MATH CIRCLE AT FAU

10/07/2017

The High School Section

Session 1

Welcome to the high school section of the FAU Math Circle. We hope that the sessions will be a mixture of fun and learning. Some of the problems may seem very hard, but you will learn something by trying to solve them. As the sessions go on, things will begin to seem easy. For some of you, they may seem easy from the start. Problems are preceded by some notes on background material; you might want to read these notes before attempting the problems. **In addition, we (the organizers) are there to provide additional hints.**

The type of problems you will encounter here are basically of three types. Problems that we (the compilers) think are interesting, problems similar (sometimes equal) to problems that appear in the diverse high school mathematics competitions, and problems that give you the background for the competition problems. Many of the problems can teach you important facts about mathematics in general and geometry in particular. Every session will have a theme or two; today's themes are *Viéte's Relations* and *Chasing Angles*.

The problems will be posted shortly after each session. Solutions will take longer to be posted; you might want to continue to work on some of these problems at home. The things that are discussed in any session may be used in later sessions.

Rules: Work the problems in any order. You may work alone or in groups. You may, of course, get up, walk about, use the whiteboard; we provide markers.

1 Viète's Relations

François Viète (1540-1603)
French Mathematician, Cryptographer. Has been called "the father of algebra."

Background Material. Suppose

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial and $x_1, \ldots, x_n$ are its zeros (real or complex); that is, the roots of the equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Assume $a_n \neq 0$. Then $P(x) = a_n(x - x_1)(x - x_2)\cdots(x - x_n)$ from which one gets the so called *Viète's Relations*, which relate roots and coefficients:

$$\begin{aligned}
x_1 + x_2 + \cdots + x_n &= -\frac{a_{n-1}}{a_n} \\
x_1x_2 + \cdots x_1x_n + x_2x_3 + \cdots x_{n-1}x_n &= \frac{a_{n-2}}{a_n} \\
x_1x_2x_3 + \cdots x_1x_2x_n + x_1x_3x_4 + \cdots + x_{n-2}x_{n-1}x_n &= -\frac{a_{n-3}}{a_n} \\
\cdots \quad & \quad \cdots \\
x_1x_2\cdots x_n &= (-1)^n \frac{a_0}{a_n}
\end{aligned}$$

For example, if $P(x) = a_2x^2 + a_1x + a_0$, so $n = 2$, we get $x_1 + x_2 = -a_1/a_2$ and $x_1x_2 = a_0/a_2$. If $P(x) = a_3x^3 + a_2x^2 + a_1x + a + 0$, we get

$$\begin{aligned}
x_1 + x_2 + x_3 &= -a_2/a_3 \\
x_1x_2 + x_1x_3 + x_2x_3 &= a_1/a_3 \\
x_1x_2x_3 &= -a_0/a_3
\end{aligned}$$

These relations are useful not only to solve a lot of problems having to do with roots of equation where the roots are very hard to determine (perhaps impossible without a computer), but also in other situations because every function of $x_1, \ldots x_n$ that is *symmetric* in $x_1, \ldots, x_n$; meaning that it doesn't change if you switch around (permute) the variables, can be expressed somehow in terms of the right hand sides of the relations. For example, if $n = 2$, $x_1^2 + x_2^2$ is symmetric in x_1, x_2, while $2x_1 + x_2$ is not. You may notice

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2.$$

The Problems.

1. The equation

$$x^5 - 11x^4 + 24x^3 + 55x^2 - 224x + 175 = 0$$

has five distinct real roots x_1, x_2, x_3, x_4, x_5. Find: $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$.

2. The roots of the equation $2x^3 - 9x^2 - 11x + c = 0$ are in arithmetic progression. Determine c and solve the equation.

