The Bieberbach Conjecture Solved

1 Notation and some terminology

If $z \in \mathbb{C}$ and $r > 0$, then $D(z, r)$ denotes the open disc of center z radius r; $D(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$. We write D for $D(0, 1)$. An analytic function $f : U \to \mathbb{C}$, where U is an open subset of $\mathbb{C}$, is said t be univalent iff it is injective. Functions declared univalent will always be assumed to be analytic. We set

$$S = \{f : D \to \mathbb{C} : f \text{ is univalent}, f(0) = 0, f'(0) = 1\}.$$

The Koebe function is the function $k : D \to \mathbb{C}$ defined by

$$k(z) = \frac{z}{(1-z)^2}.$$

It is easily seen to be in S. Moreover

$$k(z) = \sum_{n=1}^{\infty} nz^n = z + 2z^2 + 3z^3 + \cdots$$

for $|z| < 1$.

If $f \in S$, then $f(z) = z + \sum_{n=2} a_n z^n$. The Bieberbach conjecture states *if* $f(z) = z + \sum_{n=2} a_n z^n \in S$, *then* $|a_n| \leq n$ *for all* $n \geq 2$. A strong version adds: If equality holds for some $n \geq 2$; i.e., if $|a_n| = n$ for some $n \geq 2$, then f is essentially the Koebe function, specifically

$$f(z) = e^{-i\alpha} k(e^{i\alpha} z) = \frac{z}{(1 - e^{i\alpha} z)^2}$$

for all $z \in D$, where $\alpha \in \mathbb{R}$.

A *path* is a continuous map $\gamma : I \to \mathbb{C}$, where I is an interval in $\mathbb{R}$; not necessarily a closed or bounded interval. We denote by γ^* the points on the path; $\gamma^* = \{\gamma(t) : t \in I\}$

2 A Sketch of the Proof.

I will sketch here the proof. This could be most of the presentation. The section that follow contain the needed proofs of details, so most references are to results in the following sections.

I stated the Bieberbach conjecture in the previous section. In 1936 M.S. Robertson made the following stronger conjecture: *If* $f(z) = \sum_{k=1}^{\infty} b_{2k+1} z^{2k+1}$

is an odd function in S, then $\sum_{k=0}^{n-1} |b_{2k+1}|^2 \le n$ *for $n \in \mathbb{N}$.* This conjecture, if true, implies the Bieberbach conjecture. In fact, assuming it true, we can apply it to the odd function F coming from f by Lemma 8, if $F(z) = \sum_{k=0}^{\infty} c_k z^{2k+1}$, the Robertson conjecture implies

$$\sum_{k=0}^{n-1} |c_k|^2 \le n$$

for all $n \in \mathbb{N}$. By (7), if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ $(a_1 = 1)$, then (by the Cauchy Schwarz inequality)

$$\begin{aligned} a_n &= \sum_{k=0}^{n-1} c_k c_{n-k-1} \le \left(\sum_{k=0}^{n-1} |c_k|^2\right)^{1/2} \left(\sum_{k=0}^{n-1} |c_{n-k-1}|^2\right)^{1/2} \\ &= \left(\sum_{k=0}^{n-1} |c_k|^2\right)^{1/2} \left(\sum_{k=0}^{n-1} |c_k|^2\right)^{1/2} = \sum_{k=0}^{n-1} |c_k|^2 \le n. \end{aligned}$$

In 1971 N.A. Lebedev and I.M. Milin made an even stronger conjecture, and that was what De Branges proved: Let $f \in S$ and define $c_k \in \mathbb{C}$ by

$$\log \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^k$$

for $z \in D$. Then

$$\sum_{k=1}^{n} k(n+1-k)|c_k|^2 \le 4 \sum_{k=1}^{n} \frac{n+1-k}{k}. \tag{1}$$

As will be used perhaps more than once, $f(z)/z$ is analytic in D since $f(0) = 0$. Moreover $f(z)/z \ne 0$ for $z \in D$; it can't be 0 at 0 since $f'(0) = 1$; it can't be 0 anywhere else since f is univalent and already 0 at 0. Thus there is an analytic determination of $\log(f(z)/z)$; we choose the one satisfying $\log 1 = 0$. The proof that the Lebedev Milin conjecture implies the Robertson conjecture is not easy. It depends on the following inequality known as *the second Lebedev-Milin inequality.*

Theorem 1 *Let $\varphi(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ be analytic in some neighborhood of the origin (notice $\varphi(0) = 0$) and let $\psi(z) = e^{\varphi(z)} = \sum_{n=0}^{\infty} \beta_n z^n$. Then*

$$\sum_{k=0}^{n-1} |\beta_k|^2 \le n \exp\left(\frac{1}{n}\left(\sum_{k=1}^{n-1} k(n-k)|\alpha_k|^2 - \sum_{k=1}^{n-1} \frac{n-k}{k}\right)\right) \tag{2}$$

for $n = 2, 3, \ldots$. Equality occurs if and only if there exists $\gamma \in \mathbb{C}$, $|gm| = 1$ such that $\alpha_k = \frac{1}{k}\gamma^k$ for $k = 1, 2, \ldots, n-1$.

