MATH DAY 2008 at FAU

Competition A–Individual

NOTE:

1. Enter your name on the answer sheet. Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.
2. **Starred Problems** Twenty of the problems are multiple choice. For the other five problems (identified with a star beside their number) the answer is **in every case** a positive integer which you enter directly beside the problem number on the answer sheet. Make sure you write clearly.
3. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."
4. In all questions, i stands for the imaginary unit; $i^2 = -1$.
5. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.
6. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

THE QUESTIONS

(25 problems; 4 pages not counting the cover sheet nor the answer sheet)

1. One of the factors of $x^4 + 4$ is

   (A) $x^2 + 2$ **(B)** $x + 1$ **(C)** $x^2 - 2x + 2$ **(D)** $x^2 - 4$ **(E)** NA

2. The degree of $(x^4 + x + 2)^5 - (x^4 + x - 3)^5$ as a polynomial in x is

   (A) 20 **(B)** 19 **(C)** 18 **(D)** 17 **(E)** 16 **(F)** NA

3. What is the coefficient of x in the polynomial $(1 + x) + (1 + x)^2 + (1 + x)^3 + \cdots + (1 + x)^n$?

   (A) $n(n + 1)$ **(B)** $n(n + 1)/2$ **(C)** n **(D)** $n + 1$ **(E)** NA

4. The symbol $|x|$ means x if x is not negative and $-x$ if x is not positive. The **number** of solutions of the equation

   $$|x|^2 + |x| - 6 = 0$$

   is

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

5. If $x + y = 10$; and $xy = 20$, then $\frac{1}{x} + \frac{1}{y}$ is:

   (A) $\frac{1}{10}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** 4

6. If the points $(2, -3)$, $(4, 3)$, and $(5, k/2)$ are on the same straight line, then k equals:

   (A) 12 **(B)** -12 **(C)** ± 12 **(D)** 6 **(E)** -6

7. The arithmetic mean (average) of a set of 50 numbers is 38. If two numbers, namely 45 and 55, are discarded, the mean of the remaining set of numbers is:

   (A) 36.5 **(B)** 37 **(C)** 37.2 **(D)** 37.5 **(E)** 37.52

8. How many positive integers, including 1 and 756, are there that exactly divide 756?

(A) 6 **(B)** 18 **(C)** 24 **(D)** 36 **(E)** NA

9*. Determine the smallest positive number that has a remainder of 1 when divided by 3, of 3 when divided by 4, and of 4 when divided by 5. Enter your answer directly on the answer sheet.

10*. The sum of three numbers is 98. The ratio of the first to the second is 2/3, and the ratio of the second to the third is 5/8. Determine the value of the **second** number. Enter your answer directly on the answer sheet.

11. The expression $2+\sqrt{2}+\dfrac{1}{2+\sqrt{2}}+\dfrac{1}{\sqrt{2}-2}$ is equal to:

(A) 2 **(B)** $2-\sqrt{2}$ **(C)** $2+\sqrt{2}$ **(D)** $2\sqrt{2}$ **(E)** NA

12. By adding the same constant to each of 20, 50, 100 a geometric progression results. The common ratio is:

(A) 5/3 **(B)** 4/3 **(C)** 3/2 **(D)** 1/2 **(E)** 1/3 **(F)** NA

13. Charles has $5q+1$ quarters and Richard has $q+5$ quarters. The difference in their money in dimes is:

(A) $10(q-1)$ **(B)** $2(4q-4)/5$ **(C)** $5(q-1)/2$ **(D)** $2(q-1)/5$ **(E)** NA

14. Peter takes 4 hours to paint a wall of a certain length. Paul takes 6 hours to do the same job. If both are to paint a wall twice as long, Paul beginning at one end, Peter at the other, how much time would it take them to complete the job?

(A) 5 hours **(B)** $5\frac{1}{2}$ hours **(C)** $4\frac{1}{2}$ hours **(D)** 4 hours 48 minutes **(E)** 5 hours 12 minutes

15. A wheel with a rubber tire has an outside **diameter** of 25 inches. When the **radius** has been decreased a quarter of an inch, the number of revolutions of the wheel in one mile will:

(A) be increased about 2% **(B)** be increased about 1% **(C)** be increased about 20%
(D) be increased about 0.5% **(E)** remain the same

16. In the puzzle sudoku, one fills in a 9 by 9 square grid with numbers from the set 1,2,3,4,5,6,7,8,9, with exactly one occurrence of each of these numbers in each row, one in each column, and one in each of the nine 3 by 3 boxes. Suppose that you are constructing such a puzzle, and have filled in the upper left 3 by 3 box completely. How many possible arrangements of numbers remain for the upper right 3 by 3 box?

(A) $1,082$ **(B)** $12,096$ **(C)** $40,324$ **(D)** $362,880$ **(E)** NA

17. 10 boys and 10 girls are to be seated in a row. What is the probability that no two boys or two girls seat next to each other?