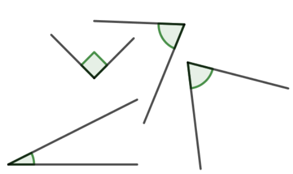
We interrupt this set of problems to introduce some notation. The left hand sides in the Viéte relations are known as the *elementary symmetric polynomials*. We'll use the following notation:

$$\begin{array}{rcl} s_1(x_1,\ldots,x_n) & = & x_1+x_2+\cdots+x_n \\ s_2(x_1,\ldots,x_n) & = & x_1x_2+\cdots x_1x_n+x_2x_3+\cdots x_{n-1}x_n \\ s_3(x_1,\ldots,x_n) & = & x_1x_2x_3+\cdots x_1x_2x_n+x_1x_3x_4+\cdots+x_{n-2}x_{n-1}x_n \\ \cdots & & \cdots \\ s_n(x_1,\ldots,x_n) & = & x_1x_2\cdots x_n \end{array}$$

3. It was mentioned that all symmetric functions can be written in terms of the elementary symmetric functions. There is a converse to this. All symmetric functions can be written in terms of sums of (equal) powers of the variables. For example, if $n=2$,
$$s_1(x_1,x_2)=x_1+x_2$$
is already so expressed; the power is 1.
$$s_2(x_1,x_2)=x_1x_2=\frac{1}{2}\left((x_1+x_2)^2-(x_1^2+x_2^2)\right).$$
Do the same for $n=3$. Of course s_1 is already a sum of powers, so concentrate on s_2, s_3. The objective is to express $s_2(x_1,x_2,x_3)=x_1x_2+x_1x_3+x_2x_3$, $s_3(x_1,x_2,x_3)=x_1x_2x_3$ in terms of $x_1+x_2+x_3$, $x_1^2+x_2^2+x_3^2$, $x_1^3+x_2^3+x_3^3$. The sums of powers may appear in any way; they may appear squared or cubed, or multiplied with each other. Etc.

4. Find all triples (x,y,z) such that $x\le y\le z$ and
$$\begin{array}{ccccccr} x & + & y & + & z & = & 4 \\ x^2 & + & y^2 & + & z^2 & = & 46 \\ x^3 & + & y^3 & + & z^3 & = & 190 \end{array}$$
Hint: Suppose $P(t)=t^3+at^2+bt+c$ is a polynomial with zeros x,y,z. Use the results from Problem 3

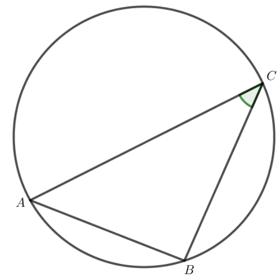
2 Chasing Angles



Background Material. The background for all classical plane geometry is the content of Volume I of the Elements. Here are some basic results you probably know already:

- The sum of the angles of a triangle equals 180°.
- The angles at the base of an isosceles triangle are equal. (A theorem known as the "bridge of donkeys" (or asses, *pons asinorum* in Euclid)).

Here is your first "angle chasing" assignment. In a circle, as you probably know, a *chord* is any segment joining two points of the circumference. The largest chord is the diameter of the circle. Given a chord of endpoints A, B and a point C on the circle, $C\neq A, C\neq B$, we say that the angle $\angle ACB$ *subtends* the chord AB.



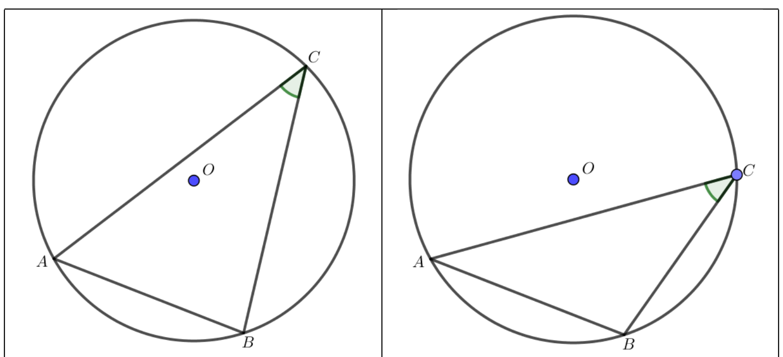
A chord divides the disc bounded by the circle into two parts. If the chord is not the diameter, one part contains the center, the other does not. Here is the assignment.

