FAU Math Circle

11/04/2017

SOLUTIONS

1. Every day Mr. Wooster returns from his club in the city (were he spends the day drinking tea and gossiping) by train to the train station of the town were he lives. His butler Jeeves is supposed to pick him up by car and drive him home. No time is to be wasted; Jeeves is to leave the house so he exactly arrives at the train station the moment the train pulls in, Mr. Wooster jumps into the car and they drive immediately home. One day the train arrives early, Jeeves isn't there yet, and Mr. Wooster decides to walk home. After walking for half an hour he meets Jeeves on the way to pick him up. He gets into the car and they arrive at the home 20 minutes earlier than usual. How many minutes early was the train?

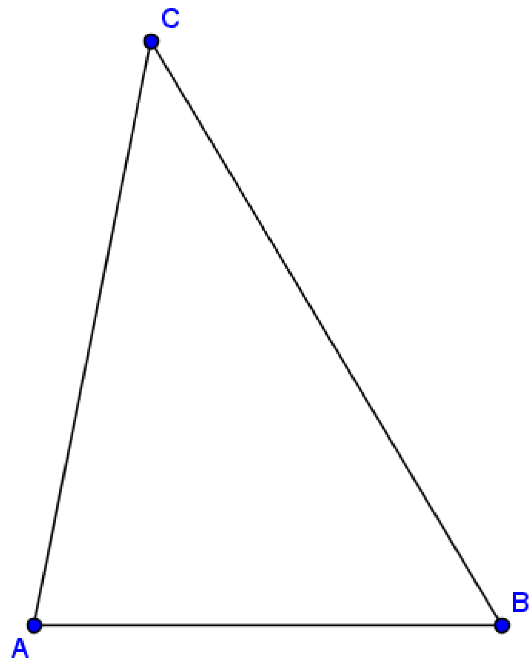
   Solution. Since Jeeves and Mr. Wooster (Bertie to his friends) got home 20 minutes early, Jeeves spent 10 minutes less in each way so he picked up Bertie at 10 minutes driving time from the train station. Let us suppose that Bertie had not walked, just stayed at the station. After 30 minutes Jeeves would be at the pickup point, it would take him 10 extra minutes to get to the station. The answer is 30+10=40 minutes.

2. There are 2017 representatives of four of the Middle-Earth races sitting around a round table, dwarves, elves, gnomes,and humans. Elves never sit next to dwarves, and humans cannot sit next to gnomes. Prove that at least one pair of representatives sitting next to each other must be of the same race.

   Solution. One might want to try with fewer than 2017 representatives, say with 6 and then with 7. The fact that 2017 is odd plays a big role. Suppose the representatives are seated so that no two dwarves, no two elves, no two gnomes, no two humans are seated next to each other. We will see this is impossible. A nice way of seeing this is by removing for a moment two of the warring races, say we remove all humans and all gnomes, leaving only the elves and the dwarves. Between every elf and every dwarf there has to be at least one empty chair. Similarly between elf and elf and dwarf and dwarf, since we assume no two members of the same race are seated next to each other. If there is more than one empty chair between two of the remaining races, we must either have had a human seated next to a human, a gnome next to a gnome, or a human next to a gnome. The conclusion is that once we remove humans and gnomes, there is exactly one empty chair between any two of the remaining representatives. But this being a round table, there will also be the same number of occupied chairs, so the total number of chairs must be even. And 2017 is odd.

3. Suppose there are exactly 9 towns in a very small country and **all distances between the towns are different**. A person starts in each town and walks towards the **nearest** town. Prove: (a) There are two towns A and B such that a person from A walks to B, and a person from B walks to A. (b) There is a town that nobody walks to.

   Solution. This problem can be seen as a nice illustration of the concept of mathematical induction. As in all problems, one might want to start with a simpler example, say with three towns A, B, C. If they are disposed as in the picture below.

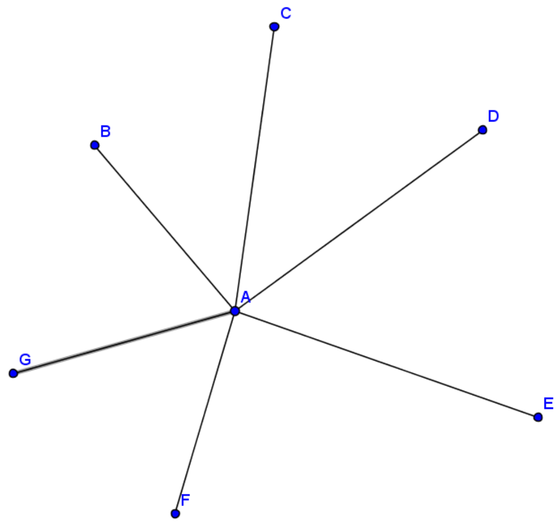


Then there will be a person going from A to B, one from B to A, one from C to A, and none to C.

