MATH CIRCLE AT FAU

10/28/2017

Middle School Version

Session # 3

RULES

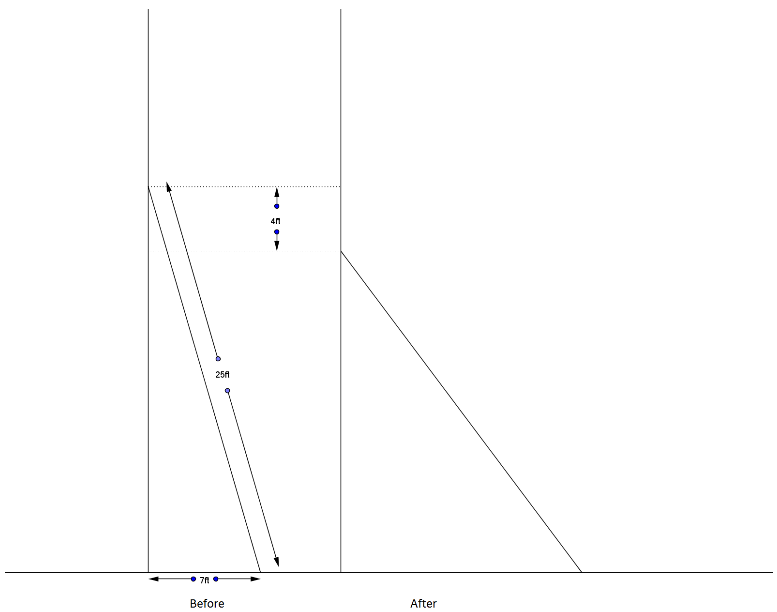
- Work the problems in any order. Some problems are harder than others; do what you can. **If later on you take the time to figure out how to solve the problems you could not solve, you will learn more from what you could not do, than from what you could do.**
- If you think you have finished a problem correctly, tell one of the organizers. If it is really correct, he or she will certify that it is correct.
- Don't feel shy about asking for hints.
- Don't feel shy about getting up, walking around, or talking with anybody you want to talk to.
- If you want to write on one of the whiteboards, we have markers available.

1. A boy buys trumples at 3 for 10 cents. He sells them at 5 for 20 cents. He makes a profit of 1 dollar. How many trumples did he sell? (A *trumple* is a musical instrument made by elves).

   Solution. The boy paid $10/3$ cents per trumple, sold each at $20/5 = 4$ cents; his per trumple profit is $4 = 10/3 = 2/3$ cents. So if he sells x trumples, his profit is $\frac{2}{3}x$ cents. To get this to be 100 cents, you need $x = \frac{3}{2} \cdot 100$ so **150 trumples**.

2. Helen Pythagoras placed a 25 foot ladder against a vertical wall of a building. The foot of the ladder is 7 feet from the base of the building. If the top of the ladder slips four feet, how much will the foot of the ladder slip?

   Solution.



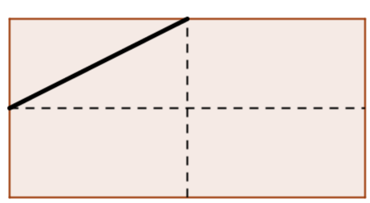
   By Pythagoras, the top of the ladder is $\sqrt{25^2 - 7^2} = 24$ feet above the ground. If it slips four feet, the top is now at 20 feet above the ground, so the foot is $\sqrt{25^2 - 20^2} = 15$ feet from the base of the building; $15 - 7 = 8$; the **foot of the ladder slipped** 8 **feet**.

3. A grasshopper jumping along a straight line can jump 6 or 8 inches in either direction. Can it reach a point that is **(a)** 1.5 inches away from its original position; **(b)** 7 inches away; **(c)** 4 inches away?

   Solution. **(a)** Clearly impossible. **(b)** No. The grasshopper can only jump an even number of inches so its distance from the original position will be even. **(b)** Yes. For example, two 6 inch jumps to the right, followed by one 8 inch jump to the left.

4. A cardboard rectangle of area 1 is cut into two pieces along a line segment that connects the midpoints of two adjacent sides. Find the areas of the two pieces. JUSTIFY YOUR ANSWER.

   Solution. The picture below shows that the answer is 1/8 and 7/8.



5. The numbers 2^{2017} and 5^{2017} are expanded and their digits are written out consecutively on one page. How many total digits are on the page.

