



# The Bieberbach Conjecture

2016/9/1

# How This Will Proceed

Most of the talk will be by chalk. This includes all proofs. The “beamer” document contains the definitions, statements of theorems; etc, and will be available, together with other material, at <http://cosweb1/faumath/Bieberbach/>

# Outline

- 1 BASICS
- 2 The Case  $n = 2$ ; Bieberbach's Proof
- 3 The Work of Charles Löwner
- 4 The Conjectures of Robertson and Lebedev-Milin
- 5 de Branges' Proof; a Minimalist Sketch
- 6 Landau, Bieberbach and Hardy

# Univalence

- **Definition.** Let  $U$  be open in  $\mathbb{C}$ . An **analytic** function is said to be *UNIVALENT* (German: *slicht*) iff it is one-to-one. Recall that  $f$  is analytic in  $U$  if and only if it has a Taylor expansion around all points of  $U$ . The Taylor series converges to the function in the largest open disc that can be included in  $U$ .

# The set $S$

- **Notation.**  $D = \{z \in \mathbb{C} : |z| < 1\}$ ,
- **The main set of functions** Let

$$S = \{f : D \rightarrow \mathbb{C} : f \text{ is univalent, } f(0) = 0, f'(0) = 1\}.$$

- **Easy fact.** If  $f \in S$ , then  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  for some coefficients  $a_2, a_3, a_4 \dots \in \mathbb{C}$ .

# The Bieberbach Conjecture

- **The Bieberbach Conjecture.** If  $f = z + \sum_{n=2}^{\infty} a_n z^n \in S$ , then  
 $|a_n| \leq n$  for  $n = 2, 3, \dots$

*Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, Sitzungsberichte Preußische Akademie der Wissenschaften, 1916, 940-955.



Ludwig Georg Elias Moses Bieberbach  
1886-1982

# The Prototype

- The prototype of a function in  $S$  is the *Koebe function* defined by

$$k(z) = \frac{z}{(1-z)^2}.$$

- One sees that

$$k(z) = z + 2z^2 + 3z^3 + \dots$$



Paul Koebe  
1882-1945

# The Strong Version

- If  $f(z) = z + \sum_{n=2} a_n z^n \in S$ , then  $|a_n| \leq n$  for all  $n \geq 2$ .  
Moreover, if equality holds for some  $n \geq 2$ ; i.e., if  $|a_n| = n$  for some  $n \geq 2$ , then  $f$  is essentially the Koebe function, specifically

$$f(z) = e^{-i\alpha} k(e^{i\alpha} z) = \frac{z}{(1 - e^{i\alpha} z)^2}$$

for all  $z \in D$ , where  $\alpha \in \mathbb{R}$ .

# Bieberbach's 1916 Theorem

## Theorem

*Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ . Then  $|a_2| \leq 2$  with equality if and only if  $f$  is the Koebe function, or the Koebe function composed with rotations; i.e.,*

$$f(z) = e^{-i\alpha} k(e^{i\alpha} z)$$

*for all  $z \in D$ , some  $\alpha \in \mathbb{R}$ .*

- Bieberbach's Theorem is an easy consequence of the following result due to Grönwall (*Some Remarks on Conformal Representations*, Ann. Math, 16 (1914-15), 72-76).

## Theorem

Let  $f \in S$  and assume  $\tilde{f}(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ . Then  $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$ , with equality if and only if  $\mathbb{C} \setminus \tilde{f}(\Omega)$  is a null set, where  $\Omega = \mathbb{C} \setminus \bar{D} = \{z \in \mathbb{C} : |z| > 1\}$ .

- **Definition** If  $f \in S$  we define  $\tilde{f} : \mathbb{C} \setminus \bar{D}$  by

$$\tilde{f}(z) = \frac{1}{f(1/z)}.$$



Thomas Hakon Grönwall

1877-1932

- Proof of both theorems, and more, by chalk.



