MATH DAY 2008 at FAU

Competition A–Individual

SOLUTIONS

THE QUESTIONS

1. One of the factors of $x^4 + 4$ is

$$\textbf{(A)}\ x^2+2 \qquad \textbf{(B)}\ x+1 \qquad \textbf{(C)}\ x^2-2x+2 \qquad \textbf{(D)}\ x^2-4 \qquad \textbf{(E)}\ \text{NA}$$

SOLUTION. $x+1$ and x^2-4 can be excluded at once; both have real zeroes and x^4+4 does not. One can exclude x^2+2 for a similar reason; it has purely imaginary zeroes and x^4+4 does not. C seems like the most likely answer. To be absolutely sure that the answer is not E, we divide x^4+4 by x^2-2x+2 getting a quotient of x^2+2x+2 and a remainder of 0.

Alternatively, we can notice directly that

$$x^4+4 = x^4+4x^2+4-4x^2 = (x^2+2)^2-(2x)^2 = (x^2-2x+2)(x^2+2x+2)$$

The correct solution is **C.** ■

2. The degree of $(x^4+x+2)^5-(x^4+x-3)^5$ as a polynomial in x is

$$\textbf{(A)}\ 20 \qquad \textbf{(B)}\ 19 \qquad \textbf{(C)}\ 18 \qquad \textbf{(D)}\ 17 \qquad \textbf{(E)}\ 16 \qquad \textbf{(F)}\ \text{NA}$$

SOLUTION. Recall (using Pascal's triangle, for example) that

$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$$

Expanding $(x^4+x+2)^5-(x^4+x-3)^5 = ((x^4+x)+2)^5-((x^4+x)-3)^5$ one gets

$$\begin{array}{rl} & (x^4+x)^5+10(x^4+x)^4+40(x^4+x)^3+\cdots \\ - & (x^4+x)^5+15(x^4+x)^4-90(x^4+x)^3+\cdots \end{array}$$

The power 5 terms cancel. One sees that the highest remaining power of x is 16, with a coefficient of 25.

The correct solution is **E.** ■

3. What is the coefficient of x in the polynomial $(1+x)+(1+x)^2+(1+x)^3+\cdots+(1+x)^n$?

$$\textbf{(A)}\ n(n+1) \qquad \textbf{(B)}\ n(n+1)/2 \qquad \textbf{(C)}\ n \qquad \textbf{(D)}\ n+1 \qquad \textbf{(E)}\ \text{NA}$$

SOLUTION. The coefficient of x in $(1+x)^k$ is $\binom{k}{1} = k$, thus the coefficient of x in the given polynomial is

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

Alternatively

$$(1+x)+(1+x)^2+(1+x)^3+\cdots+(1+x)^n = \frac{(1+x)^{n+1}-1}{x}$$

and the question can be reformulated to: What is the coefficient of x^2 in $(1+x)^{n+1}$? The answer is $\binom{n+1}{2} = n(n+1)/2$.

The correct solution is **B.** ■

4. The symbol $|x|$ means x if x is not negative and $-x$ if x is not positive. The **number** of solutions of the equation

$$|x|^2 + |x| - 6 = 0$$

is

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

SOLUTION. By the quadratic formula, or by inspection, one sees that the roots of the equation $r^2 + r - 6 = 0$ are 2 and -3. Of these, only 2 can be the absolute value of a number. Thus $|x| = 2$, hence $x = 2$ and $x = -2$ are solutions (and the only solutions).

The correct solution is **C.**

5. If $x + y = 10$; and $xy = 20$, then $\frac{1}{x} + \frac{1}{y}$ is:

(A) $\frac{1}{10}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** 4

SOLUTION. If $x + y = 10$ and $xy = 20$, then

$$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{10}{20} = \frac{1}{2}$$

The correct solution is **B.**

6. If the points $(2, -3)$, $(4, 3)$, and $(5, k/2)$ are on the same straight line, then k equals:

(A) 12 **(B)** -12 **(C)** ± 12 **(D)** 6 **(E)** -6

SOLUTION. The slope of the line determined by $(2, -3)$ and $(4, 3)$ is $(3 - (-3))/(4 - 2) = 3$; the slope of the line determined by $(4, 3)$ and $(5, k/2)$ is $(k/2 - 3)/(5 - 4) = (k - 6)/2$ Since both lines have to be the same, $(k - 6)/2 = 3$. Thus $k = 12$.

