answer directly on the answer sheet.

19*. Determine the value of x in degrees such that $0 < x < 36$ and x solves the equation $\sin x + \sin 5x = \cos x + \cos 5x$. Enter your answer directly on the answer sheet.

20. Determine a so that the identity $\cos^4\theta = a + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$ is valid for all θ.

(A) $\frac{1}{4}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{3}{8}$ **(D)** 1 **(E)** NA

21. A quadrilateral has sides of 2, 5, 10 and 11. What is the largest possible area?

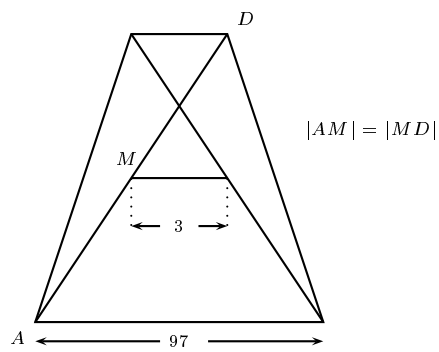
(A) 30 **(B)** 36 **(C)** 44 **(D)** 55 **(E)** NA

22. AB is the hypotenuse of a right triangle ABC. If the median AD is 8 and median BE is 6, then the length of AB is

(A) 5 **(B)** $\sqrt{5}$ **(C)** $2\sqrt{5}$ **(D)** $4\sqrt{5}$ **(E)** 10 **(F)** NA

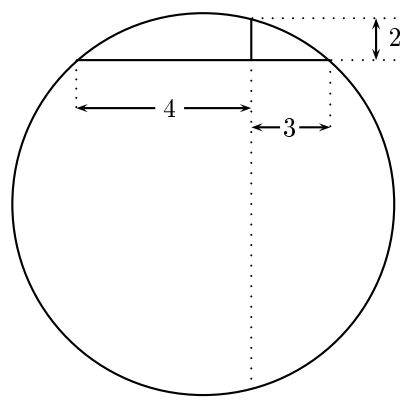
(The *medians* of a triangle are the segments joining a vertex to the midpoint of the opposite side. In our problem, D is the midpoint of the side BC, E is the midpoint of AC.)

23. The line joining the midpoints of the diagonals of a trapezoid has length 3. If the longer base is 97, determine the length of the shorter base.



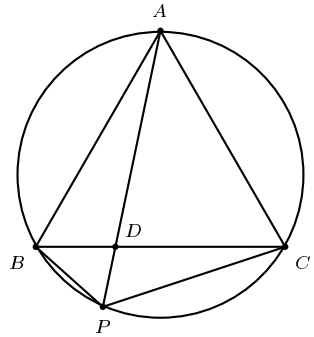
(A) 94 **(B)** 92 **(C)** 91 **(D)** 90 **(E)** 89 **(F)** NA

24. Two perpendicular chords intersect in a circle. The segments of one chord are 3 and 4; one segment of the other has length 2. Determine the diameter of the circle.



(A) $\sqrt{89}$ **(B)** $\sqrt{56}$ **(C)** $\sqrt{61}$ **(D)** $\sqrt{75}$ **(E)** $\sqrt{65}$ **(F)** NA

25.* The equilateral triangle ABC is inscribed in a circle. The point P is chosen on the arc BC and the lines AP, BP, and CP are drawn with $PB = 5$ and $PC = 20$. If AP intersects BC at point D, what is the length of AD?



Enter your answer directly on the answer sheet.