Turning to the case of nine towns, there will be two at minimum distance. Call them A, B. Then a person will walk from A to B one from B to A, so this takes care of proving (a). Is anybody else walking to A or to B?. If somebody from another town is walking to A or B, then since we have nine towns and nine walkers, there has to be a town to which nobody walks. And we are done. If nobody else walks to A or B, we can concentrate in the seven remaining towns, and do the same analysis. We now have seven towns and seven walkers; two of these towns are closest to each other, call them C, D then the walker from C goes to D, the one from D to C. If any other walker ends at C or D, there must be a town to which no walker goes. Otherwise we can remove C, D and have the same problem with 5 towns and 5 walkers. Once more reduces it to three towns and three walkers, and we know that then there is a town not walked to.

4. A certain country has several airfields with lots of planes in each. The distances between all of the airfields are different; no two are at the same distance. One day an airplane takes off from each field and lands on the closest airfield. Prove that at most 5 airplanes will land on each airfield.

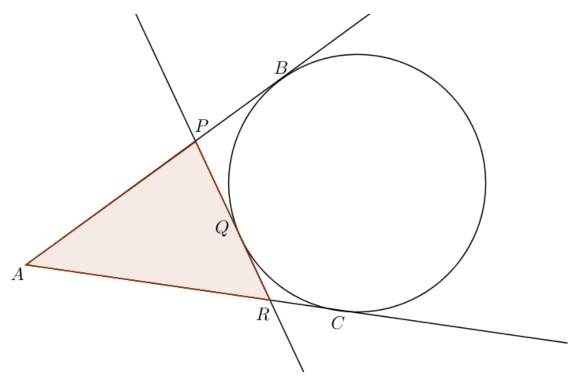
   Solution. This is a very pretty problem. First of all there must be more than 6 airfields for the question not to be what in mathematics we call trivial. So assume now we have at least 7 airfields and there actually is an airfield on which more than 5 planes land. Let us call this airfield A and let B, C, D, E, F, G be six airfields landing planes at A. Since all are at different distances we can have a picture that looks as follows



Now if we look at all the angles formed by these segments, one at least of these angles has to be less than $60°$. Say $\angle CAB < 60°$. Consider then the triangle CAB, since $\angle CAB < 60°$ and the sum of the three angles of a triangle equals $180°$ we conclude that $\angle ACB + \angle ABC > 120°$. Then one of $\angle ACB, \angle ABC$ is larger than $60°$. Say $\angle ABC$ is the largest of these two angles; being larger than $60°$ it is also larger than $\angle CAB$. Now the side opposite to the largest angle of a triangle is the largest side, so that side AC is the largest side of $\triangle ABC$ and the plane from C should not have flown to A since B is nearer.

There is one imprecision in this argument, what if $\angle CAB = 60°$? Then we would have $\angle ACB + \angle ABC = 120°$; but now one will have one of these two angles adding up to 120 degrees having more than 60 degrees; otherwise $\triangle ABC$ is equilateral, and all distances were supposed to be different.

5. From a point A outside of a circle we draw two tangents touching the circle at points B, C, respectively. We then draw a third tangent intersecting segment AB at P, segment AC at R and touching the circle at Q. If $|AB| = 20$, what is the perimeter of triangle APR? Can one even determine it from the provided data?



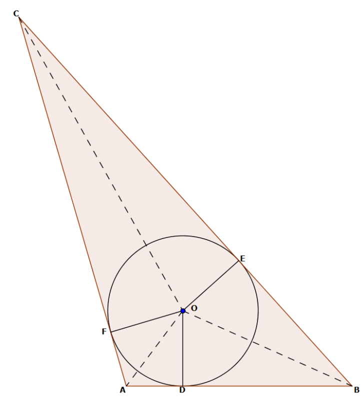
Solution. There is an honest solution and a cheating one. The cheating one has, however, a defect; it assumes there is a solution. In addition, the honest solution emphasizes a simple, though important, fact: if we draw tangents from a point X outside of a circle and these tangents touch the circle at Y, Z, respectively, then $|XY| = |XZ|$. The honest solution first. By the fact that we mentioned, $|AB| = |AC|$, but also $|PQ| = |PB|$ and $|RQ| = |RC|$. If $\mathcal{P}$ is the perimeter of triangle APR then

$$\mathcal{P} = |AP|+|PR|+|RA| = |AP|+|PQ|+|QR|+|RA| = |AP|+|PB|+|RC|+|RA| = |AB|+|AC| = 2|AB| = 40.$$

The cheating solution assumes there is a solution and since nothing is said about Q it has to be the same wherever Q. A lot of assumptions, of course! So we move Q in the direction of B. As this happens, side PR

gets closer and closer to side AB, while side AR becomes negligible; in the end Q, R, B become indistinguishable and R collapses onto A. So in the limit we have a "triangle" with two sides equal to AB and one side equal to 0, so its perimeter is $2 \cdot 20 = 40$.

6. The ideas in the solution of the previous problem play a role here. The perimeter of a triangle is 75 inches, the radius of the inscribed circle is 5 inches. What is the area of the triangle?