The correct solution is **A.**

7. The arithmetic mean (average) of a set of 50 numbers is 38. If two numbers, namely 45 and 55, are discarded, the mean of the remaining set of numbers is:

(A) 36.5 **(B)** 37 **(C)** 37.2 **(D)** 37.5 **(E)** 37.52

SOLUTION. Let S be the sum of the 50 numbers. Since the average is 38, we must have $S = 38 \cdot 50 = 1900$. Discarding 45 and 55 leaves us with 48 numbers adding up to $1900 - (45 + 55) = 1800$. The average is $1800/48 = 37.5$.

The correct solution is **D.**

8. How many positive integers, including 1 and 756, are there that exactly divide 756?

(A) 6 **(B)** 18 **(C)** 24 **(D)** 36 **(E)** NA

SOLUTION. One way of getting an answer of which one can feel sure is to notice that if we factor 756 into primes we get

$$756 = 2^2 \cdot 3^3 \cdot 7.$$

It follows that a number is a divisor of 756 if and only if it is of the form $2^a \cdot 3^b \cdot 7^c$ where $a = 0, 1$, or 2; $b = 0, 1, 2$ or 3; $c = 0$ or 1. That is, there are 3 possible values for a, 4 for b and 2 for c. The total number of divisors is thus $3 \times 4 \times 2 = 24$.

The correct solution is **C.**

9*. Determine the smallest positive number that has a remainder of 1 when divided by 3, of 3 when divided by 4, and of 4 when divided by 5. Enter your answer directly on the answer sheet.

SOLUTION. If you know how to work this out with congruences, that is the best way of doing it. However, this problem has a small enough solution and it can be found by guessing or inspection. If we simply write out numbers that have a remainder of 1 when divided by 3, of 3 when divided by 4, of 4 when divided by 5, we see that 19 is the first number in all three lists.

The number to be entered should be **19.** ■

10*. The sum of three numbers is 98. The ratio of the first to the second is 2/3, and the ratio of the second to the third is 5/8. Determine the value of the **second** number. Enter your answer directly on the answer sheet.

SOLUTION. If the numbers are a, b, c then $a + b + c = 98$, $a/b = 2/3$, and $b/c = 5/8$. Then $a = (2/3)b$, $c = (8/5)b$ and

$$\frac{2}{3}b + b + \frac{8}{5}b = 98;$$

solving, $b = 30$.

The number to be entered should be **30.** ■

11. The expression $2 + \sqrt{2} + \dfrac{1}{2+\sqrt{2}} + \dfrac{1}{\sqrt{2}-2}$ is equal to:

(A) 2 **(B)** $2 - \sqrt{2}$ **(C)** $2 + \sqrt{2}$ **(D)** $2\sqrt{2}$ **(E)** NA

SOLUTION. Notice $(2+\sqrt{2})(\sqrt{2}-2) = -2$. Thus

$$\begin{aligned} 2 + \sqrt{2} + \frac{1}{2+\sqrt{2}} + \frac{1}{\sqrt{2}-2} &= \frac{(2+\sqrt{2})(2+\sqrt{2})(\sqrt{2}-2) + \sqrt{2} - 2 + 2 + \sqrt{2}}{(2+\sqrt{2})(\sqrt{2}-2)} \\ &= \frac{(-2)(2+\sqrt{2}) + 2\sqrt{2}}{-2} = 2. \end{aligned}$$

The correct solution is **A.** ■

12. By adding the same constant to each of 20, 50, 100 a geometric progression results. The common ratio is:

(A) 5/3 **(B)** 4/3 **(C)** 3/2 **(D)** 1/2 **(E)** 1/3 **(F)** NA

SOLUTION. Let a be the constant added to each number, r the ratio of the three term geometric progression. Then

$$\frac{50+a}{20+a} = r = \frac{100+a}{50+a}.$$

From $(50+a)/(20+a) = (100+a)/(50+a)$ we get $a = 25$, hence $r = (50+a)/(20+a) = 5/3$.

