MATH CIRCLE AT FAU

11/18/2017

Middle School Version

Session # 5

Solutions

1. This is a classic question. A census-taker knocks on a door and asks the woman inside how many children she has and how old they are.

   "I have three daughters, their ages are whole numbers, and the product of their ages is 36."

   "That's not enough information," responds the census-taker.

   "I'd tell you the sum of their ages, but you'd still be stumped."

   "I wish you'd tell me something more."

   "O.K., my oldest daughter Anne likes dogs."

   How old are the daughters?

   Solution. The table below shows all triples multiplying to 36 and their sum

1	1	36	38
1	2	18	21
1	3	12	16
1	4	9	14
1	6	6	13
2	2	9	13
2	3	6	11
3	3	4	10

   If we look at the sums of ages, we see that the only repetition is if the ages add up to 13; so that's the only situation in which the sum of ages doesn't give away the ages. Because the mother refers to an oldest daughter, she is excluding having two oldest daughters. The answer is that she has twins aged 2, and a 9 year old daughter.

2. A farmers market sells only baskets of apples and baskets of oranges, each for a fixed whole number of dollars. Abigail paid \$23 for five baskets of fruit. Benedict paid \$24 for four baskets of fruit. If Charles paid \$20, how many baskets of fruit did he buy?

Solution. One can write up a whole bunch of equations, but the numbers are small enough so that trial and error will work. Intelligent trial and error. But we should have some equations, so let us call x the price of a bag of apples and y the price of a bag of oranges. Let us concentrate on Abigail first. She bought five bags, of these either 0, 1, 2, 3, 4, or 5 are bags of apples. Because 23 is prime, there is no way she had to pay \$23 if she only bought apples or only oranges. So we can rule 0, 5 out and we have the following equations:

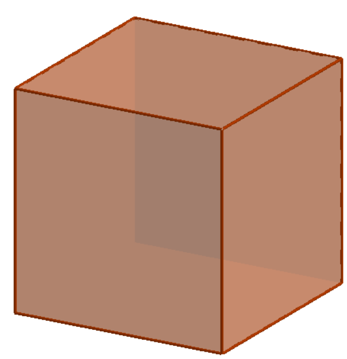
1 basket of apples, 4 of oranges	$x + 4y = 23$
2 baskets of apples, 3 of oranges	$2x + 3y = 23$
3 baskets of apples, 2 of oranges	$3x + 2y = 23$
4 baskets of apples, 1 of oranges	$4x + 4 = 23$

The assumption is that x, y are whole numbers and, of course, they are positive. It is not hard to see all possible such values of x, y. Listing all possible x, y in the same order as the equations:

Eq 1	$x = 3, y = 5$	$x = 7, y = 4$	$x = 11, y = 3$	
Eq 2	$x = 1, y = 7$	$x = 4, y = 5$	$x = 7, y = 3$	$x = 10, y = 1$
Eq 3	$x = 1, y = 10$	$x = 3, y = 7$	$x = 5, y = 4$	$x = 10, y = 1$
Eq 4	$x = 3, y = 11$	$x = 4, y = 7$	$x = 5, y = 3$	

Concerning Benedict, he could have bought 1, 2, or 3 baskets of apples. His equations are $x + 3y = 24$, $2x + 2y = 24$ and $3x + y = 24$. Analyzing his solutions as we did for Abigail, we can simplify by ignoring any combination of (x, y) that did not appear in the table above. We then see: For the case of one basket of apples, $x + 4y = 24$, the only solution appearing in Abigail's table is $x = 3, y = 7$. For the case of two baskets of apples, two of oranges, no solution coincides with a solution valid for Abigail. For 3 baskets of apples the equation is $3x + y = 24$; the only solution consistent with Abigail's table is $x = 7, y = 3$. So we can now say that one of the produce (apples or oranges) sold at \$3 the basket, the other one at \$7. The only way we can get 20 using 3 and 7 is by $20 = 2 \cdot 3 + 2 \cdot 7$. $\boxed{\text{The answer is 4 baskets..}}$