1. Suppose AB is a chord of a circle, not the diameter, and suppose C is on the circle, on the same side of the line determined by A, B as the center O of the circle. Show that $\angle ACB = \frac{1}{2}\angle AOB$.

   The pictures below should help. More help is available for the asking. This problem has as a consequence an amazing result: *If two angles subtend the same chord AB, and are on the same side of the line through AB, then they are equal.*

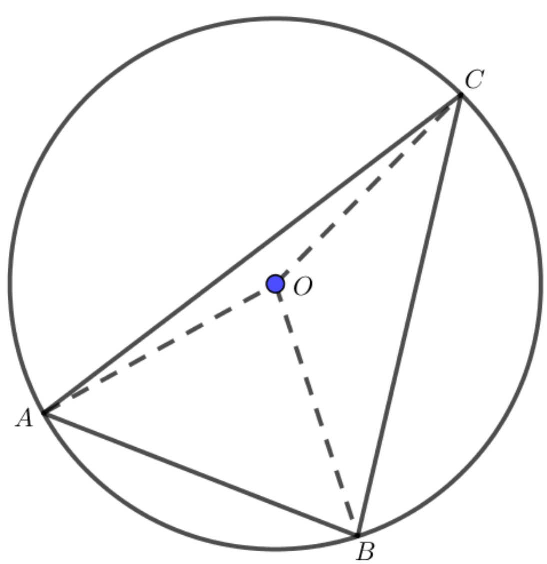
Pictures for the problem.

There are actually two possibilities: The center is inside the triangle ACB, or outside of it. Let's ignore the limiting case of the center on the segment AC or BC.



Case 1 | Case 2

I'll give you some help for case 1. Case 2 is similar, perhaps even easier. Here is the picture for Case 1, with a few more segments marked. Perhaps you can answer the questions below.



Hints and Questions

Hint. Angles $\angle OCA$ and $\angle CAO$ look equal. **Prove** they are equal. Same for $\angle OCB$ and $\angle CBO$.

Question. What can you say about $\angle AOC + 2\angle OCA = \angle AOC + \angle OCA + \angle CAO$?

Question. What can you say about $\angle BOC + 2\angle OCB = \angle BOC + \angle OCB + \angle CBO$?

Question. Noticing that $\angle ACB = \angle OCA + \angle OCB$, can you express $\angle ACB$ in terms of $\angle AOC + \angle BOC$?

Question. How is $\angle AOC + \angle BOC$ related to $\angle AOB$?

2. Suppose the point C on the circle is on the other side of the line through AB as the center O of the circle. Show that

$$\angle ACB = 180^\circ - \frac{1}{2}\angle AOB.$$

3. Show that all angles subtended by the diameter are right angles. That is, if AB is a diameter of the circle, C a point on the circle, $C \neq A, B$, then $\angle ACB = 90^\circ$. Conversely, show that if A, B, C are distinct (no two equal) points on the circle, and ACB is a right angle, then AB is a diameter of the circle.

4. A *cyclic quadrilateral* is a quadrilateral that can be inscribed in a circle. Assume $ABCD$ is a cyclic quadrilateral. Show that opposite angles add up to 180°. That is, the angle at A plus the angle at C add up to 180°, and the same is true for the angles at B, D.

The converse of this result is also true:

5. Prove: If a quadrilateral $ABCD$ satisfies that a pair of opposite angles (say at A and at C) add up to 180°, then it is cyclic.

   Hint: Consider the circle through A, B, C. If D is not on the circle, where can it be?

All these are very useful results for proving geometry theorems and solving geometry problems. We expect to continue chasing angles in future sessions. With a bit of luck, we'll catch some.