The correct solution is **A.** ■

13. Charles has $5q + 1$ quarters and Richard has $q + 5$ quarters. The difference in their money in dimes is:

(A) $10(q-1)$ **(B)** $2(4q-4)/5$ **(C)** $5(q-1)/2$ **(D)** $2(q-1)/5$ **(E)** NA

SOLUTION. The difference , in quarters, is $4q - 4 = 4(q-1)$. Since one quarter equals 2.5 dimes, in dimes the difference is $2.5 \cdot 4(q-1) = 10(q-1)$ dimes.

The correct solution is **A.** ■

14. Peter takes 4 hours to paint a wall of a certain length. Paul takes 6 hours to do the same job. If both are to paint a wall twice as long, Paul beginning at one end, Peter at the other, how much time would it take them to complete the job?

(A) 5 hours **(B)** $5\frac{1}{2}$ hours **(C)** $4\frac{1}{2}$ hours **(D)** 4 hours 48 minutes **(E)** 5 hours 12 minutes

Say that the wall that Peter paints in 4 hours, Paul in 6, has length L. Let T be the time it took Peter and Paul to finish painting a wall of length $2L$. If Peter painted a length a of that double wall, Paul a length b ($a + b = 2L$), then $T/4 = a/L$, $T/6 = b/L$; adding $T/4 + T/6 = (a + b)/L = 2$. It follows that

$$\frac{1}{T} = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{6}\right) = \frac{5}{24}; \qquad T = \frac{24}{5} = 4.8.$$

(T is the *harmonic mean* of $4, 6$). Since 0.8 of an hour equals 48 minutes, the time is 4 hours and 48 minutes.

The correct solution is **D.** ■

15. A wheel with a rubber tire has an outside **diameter** of 25 inches. When the **radius** has been decreased a quarter of an inch, the number of revolutions of the wheel in one mile will:

(A) be increased about 2% **(B)** be increased about 1% **(C)** be increased about 20%
(D) be increased about 0.5% **(E)** remain the same

SOLUTION. A wheel of radius r rolling (without slipping, of course) for a distance d will have completed $d/2\pi r$ revolutions. The percentage increase in the number of revolutions if the radius is decreased by a (measured in the same units as r) is thus

$$100\,\frac{\dfrac{d}{2\pi(r-a)} - \dfrac{d}{2\pi r}}{\dfrac{d}{2\pi(r-a)}} = 100\frac{a}{r}.$$

In our case the $r = 25/2 = 12.5$; this works out to $100 \cdot (0.25)/12.5 = 2$.

The correct solution is **A.** ■

16. In the puzzle sudoku, one fills in a 9 by 9 square grid with numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, with exactly one occurrence of each of these numbers in each row, one in each column, and one in each of the nine 3 by 3 boxes. Suppose that you are constructing such a puzzle, and have filled in the upper left 3 by 3 box completely. How many possible arrangements of numbers remain for the upper right 3 by 3 box?

(A) $1,082$ **(B)** $12,096$ **(C)** $40,324$ **(D)** $362,880$ **(E)** NA

SOLUTION.

We may as well assume that upper left square is filled with the numbers in order:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

In filling the upper right square we have the following options:

(a) We can use the numbers 4, 5, 6 in the first row. This forces us to use 1, 2, 3 in the third row and 7, 8, 9 in the second row. Since each one of these sets of three numbers ($\{4, 5, 6\}$, $\{1, 2, 3\}$, $\{7, 8, 9\}$ can be arranged in $3! = 6$ different ways, we have $6^3 = 216$ choices.

(b) The same analysis applies if we use 7, 8, 9 for the first row. Again we have 216 choices.

(c) We can select two numbers from the set $\{4,5,6\}$ and one from $\{7,8,9\}$ for the first row. This can be done in $\binom{3}{2}\cdot\binom{3}{1}=9$ different ways. Say we select $4,5,7$. As before these can be arranged in $3!=6$ different ways as entries for the first row. The two remaining elements of the set $\{7,8,9\}$ (8, 9 in the example at hand) must be placed in the second row, plus one additional element from the set $\{1,2,3\}$. This element can be chosen in $\binom{3}{1}=3$ different ways. After this, the elements of each row are decided; the only question is which of the 6 possible arrangements we use. The total number of choices is thus:

$$\binom{3}{2}\cdot\binom{3}{1}\cdot 3!\cdot\binom{3}{1}\cdot 3!\cdot 3!=5832$$

choices.