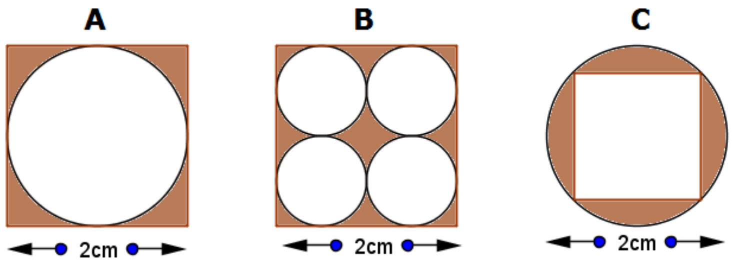
But, as I said earlier, it is quite easy to just guess the answer with a bit of trial and error.

3. 

Jamie counted the number of edges of a cube, Jimmy counted the number of corners, and Judy counted the number of faces. They then added the three numbers. What was the resulting sum?

Solution. The cube has 12 edges, 8 corners and 6 faces, so $\boxed{\text{The answer is 26.}}$.

4. The following figures are composed of squares and circles. Which figure has a shaded region with largest area?



(A) A only **(B)** B only **(C)** C only **(D)** both A and B **(E)** all are equal

Solution. We see that the shaded areas in A and B are equal, and equal to $4 - \pi$. The shaded area in C is $\pi - 2$. Because $2\pi > 6$, we get that $4 - \pi < \pi - 2$. $\boxed{\text{The answer is C.}}$

5. If 20% of a number is 12, what is 30% of that number?

   Solution. One way of proceeding is to figure out first what the number is. So if $0.20x = 12$, then $x = 12/0.20 = 60$. Finally, 30% of 60 is 18.

6. A group of children riding on bicycles and tricycles rode past Billy Bob's house. Billy Bob counted 7 children and 19 wheels. How many tricycles were there?

   Solution. Suppose t is the number of tricycles and b the number of bicycles. Then $b = 7 - t$ and the total number of wheels is $3t + 2(7 - t) = t + 14$. So $t + 14 = 19$, giving $t = 5$.

7. Ali, Bonnie, Carlo and Dianna are going to drive together to a nearby theme park. The car they are using has four seats: one driver's seat, one front passenger seat and two back seats. Bonnie and Carlo are the only two who can drive the car. How many possible seating arrangements are there?

   Solution. Suppose Bonnie drives. Then there are three seats for the other three that can be occupied in all possible ways, namely $3! = 6$ ways. The same is true when Carlo drives. Thus $\boxed{\text{12 arrangements.}}$

8. A *palindrome* is word, phrase or number that reads the same forward as backwards. Two famous palindromes are:

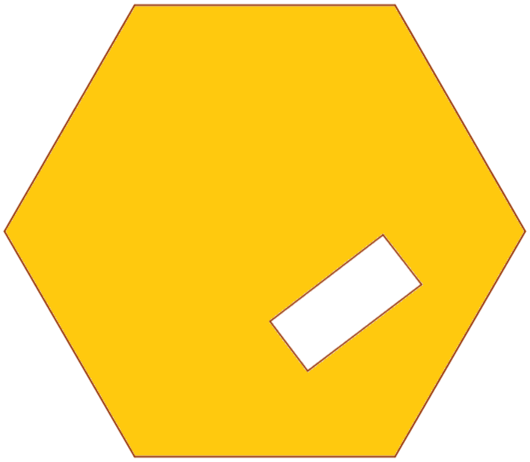
   Able was I ere I saw Elba

   A man, a plan, a canal, Panama.