## Charles Löwner

- Charles (born Karel, then Karl, finally Charles) Löwner (1893-1968) proved the case  $n = 3$  of the Bieberbach conjecture. This hardly does him any justice because his approach, the tools he set up, were absolutely essential to De Branges. One can say that the most important work on the conjecture done in the years between its formulation and De Branges solution is the work of Löwner.

# Slit Maps

- A *slit map* is an element  $f \in S$  such that  $f(D) = \mathbb{C} \setminus \Gamma^*$ , where  $\Gamma : [0, \infty) \rightarrow \mathbb{C}$  is a Jordan arc going to infinity; that is  $\Gamma$  is continuous and injective and  $\lim_{t \rightarrow \infty} |\Gamma(t)| = \infty$ .
- The set of slit maps is dense in  $S$  in the topology of uniform convergence over compact subsets of  $D$  (an absolutely non-trivial fact), so it suffices to prove the conjecture for slit maps.

## Theorem

Let  $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$  be a slit map. There exists a continuous  $g : D \times [0, \infty) \rightarrow \mathbb{C}$ , univalent in  $z$  for all  $t \geq 0$  and such that

- ①  $g(z, t) = e^t (z + \sum_{n=2}^{\infty} a_n(t) z^n)$  where  $a_n : [0, \infty) \rightarrow \mathbb{C}$  are differentiable,  $a_n(0) = a_n$  for each  $n \in \mathbb{N}$ . In particular,  $g(\cdot, 0) = f$ .
- ②  $g$  satisfies the following differential equation:  
$$\frac{\partial g}{\partial t}(z, t) = z \frac{1 + \kappa(t)z}{1 - \kappa(t)z} \frac{\partial g}{\partial z}(z, t),$$
 where  $\kappa : [0, \infty) \rightarrow \mathbb{C}$  is continuous and  $|\kappa(t)| = 1$  for all  $t \in [0, \infty)$ .

(Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I, Math. Ann., 89 (1923), 103121)

# The conjecture of M.S. Robertson

- If  $f(z) = \sum_{k=1}^{\infty} b_{2k+1} z^{2k+1}$  is an odd function in  $S$ , then

$$\sum_{k=0}^{n-1} |b_{2k+1}|^2 \leq n \text{ for } n \in \mathbb{N}.$$

(A remark on the odd *slicht* functions, B.A.M.S., 42 (1936), 366-370.

- It is not terribly hard to see that if Robertson's conjecture is true, so is Bieberbach's.

Malcolm Irving Slingsby Robertson (1906-1998)

# The conjecture of Lebedev-Milin (1971)

- This conjecture is quite deep, an important step in reformulating the Bieberbach conjecture.
- It states: Let  $f \in S$  and define  $c_k \in \mathbb{C}$  by

$$\log \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^k \quad (z \in D).$$

Then

$$\sum_{k=1}^n k(n+1-k)|c_k|^2 \leq 4 \sum_{k=1}^n \frac{n+1-k}{k}.$$

(Nikolai Andreevich Lebedev, (1919-1982))(Isaak Moiseevich Milin, (1919-1992))

## Other Developments

- Garabedian and Schiffer proved  $|a_4| \leq 4$  (1955).
- Ozawa and Pederson proved  $|a_6| \leq 6$  (1968).
- Pederson and Schiffer proved  $|a_5| \leq 5$  (1972).
- Hayman proved  $L = \lim_{n \rightarrow \infty} a_n/n$  exists;  $|L| \leq 1$  and  $|L| = 1$  iff  $f$  is the Koebe function. Shows that for any  $f$  there can only be a finite number of exceptions to the conjecture. (1955)
- Numerous results of the form  $|a_n| \leq cn$  where  $c$  gets closer and closer to 1.

**Source:** The fount of all human knowledge, *Wikipedia*.

# Louis de Branges de Bourcia- b. 1933



# Löwner Recalled

- To prove  $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$  satisfies the Lebedev-Milin conjecture, de Branges brings in Löwner's function  $g(z, t)$ , solving the Löwner differential equation.
- Let  $c_k(t)$  for  $t > 0$  be defined by  $\log \frac{g(z, t)}{z} = \sum_{k=1}^{\infty} c_k(t) z^k$ .
- Since  $g(z, 0) = f(z)$ ; the coefficients  $c_k = c_k(0)$  are the ones in the Lebedev-Milin conjecture.