(d) The same analysis applies if we select two numbers from $\{7,8,9\}$, one from $\{4,5,6\}$ for the first row. Another 5832 choices.

This exhausts all possible choices. Counting:

$$216+216+5832+5832=12096.$$

The correct solution is **B.** ■

17. 10 boys and 10 girls are to be seated in a row. What is the probability that no two boys or two girls seat next to each other?

(A) $\frac{10!}{20!}$ **(B)** $\frac{10!\cdot 10!}{20!}$ **(C)** $\frac{10!+10!}{20!}$ **(D)** $\frac{2\cdot 10!\cdot 10!}{20!}$ **(E)** NA

SOLUTION. The total number of different ways we can arrange the 10 boys and the 10 girls is $20!$. To count the number of ways they can be seated so that no boy nor girl sits next to a boy or girl, let us number the seats from 1 to 20. We can then seat all the girls in odd numbered seats, the boys in even numbered seats, or vice-versa. There are $10!$ different ways of seating all the girls in odd numbered seats, $10!$ ways of seating all the boys in even numbered seat, so that if we go that route we have $10!\cdot 10!$ different seatings. We have the same number of different seatings if we decide to seat all boys in odd numbered seats, the girls in the even numbered ones. In all, we have $2\cdot(10!)^2$ different ways of accomplishing the objective.

The correct solution is **D.** ■

18. If $4^x-4^{x-1}=24$, then $(2x)^x$ equals:

(A) $5\sqrt{5}$ **(B)** $\sqrt{5}$ **(C)** $25\sqrt{5}$ **(D)** 125 **(E)** 25 **(F)** NA

SOLUTION. $4^x-4^{x-1}=4^{x-1}(4-1)=3\cdot 4^{x-1}$ thus $4^x-4^{x-1}=24$ implies $4^{x-1}=8$. Since $8=2^3$ it is convenient to write $=2^2$, getting $2^{2(x-1)}=2^3$, hence $2(x-1)=3$. From this we conclude that $2x=5$, $x=5/2=2\frac{1}{2}$, thus

$$(2x)^x=5^{5/2}=25\sqrt{5}.$$

The correct solution is **C.** ■

19*. If α is an acute angle and $\tan\alpha=4$, then $\sin(2\alpha)$ can be written in the form a/b, where a,b are positive integers with no common divisor other than 1. Enter the value of $a+b$ in the answer sheet.

SOLUTION. We find first the values of $\cos\alpha$, $\sin\alpha$, for example by drawing a right triangle with angle α and horizontal leg of length 1, vertical length of length 4. The hypotenuse is then of length $\sqrt{17}$ so that $\cos\alpha=1/\sqrt{17}$, $\sin\alpha=4/\sqrt{17}$. Now

$$\sin(2\alpha)=2\sin\alpha\cos\alpha=2\cdot\frac{1}{\sqrt{17}}\cdot\frac{4}{\sqrt{17}}=\frac{8}{17}.$$

The number to be entered should be **25.** ■

20.* What is the last digit (unit digit) of 2^{2008}? Enter your answer directly on the answer sheet.

SOLUTION. looking at the first few powers of 2, we see that the last digits display a clear pattern:

$$2,\ 4,\ 8\ 16,\ 32,\ 64\ 128,\ 256\ \ldots;$$

the pattern is $2, 4, 8, 6$ repeated again and again. When the power is a multiple of 4, as 2008 is, the last digit is 6.

The number to be entered should be **6.** ■

21.* The **sum** of the roots of the equation $x^{\log_5 x} = 625$ can be written as a fraction a/b where a and b have no common divisor other than 1. Enter the **numerator** a directly on the answer sheet.

SOLUTION. Taking $\log_5$ of both sides of the equation, we get $(\log_5 x)^2 = \log_5 625 = 4$, thus $\log_5 x = \pm 2$, so that the roots are $5^2 = 25$ and $5^{-2} = 1/25$ and $25 + (1/25) = 626/25$.