Solution. If a number x has k digits then $10^{k-1} \leq x < 10^k$. If it isn't a power of 10 we may even say $10^{k-1} < x < 10^k$. Let m be the number of digits of 2^{2017}, m the number of digits of 5^{2017}. Then

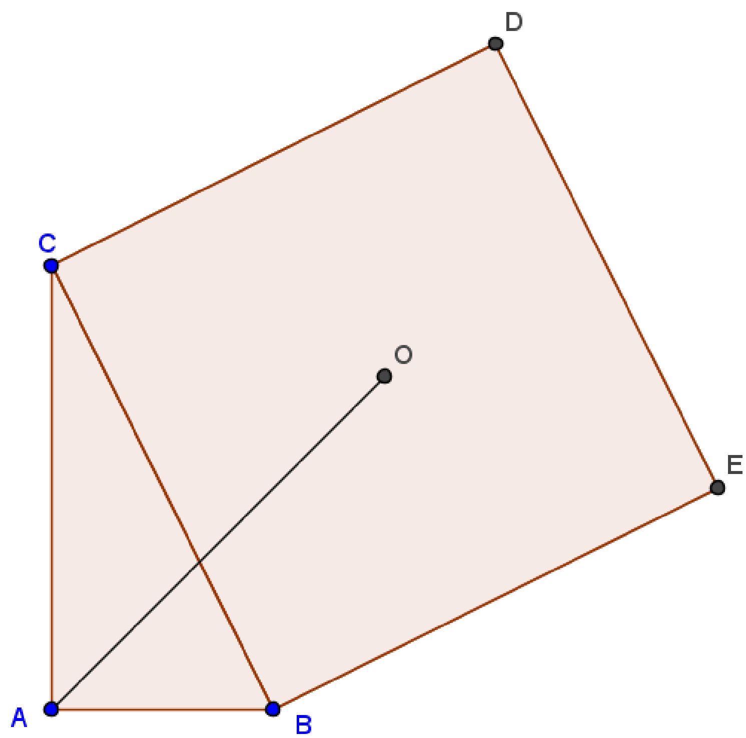
$$10^{m-1} < 2^{2017} < 10^m$$
$$10^{n-1} < 5^{2017} < 10^n.$$

Multiplying the two inequalities together and using that $2^{2017}5^{2017} = 10^{2017}$, we get

$$10^{m+n-2} < 10^{2017} < 10^{m+n},$$

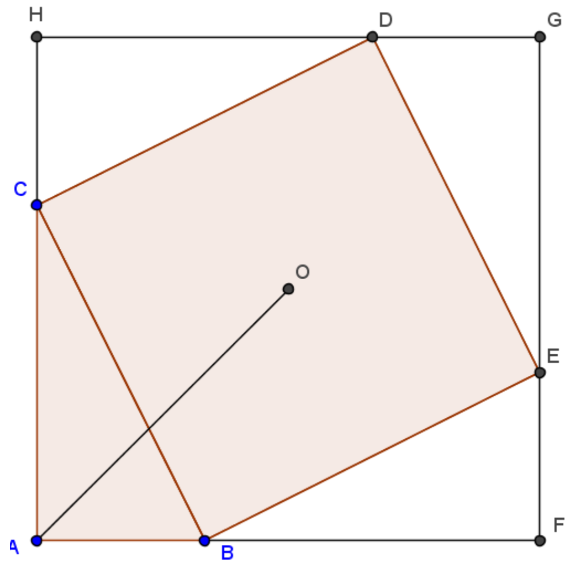
so that $m+n-2 < 2017 < m+n$. This forces $m+n-1 = 2017$; that is $m+n = \boxed{2018}$.

6.



The square $BCDE$ has one side coinciding with the hypotenuse of the right triangle ABC. If $|AB| = a$ and $|AC| = b$, determine the length of the segment AO from A to the center O of the square.

Solution. Add three more corner triangles to get the large square $AFGH$.