(A) $\dfrac{10!}{20!}$ **(B)** $\dfrac{10!\cdot 10!}{20!}$ **(C)** $\dfrac{10!+10!}{20!}$ **(D)** $\dfrac{2\cdot 10!\cdot 10!}{20!}$ **(E)** NA

18. If $4^x-4^{x-1}=24$, then $(2x)^x$ equals:

(A) $5\sqrt{5}$ **(B)** $\sqrt{5}$ **(C)** $25\sqrt{5}$ **(D)** 125 **(E)** 25 **(F)** NA

19*. If α is an acute angle and $\tan\alpha=4$, then $\sin(2\alpha)$ can be written in the form a/b, where a,b are positive integers with no common divisor other than 1. Enter the value of $a+b$ in the answer sheet.

20*. What is the last digit (unit digit) of 2^{2008}? Enter your answer directly on the answer sheet.

21*. The **sum** of the roots of the equation $x^{\log_5 x}=625$ can be written as a fraction a/b where a and b have no common divisor other than 1. Enter the **numerator** a directly on the answer sheet.

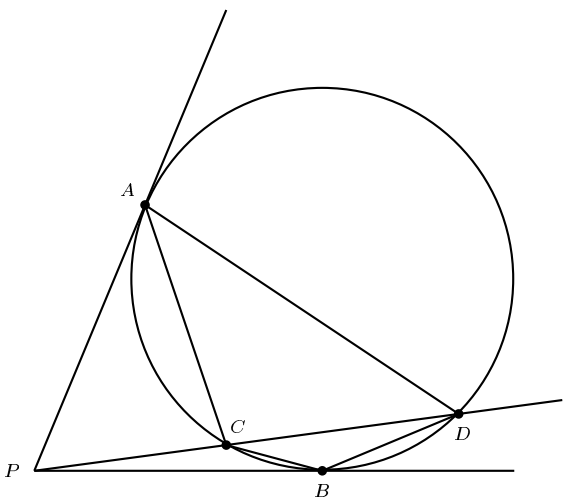
22. The area of a circle inscribed in a regular hexagon is 100π. The area of the hexagon is:

(A) 600 **(B)** 300 **(C)** $200\sqrt{2}$ **(D)** $200\sqrt{3}$ **(E)** $200\sqrt{5}$

23. The base of an isosceles triangle is 6 inches and one of the equal sides is 12 inches. The radius of the circle through the vertices of the triangle is:

(A) $8\sqrt{15}/5$ **(B)** $4\sqrt{3}$ **(C)** $3\sqrt{5}$ **(D)** $6\sqrt{3}$ **(E)** NA

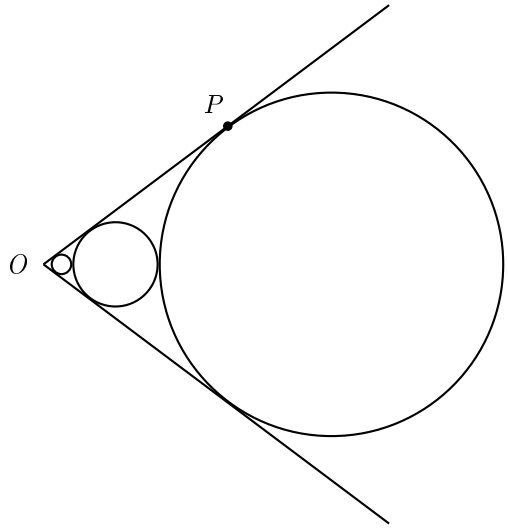
24. Three lines are drawn from a point P. Two of them are tangent to a circle at A and at B, respectively; the third one intersects the circle at C and at D. If the lengths of the segments AC, CB, DB are $|AC| = 6$, $|BC| = 3$, $|BD| = 4$, determine $|AD|$.



Hints: Two useful theorems are: *The measure in radians of an angle with its vertex on a circle and sides formed by a tangent and a secant is one half the length of the intercepted arc. The measure in radians of an inscribed angle in a circle is one half the measure of the intercepted arc.* One immediate consequence of these theorems is, for example, $\angle PAC = \angle PDA$. There are more equal angles floating about.

(A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** NA

25. Three circles are placed between two lines as in the diagram. The circles are tangent to the lines, and the middle circle is tangent to the two other circles. If $\overline{OP} = 12$ and the radius of the **largest** circle is 5, determine the radius of the smallest circle.



(A) $\frac{80}{81}$ **(B)** $\frac{81}{80}$ **(C)** $\frac{9}{\sqrt{80}}$ **(D)** $\frac{\sqrt{80}}{81}$ **(E)** NA

ANSWER SHEET

NAME: _____ (Please **print** clearly)

NAME OF YOUR SCHOOL: _____ (Please **print** clearly)

1	A	B	C	D	E	F	15	A	B	C	D	E	F
2	A	B	C	D	E	F	16	A	B	C	D	E	F
3	A	B	C	D	E	F	17	A	B	C	D	E	F
4	A	B	C	D	E	F	18	A	B	C	D	E	F
5	A	B	C	D	E	F	19	_	_	_	_	_	_
6	A	B	C	D	E	F	20	_	_	_	_	_	_
7	A	B	C	D	E	F	21	_	_	_	_	_	_
8	A	B	C	D	E	F	22	A	B	C	D	E	F
9	_	_	_	_	_	_	23	A	B	C	D	E	F
10	_	_	_	_	_	_	24	A	B	C	D	E	F
11	A	B	C	D	E	F	25	A	B	C	D	E	F
12	A	B	C	D	E	F							
13	A	B	C	D	E	F							
14	A	B	C	D	E	F							