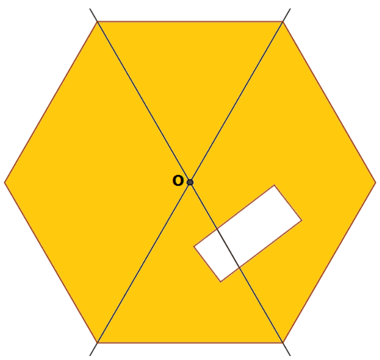
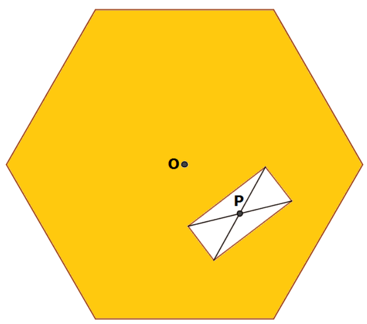
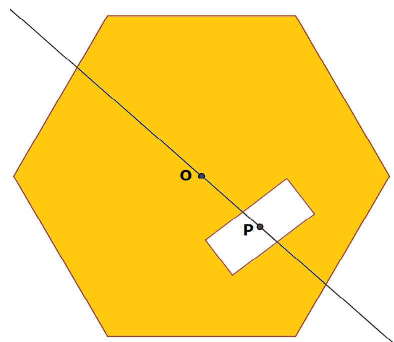
   But we are doing mathematics, so we look at numbers that are palindromic, like all one digit numbers, 33, 656, 98789, etc. Here is the question: What four digit palindrome has only one prime factor? (Only one prime number divides it).

   Solution. If only one prime divides the number, it must be a power of a prime. We can proceed by trial and error and eventually we will find it. But there is a big shortcut. A four digit palindrome looks like *abba* and because $a - b + b - a = 0$, it is divisible by 11. All four digit palindromes are divisible by 11! So now we simply compute $11^2 = 121$, too small; $11^3 = 1331$, $11^4 = 14641$ too big. $\boxed{\text{The answer is 1331.}}$

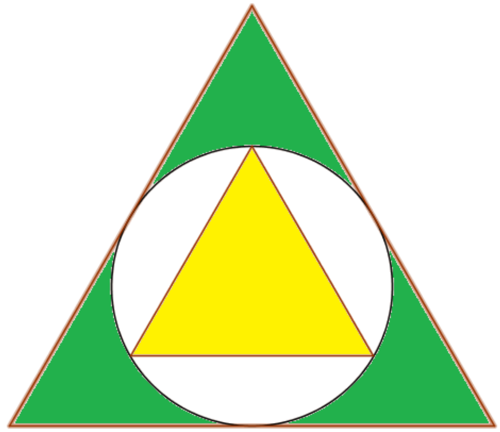
9. A rectangle has been cut out of a regular hexagon. Draw a straight line that divides the shaded figure (the hexagon minus the rectangle) into two parts of equal area.



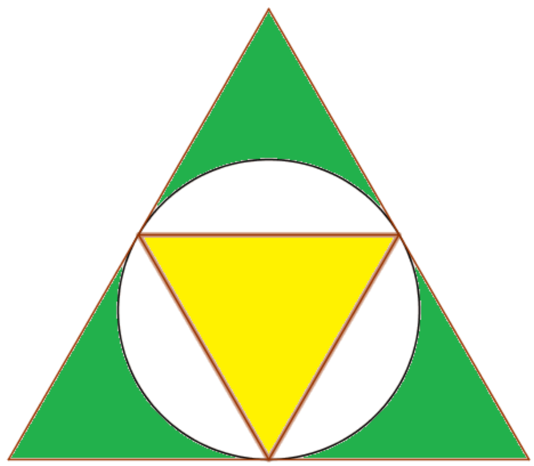
Solution.

		
Draw two lines through two pairs of opposite vertices of the hexagon. Their intersection O is the center of symmetry of the hexagon. Any line through O will divide the hexagon in half.	Draw the two diagonals of the rectangle; they intersect at P. Every line through P divides the rectangle into two equal parts.	Now draw the line through O and P. That line divides the shaded area into half the hexagon minus half the rectangle, so it is the line we are looking for.

10. The picture shows an equilateral triangle inscribed in a circle inscribed in an equilateral triangle. What is the ratio between the areas of the two triangles. If T is the area of the big triangle, and S is the area of the small triangle, what is $\frac{T}{S}$? (From Martin Gardner's *Puzzle Tales*.)



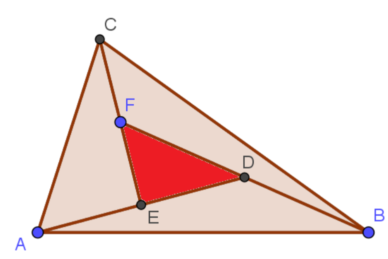
Solution. If we rotate the smaller triangle by $180°$ we get the picture:



It is clear from it that $T/S = 4$.