Solution. I drew some additional lines and gave some names to some points. Segments OD, OE, OF are radii of the circle; thus all of length $r = 5$. The angles they form with the sides of the circle are right angles. Look at triangle AOC, taking side AC as the base. Then OF is the altitude so that the area of this triangle is:

$$[AOC] = \frac{1}{2} AC \cdot OF.$$

Similarly the areas of triangles AOB and BOC are given by

$$[AOB] = \frac{1}{2} AB \cdot OD, \qquad [BOC] = \frac{1}{2} BC \cdot OE.$$

Now

$$\begin{aligned} [ABC] &= [AOC] + [AOB] + [BOC] = \frac{1}{2} AC \cdot OF + \frac{1}{2} AB \cdot OD + \frac{1}{2} BC \cdot OE = \frac{1}{2}(AC + AB + BC)\, r \\ &= \frac{1}{2}(75)(5) = 187.5 \text{ square inches.} \end{aligned}$$

7. One of the legs of a **right** triangle has length 12 and the hypotenuse has length 13. Find the area of the inscribed circle. This should be quite easy after having done the previous problem.

Solution. From the last problem we learned that the area of a triangle equals half the perimeter times the radius of the inscribed circle. By the Theorem of Pythagoras, the second leg of our triangle has length $\sqrt{13^2 - 12^2} = 5$, so its half perimeter is $(13 + 12 + 5)/2 = 15$; its area is $\frac{1}{2}(5)(12) = 30$ so that the radius r of the inscribed circle satisfies $15r = 30$. Thus $r = 2$.

8. A triangle has sides of length 9, 10, and 17. What is the radius of the inscribed circle? This can be easy if we know Heron's formula. If not, it could be hard.

Solution. Heron's formula gives the area of a triangle of sides of lengths a, b, c as

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

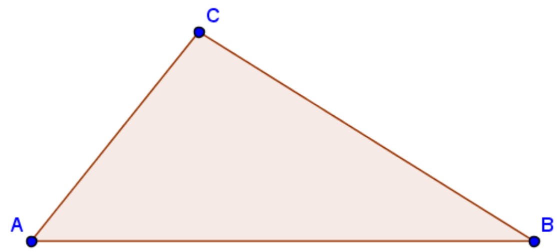
where $s = \frac{1}{2}(a + b + c)$ is the half perimeter. In our case, $s = \frac{1}{2}(9 + 10 + 17) = 18$, so that

$$A = \sqrt{18 \cdot 9 \cdot 8 \cdot 1} = 36.$$

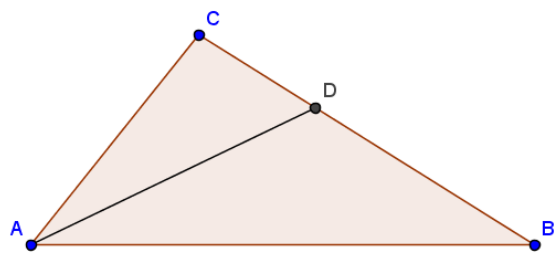
From what we learned, $r = A/s = 36/18 = 2$.

9. Using only a compass and a straightedge, how can one find the *incenter* of a triangle; the center of the inscribed circle? The incenter must be at the same distance from each one of the sides; is there always such a point?

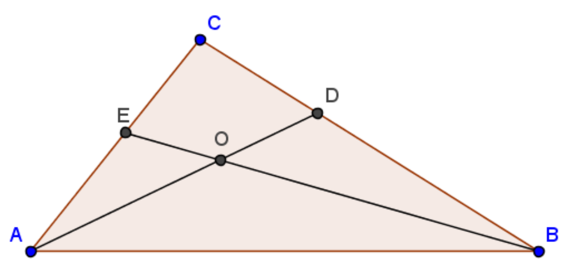
   Solution. The incenter, if it exists, has to be at the same distance from all three sides; call that distance r (it will also be the radius of the in circle. To explain this better I am drawing a triangle ABC



The points that are at same distance from sides AB and CA are precisely all the points on the bisector of the angle BAC: In the picture below it is the segment AD; all points of AD are at the same distance from AB and AB.



The points that are at the same distance from sides AB and BC will be on the bisector BE of angle ABC. The bisectors of angles BAC and ABC naturally intersect at a point O.



So for the point O, if we write $d(O, AB), d(O, BC), d(O, CA)$ for the distance of O from sides AB, BC, CA, respectively:

Because O is on AD, $d(O, AB) = d(O, CA)$.

Because O is on BE, $d(O, AB) = d(O, BC)$.

But now, SURPRISE!, we get for free that we also have $d(O, CA) = d(O, BC)$, so all three distances are the same. In particular, O has to be on the bisector of the angle BCA. So we not only see that the point O is equidistant from all three sides, thus it is the incenter; in addition we see that all three angular bisectors must intersect! That intersection (which can be determined by just finding the intersection of two of the angular bisectors) is the incenter.

If we now want to find the circle itself, we can simply draw from the point O a perpendicular to one of the sides, say to side AB. One way to do this is to first draw a circle with center O and a radius large enough so it intersects AB (or an extension of AB) at two points P, Q. Then draw the perpendicular bisector of PQ; it intersects AB at some point S. The incircle is the circle of center O, radius S.

10. The cross pictured below, having all arms of the same length, can be divided by two cuts into four pieces that can be assembled to form a square. The picture below shows the cross and the cuts that accomplish this task.