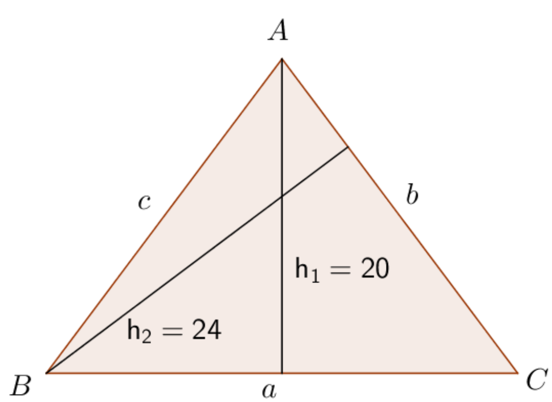


The segment AO is half the diagonal of this larger square. By Pythagoras, $(2|AO|)^2 = |AF|^2 + |AF|^2$, so $4|AO|^2 = 2|AF|^2$, $2|AO| = \sqrt{2}|AF|$,

$$|AO| = \frac{1}{2}\sqrt{2}|AF| = \frac{1}{\sqrt{2}}(a+b).$$

7. A triangle has altitudes of lengths 24, 24, and 20, respectively.
   (a) Explain why this triangle must be isosceles.
   (b) Find the radius of the inscribed circle.

   Solution. The picture below shows the triangle ABC of sides (as marked) a, b, c; h_1 is the length 20 altitude and h_2 one of the the length 24 altitudes.



   Because the area of ABC is equal to $\frac{1}{2}ah_1$ and to $\frac{1}{2}bh_2$ we must have $ah_1 = bh_2$, tus (using the values of h_1, h_2, $a = 1.2b$. By the Theorem of Pythagoras, $(a/2)^2 + h_1^2 = b^2$ or $a^2/4 + 400 = b^2$. From these equations one gets $a = 30, b = 25$. The area of the triangle works out to $\frac{1}{2}30 \cdot 20 = 300$; the semi perimeter $s = (a+b+b)/2 = 40$. Since the area A equals rs, where r is the radius of the inscribed circle, we get that $r = 7.5$.

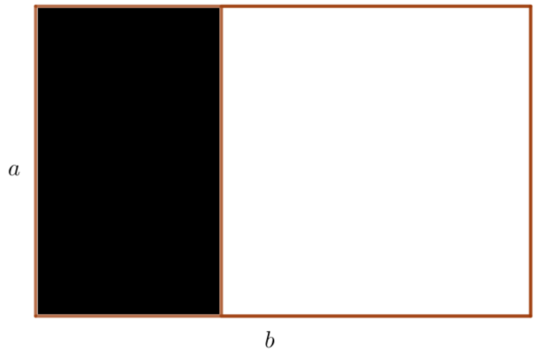
8. 852 digits are used to number the pages of a book consecutively from page 1. How many pages does the book have?

Solution. It takes 9 digits to number the first 9 pages. Then we have a total of 90 pages $(99 - 10 + 1)$ for which we need 2 digits, so an additional 180 digits to get to page 99. That is, to get to page 99 we need a total of $180 + 9 = 189$ digits. After that come 3 digit numbers and since there will be 900 of these, we will get way past 852 digits. The book must have less than 999 pages. After we used up the 189 digits to get to page 99, we have $852 - 189 = 663$ digits left to be used to form 3 digit numbers, so for a total of $663/3 = 221$ additional pages past page 99. This will take us to page $99 + 221 = 320$ pages.

9. A palindromic number is a whole number that reads the same both ways, for example 111, 45654 are palindromic. How many palindromic numbers are there between 10 and 10000?

   Solution. There are exactly 9 two digit palindromes, namely $11, 22, \ldots, 99$. For 3 digit palindromes, the first digit can be any non-zero digit, the middle digit can be any digit, so there are $9 \times 10 = 90$ palindromes. Concerning four digit palindromes, the first two digits also determine the palindrome, so we also have a total of 90 palindromes. The answer is $9 + 90 + 90 = 189$.

10. A rectangle of dimensions $a \times b$ with $b - a < a < b$ has the property that if we snip off a square whose side is the smaller side of the rectangle, we get a rectangle that is similar to the original one:



The darkly shaded small rectangle and the large rectangle are similar. What is the ratio b/a?

Solution. Let $x = b/a$, Similarity means that corresponding sides are proportional, thus

$$\frac{a}{b-a} = \frac{b}{a}$$

from which $a^2 = b^2 - ab$. Dividing both sides of this equation by a^2 gives $1 = x^2 - x$ or $x^2 - x - 1 = 0$. Solving the quadratic equation we get $x = \dfrac{1 \pm \sqrt{5}}{2}$. Since $b/a > 0$, we conclude $\dfrac{b}{a} = \dfrac{1+\sqrt{5}}{2}$, a number known as the *golden ratio*.