MATH CIRCLE AT FAU

11/04/2017

The High School Section

Session 3

The problems will be posted shortly after each session linked to our website,

$$http://cosweb1.fau.edu/\sim faumath/MathCircle/$$

Solutions will take longer to be posted; you might want to continue to work on some of these problems at home. The things that are discussed in any session may be used in later sessions.

Rules: Work the problems in any order. You may work alone or in groups. You may, of course, get up, walk about, use the whiteboard; we provide markers.

Fun with numbers

Some Background Material.

- **Primes.** I assume we know what a prime number is. The prime numbers are 1, 2, 3, 5, 7, 11, ...; they are positive integers without any positive divisors other than 1 and themselves. They are, in a way, the building blocks of numbers. The *Fundamental Theorem of Arithmetic* states that every positive integer greater than 1 can be expressed in essentially only one way as a product of primes. By essentially only one way one means that if we take an integer like 4725 (for example) we do have several ways of writing it as a product of primes, namely

$$4725 = 3 \times 3 \times 3 \times 5 \times 5 \times 7 = 3 \times 5 \times 3 \times 7 \times 5 \times 3 = 3 \times 5 \times \times 5 \times 7 \times 3 \times 3$$

  and 57 other ways. However, in every such product there will be three 3's, two 5's and one 7. This is what is meant by "there is only one way."

  A useful consequence of this theorem is the following theorem: *Suppose p is prime and p divides the product of the integers m and n; that is p divides mn. Then p divides either m or n (or both).*

- **Fermat's Little Theorem** This theorem states: If p is a prime number, if a is an integer not divisible by p, then p divides $a^{p-1} - 1$.

  We can verify this theorem with, for example $p = 7$, so $p - 1 = 6$:

$$\begin{aligned}
1^6 - 1 &= 0 = 7 \cdot 0, \\
2^6 - 1 &= 63 = 7 \cdot 9, \\
3^6 - 1 &= 728 = 7 \cdot 104, \\
4^6 - 1 &= 4095 = 7 \cdot 585, \\
5^6 - 1 &= 15624 = 7 \cdot 2232, \\
6^6 - 1 &= 46655 = 7 \cdot 6665.
\end{aligned}$$

Number Problems.

1. A number ends with the digit 2 If we move the 2 from the last place to the first, the number doubles. What is the smallest number having this property (if any)?

2. A *cyclic set of numbers* is a set of numbers consisting of a number and all the other numbers you can form cycling the digits, counting repetitions (if any). For example

$$\{134, 341, 413\}, \{56789, 67895, 78956, 89567, 95678\}, \{12442, 24421, 44212, 42124, 21244\}, \{111, 111, 111\}$$

   are cyclic sets. Given a number consisting of n digits, the cyclic set it generates consists of n numbers, with possible repetitions. Here is the problem:

   (a) Find the sum of all the numbers in the cyclic set generated by $3,451,678,992,356,478,941$. Can you find a general frmula giving this sum?

   (b) Suppose a is an n-digit positive integer and let p be a prime number, Show: If p divides $\underbrace{11\ldots11}_{n}$ and p divides one of the integers in the cyclic set generated by a, then p divides them all. What is the significance of the n-digit number all of whose digits are 1?

3. Here is a not too horribly hard one from the 2015 AIMEII.

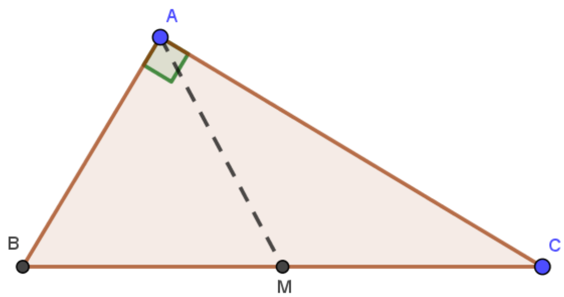
   Find the least positive integer m divisible by 17 and whose digits sum to 17.

4. I believe this problem comes from France, originally. The letters V, A, L, S, E are a permutation of the numbers $1, 2, 3, 5, 7$. Each asterisk (*) represents a digit, not necessarily from the set 1, 2, 3, 5, 7. Clear up the product

$$\begin{array}{ccccccccc}
 & & & & V & A & L & S & E \\
 & & & & S & L & A & V & E \\
\hline
 & & & * & * & * & * & * & * \\
 & & * & * & * & * & * & * & \\
 & * & * & * & * & * & & & \\
 & * & * & * & * & & & & \\
* & * & * & * & * & & & & \\
\hline
* & * & * & * & * & * & * & L & E
\end{array}$$

The Geometry Section

1. Let ABC be a right triangle, with the right angle at A, so that BC is the hypotenuse. Let M be the midpoint of the hypotenuse BC. Show that triangles AMC and AMB are isosceles; specifically show that $|AM| = |MC|$, $|AM| = |BM|$.



2. Here is a nice problem that appeared some years ago in a Canadian High School competition. Just to make sure we have all the terms understood, a *chord* of a circle is any segment that has its to endpoints on the circumference of the circle. In particular, all diameters are chords; they are the largest chords.

   Let P be a point inside a circle ω; assume P is not the center of the circle. Consider the set of chords of ω that contain P. Prove that their midpoints all lie on a circle.

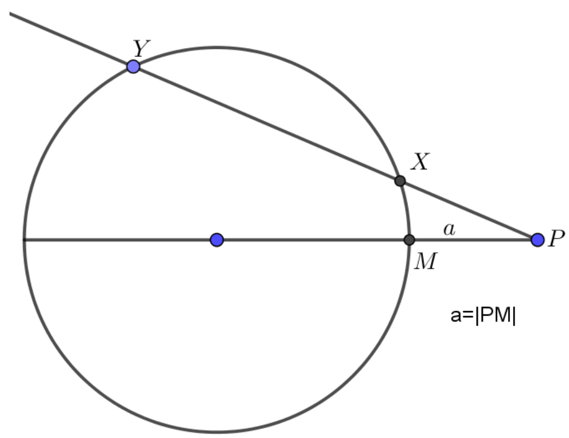
   As a hint; the result of Problem 1 can play an important role. Further hints are available for the asking.

3. **The Power of a Point**

   Suppose C is a circle and P is a point outside of the circle. A ray emanating from P intersects the circle first at X, then at Y. Surprisingly enough, the product $|PX| \cdot |PY|$ does not depend on the points X, Y and is known as the power of the point P with resepct to the circle. In particular, if r is the radius of the circle, a is the distance from P to the circle, then

   $$|PX| \cdot |PY| = a(a + 2r).$$

   Prove this.



4. This problem is a nice application of the power of a point. It is from the 2013 AMC 12 A competition. Of course, the students taking it were not told it had anything to do with power of a point.

   In $\triangle ABC$, $|AB| = 86$, and $AC = 97$. A circle with center A and radius AB intersects BC at points B and X. Moreover, BX and CX have integer lengths. What is $|BC|$?