The number to be entered should be **626.** ■

22. The area of a circle inscribed in a regular hexagon is 100π. The area of the hexagon is:

(A) 600 **(B)** 300 **(C)** $200\sqrt{2}$ **(D)** $200\sqrt{3}$ **(E)** $200\sqrt{5}$

SOLUTION. The side of a hexagon circumscribed to a circle of radius r is $2r/\sqrt{3}$, as can easily be deduced using a bit of geometry. The hexagon is thus made up of 6 equilateral triangles of sides $2r/sqrt3$ (and height r) and from this we conclude that its area is

$$6\left(\frac{1}{2}\cdot\frac{2r}{\sqrt{3}}\cdot r\right) = \frac{6r^2}{\sqrt{3}} = 2\sqrt{3}\,r^2.$$

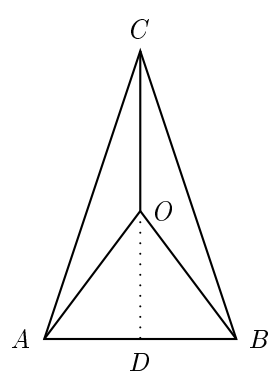
In our case, the circle has area 100π, thus $r = 10$ and $2\sqrt{3}\,r^2 = 200\sqrt{3}$.

The correct solution is **D.** ■

23. The base of an isosceles triangle is 6 inches and one of the equal sides is 12 inches. The radius of the circle through the vertices of the triangle is:

(A) $8\sqrt{15}/5$ **(B)** $4\sqrt{3}$ **(C)** $3\sqrt{5}$ **(D)** $6\sqrt{3}$ **(E)** NA

SOLUTION. The radius of the circumcircle of an isosceles triangle of base a, equal sides of length b and height h is $r = b^2/2h$. To see this, consider the following picture:



O is the center of circle, thus $r = |OA| = |OB| = |OC|$. WE also have $|AB| = a$, $|AC| = |BC| = b$, $|AD| = h$. Applying the theorem of Pythagoras to the right triangle ADO, we get

$$r^2 = |AO|^2 = |AD|^2 + |DO|^2 = \left(\frac{a}{2}\right)^2 + (h-r)^2 = \frac{a^2}{4} + h^2 - 2rh + r^2.$$

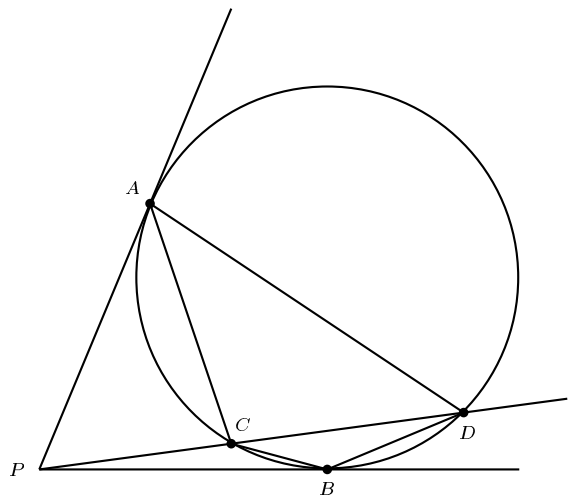
Considering that (again by Pythagoras) $h^2 + (a^2/4) = b^2$, the formula follows. In our case

$$b = 12, \quad h = \sqrt{b^2 - \frac{a^2}{4}} = \sqrt{144 - 9} = \sqrt{135} = 3\sqrt{15}.$$

We get $r = 144/(6\sqrt{15}) = 8\sqrt{15}/5$.

The correct solution is **A.** ■

24. Three lines are drawn from a point P. Two of them are tangent to a circle at A and at B, respectively; the third one intersects the circle at C and at D. If the lengths of the segments AC, CB, DB are $|AC| = 6$, $|BC| = 3$, $|BD| = 4$, determine $|AD|$.



Hints: Two useful theorems are: *The measure in radians of an angle with its vertex on a circle and sides formed by a tangent and a secant is one half the length of the intercepted arc. The measure in radians of an inscribed angle in a circle is one half the measure of the intercepted arc.* One immediate consequence of these theorems is, for example, $\angle PAC = \angle PDA$. There are more equal angles floating about.