The proof of this theorem will be postponed (perhaps forever). Assuming it, we can prove that if the Lebedev-Milin conjecture is true then so is the Robertson conjecture, as follows. Let f be an odd function in S, say

$$f(z) = \sum_{k=0}^{\infty} b_{2k+1} z^{2k+1}.$$

Then f^2 is an even analytic function in D, hence has a power series involving only even powers of z and we can define* $g : D \to \mathbb{C}$ by $g(z^2) = f(z)^2$. Then $g \in S$. In fact, $g(0) = 0$ is clear; differentiating both sides of the equality $g(z^2) = f(z)^2$ we get

$$2zg'(z^2) = 2f(z)f'(z), \quad \text{thus} \quad g'(z^2) = \frac{f(z)}{z} f'(z) \to f'(0)^2 = 1$$

as $z \to 0$; thus $g'(0) = 1$. If $g(z_1) = g(z_2)$, then let $w_1, w_2 \in D$ such that $z_1 = w_1^2$, $z_2 = w_2^2$. Then $f(w_1)^2 = f(w_2)^2$ thus $f(w_1) = \pm f(w_2)$. Since f is univalent and odd, if $f(w_1) = f(w_2)$ we get $w_1 = w_2$. If $f(w_1) = -f(w_2) = f(-w_2)$ then $w_1 = -w_2$. In either case $w_1^2 = w_2^2$; i.e., $z_1 = z_2$. Let c_k be defined by

$$\log \frac{g(z)}{z} = \sum_{k=1}^{\infty} c_k z^k.$$

Now $f(z)/z$ is an even function, so we can replace z by $\sqrt{z}$ (any determination) to get an analytic function in z; we denote it by $f(\sqrt{z})\sqrt{z}$. We have $g(z^2)/z^2 = (f(z)/z)^2$ so that

$$\frac{g(z)}{z} = \left(\frac{f(\sqrt{z})}{\sqrt{z}}\right)^2;$$

taking logs,

$$\log\left(\frac{f(\sqrt{z})}{\sqrt{z}}\right) = \frac{1}{2}\log\frac{g(z)}{z} = \sum_{k=1}^{\infty} \frac{1}{2} c_k z^k,$$

thus

$$\frac{f(\sqrt{z})}{\sqrt{z}} = e^{\sum_{k=1}^{\infty} \frac{1}{2} c_k z^k}.$$

We can now apply (2) with

$$\psi(z) = \frac{f(\sqrt{z})}{\sqrt{z}} = \sum_{k=0}^{\infty} b_{2k+1} z^k, \qquad \varphi(z) = \sum_{k=1}^{\infty} \frac{1}{2} c_k z^k$$

to get

$$\sum_{k=0}^{n-1} |b_{2k+1}|^2 \le n \exp\left(\frac{1}{n}\left(\sum_{k=1}^{n-1} k(n-k)\frac{|c_k|^2}{4} - \sum_{k=1}^{n-1} \frac{n-k}{k}\right)\right).$$

*$f(z)^2 = \sum_{n=1}^{\infty} c_n z^{2n}$ for some coefficients c_n (with $c_0 = 0$ since $f(0) = 0$); set $g(z) = \sum_{n=1}^{\infty} c_n z^n$.

By the Lebedev-Milin conjecture (1), which we assume is true, the argument inside the exponential is non-positive, thus we get

$$\sum_{k=0}^{n-1} |b_{2k+1}|^2 \leq n$$

as Robertson conjectured. A thing to notice here is that while the Robertson conjecture is a nice but mild extension of the Bieberbach conjecture, the Lebedev-Milin conjecture was a much stronger, deeper result.