11.

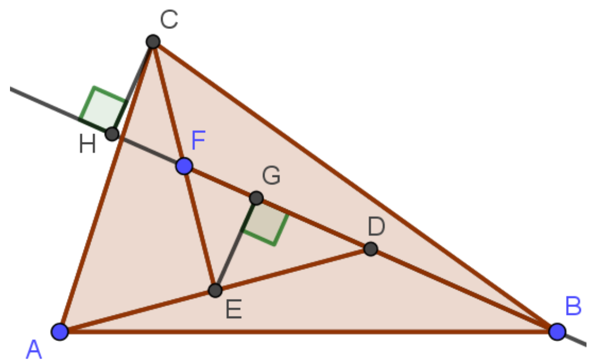
Segments are drawn in triangle ABC in such a way that D is the midpoint of BF, E is the midpoint of AD, and F is the midpoint of CE. If the area of triangle ABC is 1, what is the area of triangle DEF?



Solution. One shows that the three triangle BCF, CEA, and ABD have the same area, and that area is twice the area of triangle DEF. Denoting by $[XYZ]$ the area of a triangle of vertices X, Y, Z we show that

$$[DEF] = \frac{1}{2}[BCF].$$

Similarly one proves that $[DEF] = \frac{1}{2}[CEA]$ and $[DEF] = \frac{1}{2}[ABD]$. We compute the area of DEF using side DF as the basis, and the area of BCF with side BF as the basis. Since $|DF| = \frac{1}{2}|BF|$, our assertion reduces to proving that both triangles have the same altitudes. The picture below shows the altitudes of both triangles, EG is the altitude of DEF, CH of BCF. One sees that triangle GEF is congruent to triangle HCF; both are right triangles, $\angle GFE = \angle HFC$ and side $EF = FC$. It follows that $EG = CH$, as desired.

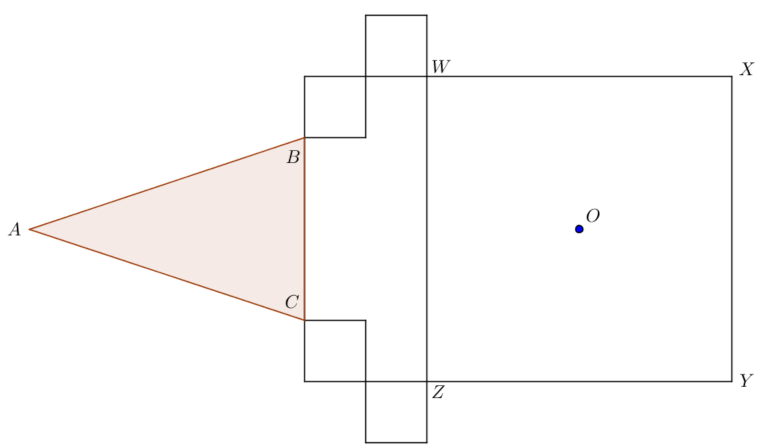


Thus

$$1 = [ABC] = [BCF] + [CEA] + [ABD] + [DEF] = 2[DEF] + 2[DEF] + 2[DEF] + [DEF] = 7[DEF].$$

$\boxed{\text{The answer is } [DEF] = 1/7.}$

12. In the figure, the area of square $WXYZ$ is 25 cm^2. The four smaller squares have sides 1 cm long, either parallel to or coinciding with the sides of the large square. In $\triangle ABC$, $|AB| = |AC|$, and when $\triangle ABC$ is folded over side BC, point A coincides with O, the center of square $WXYZ$. What is the area of $\triangle ABC$, in square centimeters?



Solution. Considering the segment BC the base of triangle ABC, the altitude is equal to the distance from O to BC, which is $2.5 + 1 + 1 = 4.5$. One also sees that $|BC| = 5 - 2 = 3$, so the area of the triangle is $\boxed{\frac{1}{2} \times 3 \times 4.5 = 6.75}$ or $\boxed{27/4 \text{ square centimeters.}}$