(A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** NA

SOLUTION. As mentioned in the hint, $\angle PAC = \angle PDA$. Obviously $\angle APC = APD$. Thus the triangles $\triangle APC$ and $\triangle DPA$ are similar. Thus

$$\frac{|AC|}{|AP|} = \frac{|DA|}{|DP|}, \quad \text{hence } \frac{|AD|}{|AC|} = \frac{|PD|}{|PA|}.$$

Similarly $\triangle BPC$ and $\triangle DPB$ are similar and

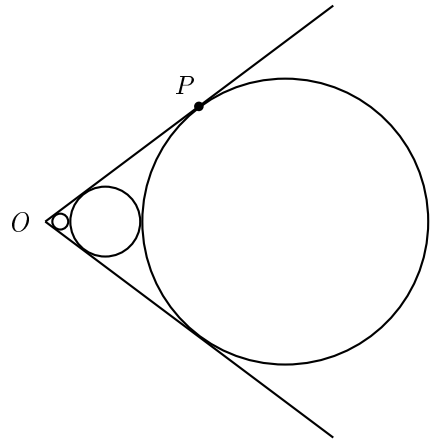
$$\frac{|BC|}{|BP|} = \frac{|DB|}{|DP|}, \quad \text{hence } \frac{|BD|}{|BC|} = \frac{|PD|}{|PB|}.$$

Since $|PB| = |PA|$, we see that $\dfrac{|AD|}{|AC|} = \dfrac{|BD|}{|BC|}$, thus

$$|AD| = \frac{|AC| \cdot |BD|}{|BC|} = \frac{6 \cdot 4}{3} = 8.$$

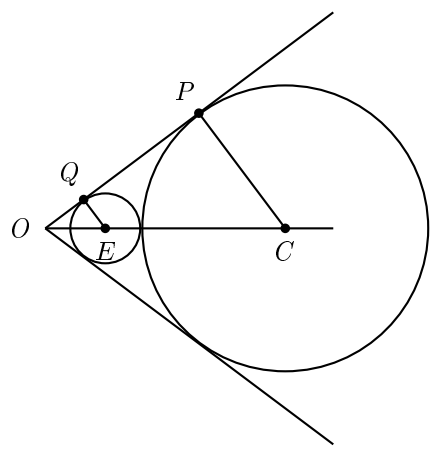
The correct solution is **B.** ■

25. Three circles are placed between two lines as in the diagram. The circles are tangent to the lines, and the middle circle is tangent to the two other circles. If $\overline{OP} = 12$ and the radius of the **largest** circle is 5, determine the radius of the smallest circle.



(A) $\frac{80}{81}$ **(B)** $\frac{81}{80}$ **(C)** $\frac{9}{\sqrt{80}}$ **(D)** $\frac{\sqrt{80}}{81}$ **(E)** NA

SOLUTION. Consider first the situation of two general circles placed between two lines, so they are tangent to each other and the lines.



In the picture, C is the center of the larger circle, E of the smaller circle. Set $R = |CP|$ the radius of the larger circle, $r = |EQ|$, the radius of the smaller circle. Let us also set $A = |OP|$ and $a = |OQ|$. Notice that $\triangle OEQ$ and $\triangle OCP$ are similar right triangles. Now

$$\begin{aligned} |OC|^2 &= A^2 + R^2, \\ \frac{|OE|}{|OC|} &= \frac{r}{R}, \\ |OE| &= |OC| - R - r. \end{aligned}$$

The first equation is due to the theorem of Pythagoras, the second to similarity of triangles, and the third one is obvious. Solving the second and third equations for r and $|OE|$ we get

$$r = \frac{R(|OC| - R)}{|OC| + R}, \quad |OE| = \frac{|OC|(|OC| - R)}{|OC| + R}.$$

Thus

$$a = \sqrt{|OE|^2 - r^2} = \frac{|OC| - R}{|OC| + R}\sqrt{|OC|^2 - R^2} = A\frac{|OC| - R}{|OC| + R}.$$

As it turns out, a is not needed.

Going armed with this to the problem at hand, taking $R = 5$ and $A = 12$, thus $|OC| = \sqrt{12^2 + 5^2} = 13$, we find that the middle circle has radius $r = 5(13-5)/(13+5) = 20/9$, and its center is at distance $13(13-5)/(13+5) = 52/9$ from P. We now simply have to switch circles, the middle circle becomes the large circle, the third circle the small circle. We find the radius of the small circle by applying the formula for r with $R = 20/9$ and $|OC| = 52/9$. We get

$$r = \frac{\frac{20}{9}\left(\frac{52}{9} - \frac{20}{9}\right)}{\frac{52}{9} + \frac{20}{9}} = \frac{80}{81}$$