And now to the sketch of how De Branges proved that the Lebedev-Milin conjecture is true. I break it up into numbered segments, or steps.
1. It suffices to prove (1) for slit maps; these are elements $f \in S$ such that $f(D) = \mathbb{C}\backslash\Gamma^*$, where $\Gamma : [0, \infty) \to \mathbb{C}$ is a Jordan arc going to infinity; that is Γ is continuous and injective and $\lim_{t\to\infty} |\Gamma(t)| = \infty$. As is proved in Theorem 19 below, slit maps are dense in S, in the topology of uniform convergence over compact subsets of D. If $\{f_n\}$ converges in this way to f, then it is clear that $\{\log(f_n/z)\}$ converges to $\log(f/z)$ uniformly over compact subsets of D. Then for each $k \in \mathbb{N} \cup \{0\}$ the sequence of k-th Taylor coefficients of $\{\log(f_n/z)\}$ converges to the k-th Taylor coefficient of $\log(f/z)$ (see Theorem 4). Since (1) only involves a finite number of coefficients, proving it for slit maps is enough. From now on assume f is a slit map, $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$, and assume c_k for $k \in \mathbb{N}$ are defined by

$$\log \frac{f(z)}{z} = \sum_{k=1}^{\infty} c_k z^k.$$

2. A construct due to Charles Löwner (1923) was one of the crucial ingredients used by De Branges in his proof. Löwner used this result to prove $|a_3| \leq 3$. A useful version of Löwner's result is the following:

Theorem 2 *Let $f \in S$ be a slit map; $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$. There exists a continuous $g : D \times [0, \infty) \to \mathbb{C}$ such that $z \mapsto g(z, t) : D \to \mathbb{C}$ is univalent for all $t \geq 0$ and $t \mapsto g(z, t) : (0, \infty) \to \mathbb{C}$ is differentiable for all $z \in D$. Moreover:*

1. *$g(z, t) = e^t \left(z + \sum_{n=2}^{\infty} a_n(t) z^n \right)$ for $z \in D$, $t \geq 0$, where $a_n : [0, \infty) \to \mathbb{C}$ are continuous, and differentiable for $t > 0$, and $a_n(0) = a_n$ for each $n \in \mathbb{N}$. In particular, $g(\cdot, 0) = f$.*

2. *g satisfies the following differential equation:*

$$\frac{\partial g}{\partial t}(z, t) = z \frac{1 + \kappa(t)z}{1 - \kappa(t)z} \frac{\partial g}{\partial z}(z, t)$$

where $\kappa : [0, \infty) \to \mathbb{C}$ is continuous and $|\kappa(t)| = 1$ for all $t \in [0, \infty)$.

For a proof, see Theorem 20 below. De Branges then defines functions $c_k : [0, \infty) \to \mathbb{C}$ for $k \in \mathbb{N}$ by

$$\log \frac{g(z,t)}{z} = \sum_{k=1}^{\infty} c_k(t) z^k;$$

since $g(z, 0) = f(z)$, it follows that $c_k(0) = c_k$. Löwner's differential equation (16) translates into ordinary differential equations for these functions, namely

$$c_k'(t) = 2\sum_{j=1}^{k-1} j c_j(t) \kappa(t)^{k-j} + k c_k(t) + 2\kappa(t)^k. \tag{3}$$

They are not exactly pretty, but they are.

3. Here is where De Branges advances past Löwner. He introduces a system of special functions. Let $n \in \mathbb{N}$, fixed the duration (the same n as in (1). Define $\tau_k : [0, \infty) \to \mathbb{R}$ for $k = 1, \ldots, n+1$ by a backward induction. Set $\tau_{n+1}(t) = 0$ for all $t \geq 0$ ad assuming τ_{k+1} defined for some $k \leq n$, define τ_k as the solution of the initial value problem

$$\begin{cases} \frac{1}{k}\tau_k' + \tau_k & = \quad \tau_{k+1} - \frac{1}{k+1}\tau_{k+1}', \qquad 0 < t < \infty, \\ \tau_k(0) & = \quad n + 1 - k, \end{cases} \tag{4}$$

for $k = n, n-1, \ldots, 1$. I'll calculate a few directly,just for the fun of it. τ_n solves $\frac{1}{n}\tau_n' + \tau_n = 0$, thus $\tau_n(t) = Ce^{-nt}$; since $\tau_n(0) = 1$, we see $\tau_n(t) = e^{-nt}$. Then τ_{n-1} solves

$$\frac{1}{n-1}\tau_{n-1}' + \tau_{n-1} = 2e^{-nt}, \qquad \tau_{n-1}(0) = 2;$$

the solution is given by $\tau_{n-1}(t) = 2ne^{-(n-1)t} - 2(n-1)e^{-nt}$. Keep in mind that n is kept fixed; τ_1 (for example) is not the same for different values of n. It is not hard to get an explicit formula for them; that is, it is not hard to prove such a formula by induction once one knows how it looks. We have

$$\tau_k(t) = k\sum_{\nu=0}^{n-k} (-1)^\nu \frac{(2k+\nu+1)_\nu (2k+2\nu+2)_{n-k-\nu}}{(k+\nu)\nu!(n-k-\nu)!} e^{-(\nu+k)t},$$

for $k = 1, \ldots, n$, where one defines $(a)_\nu$ for $a \in \mathbb{R}$ and $\nu \in \mathbb{N} \cup \{0\}$ by $(a)_0 = 1$ and $(a)_\nu = (a)_{\nu-1}(a + \nu - 1)$ if $\nu \geq 1$. This explicit expression for the τ_k's is not important; what matters is the differential equation satisfied by these functions and the following two facts: Let $1 \leq k \leq n$. Then

T1. $\lim_{t\to\infty} \tau_k(t) = 0$.