## de Branges' special functions

- Fix  $n \in \mathbb{N}$  (to prove:  $|a_n| \leq n$ , more precisely case  $n$  of LM conjecture)
- Define  $\tau_k : [0, \infty) \rightarrow \mathbb{R}$  for  $k = 1, \dots, n+1$  by a backward induction; first  $\tau_{n+1} = 0$ .
- Assuming  $\tau_{k+1}$  defined for some  $k \leq n$ , define  $\tau_k$  as the solution of the initial value problem

$$\begin{cases} \frac{1}{k}\tau_k' + \tau_k &= \tau_{k+1} - \frac{1}{k+1}\tau_{k+1}', & 0 < t < \infty, \\ \tau_k(0) &= n+1-k, \end{cases}$$

for  $k = n, n-1, \dots, 1$ .

# The main properties of these functions

- $\lim_{t \rightarrow \infty} \tau_k(t) = 0$ .
- $\tau'_k(t) < 0$  for  $t > 0$ .

Of these two facts, the first one is fairly obvious, the second one not at all obvious. It depends on a complicated inequality for Jacobi polynomials due to Askey and Gasper.

## A Final Function

- A new function is defined for  $t > 0$ , namely

$$\varphi(t) = \sum_{k=1}^n \left( k |c_k(t)|^2 - \frac{4}{k} \right) \tau_k(t), \quad (1)$$

- Using the differential equations satisfied by the  $\tau_k$ 's and the differential equations satisfied by the coefficients  $c_k$  (which come from Löwner's differential equation), and  $\tau'(t) < 0$ , one gets that  $\varphi'(t) \geq 0$  for all  $t > 0$ .
- Thus  $\varphi$  is increasing. In addition,  $\lim_{t \rightarrow \infty} \varphi(t) = 0$ , because  $\lim_{t \rightarrow \infty} \tau_k(t) = 0$ ,  $k = 1, \dots, n$ .

# The End of the Road

- A function  $\varphi(t) = \sum_{k=1}^n (k|c_k(t)|^2 - \frac{4}{k}) \tau_k(t)$  that decreases to zero must always be non-positive.
- Thus

$$\sum_{k=1}^n \left( k|c_k|^2 - \frac{4}{k} \right) (n+1-k) = \varphi(0) \leq 0.$$

- This is the inequality conjectured by Milin, now proved.

# Not All Great Mathematicians Are Great People

- It is sad to say that Bieberbach became a big Nazi once Hitler came to power.
- In one famous incident, in 1934, the mathematician Edmund Landau (1877-1938) defined  $\pi$  as twice the first positive zero of cosine. This was used as an excuse to fire him from his position at Göttingen; it was a non germanic definition of  $\pi$ .
- Actually, he would have lost the position anyway; laws barring Jews from the University had already passed.

- Bieberbach wrote about the incident: *Thus the valiant rejection by the Göttingen student body which a great mathematician, Edmund Landau, has experienced is due in the final analysis to the fact that the un-German style of this man in his research and teaching is unbearable to German feelings. A people who have perceived how members of another race are working to impose ideas foreign to its own must refuse teachers of an alien culture.*

## Hardy on Bieberbach

- Geoffrey Harold Hardy replied to this: *There are many of us, many Englishmen and many Germans, who said things during the War which we scarcely meant and are sorry to remember now. Anxiety for one's own position, dread of falling behind the rising torrent of folly, determination at all cost not to be outdone, may be natural if not particularly heroic excuses. Professor Bieberbach's reputation excludes such explanations of his utterances, and I find myself driven to the more uncharitable conclusion that he really believes them true.*

(Source: Mactutor History of Mathematics,  
<http://www-history.mcs.st-and.ac.uk/>)

# THANKS FOR LISTENING!