T2. $\tau_k'(t) < 0$ for $t > 0$.

Of these two facts, the first one is fairly obvious, the second one not at all obvious. It depends on a complicated inequality for Jacobi polynomials due to Askey and Gasper.

4. Here is where all is put together. A new function is defined, namely

$$\varphi(t) = \sum_{k=1}^{n} \left(k|c_k(t)|^2 - \frac{4}{k} \right) \tau_k(t), \tag{5}$$

for $0 \le t < \infty$. Using the differential equations (4) satisfied by the τ_k's and the differential equations satisfied by the coefficients c_k (which come from Löwner's differential equation), one gets

$$\varphi'(t) = -\sum_{k=1}^{n} |b_{k-1}(t) + b_k(t) + 2|^2 \frac{\tau_k'(t)}{k}, \tag{6}$$

where to keep the notation from getting exceedingly horrible, one abbreviated certain expressions involving the coefficients c_k by defining

$$b_k(t) = \sum_{j=1}^{k} j c_j(t) \kappa(t)^{-1}, \qquad k = 1, 2, 3, \ldots,$$

and $b_0(t) = 0$. If we consider fact **T2.** above; namely that τ_k' is always negative for $t > 0$, $1 \le k \le n$, one concludes from (6) that $\varphi'(t) \ge 0$ for all $t > 0$. Thus φ is increasing.

We notice next that $|c_k(t)| \le C_k$ for $k = 1, 2, 3, \ldots$, $t \ge 0$, where C_k are the constants given by Corollary 16. In fact, for each $t \in [0, \infty)$, $g(\cdot, t)$ is univalent and 0 at 0; the only thing stopping it from being in S is that $g(0,) = e^t$. But this is easily remedied, $e^{-t} g(\cdot, t) \in S$. Now

$$\log \frac{g(z,t)}{z} = \log \left(e^t \frac{e^{-t} g(z,t)}{z} \right) = t + \log \frac{e^{-t} g(z,t)}{z},$$

and it follows that if $k \ge 1$, $c_k(t)$ is also the coefficient of z^k of $\log(e^{-t} g(z,t)/z)$. The assertion follows. If we now bring in the fact called **T1.**, namely that each $\tau_k(t) \to 0$ as $t \to \infty$, and the definition of φ, we see that $\lim_{t \to \infty} \varphi(t) = 0$. A function that **increases** to 0 must be non-negative for finite values of t; in particular $\varphi(0) \le 0$. In other words,

$$\sum_{k=1}^{n} \left(k|c_k|^2 - \frac{4}{k} \right) (n + 1 - k) \le 0.$$

This inequality is the same as (1), proving the theorem.

3 Some Basics from Complex Analysis.

(Some with, others without, proof. Proofs can be found in most, probably all, complex analysis textbooks)

Proposition 3 *Let U be open in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be analytic. Then f is an open map. If in addition f is univalent, then $f^{-1} : f(D) \to U$ is analytic and*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$$

for all $w \in f(D)$. In particular, if f is univalent then $f'(z) \neq 0$ for all $z \in U$.

Proposition 4 *Let $f_n : D \to \mathbb{C}$ be analytic for each $n \in \mathbb{N}$ and assume the sequence $\{f_n\}$ converges uniformly over compact subsets of D to f. If*

$$f_n(z) = \sum_{k=0}^{\infty} c_{nk} z^k$$

for $n = 1, 2, 3, \ldots$, then $\lim_{n\to\infty} c_{nk} = c_k$ for each $k \in \mathbb{N} \cup \{0\}$, where $f(z) = \sum_{k=0}^{\infty} c_k z^k$.

Proof. Let $r \in (0, 1)$, then

$$c_{nk} = \frac{1}{2\pi i} \int_{|z|=r} \frac{f_n(z)}{z^{k+1}}\, dz$$

for $n \in \mathbb{N}$, and we also have

$$c_k = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}}\, dz.$$

Since $\{z \in \mathbb{C} : |z| = r\}$ is a compact subset of D, the result follows. ∎

The usual topology for analytic functions is uniform convergence on compact subsets of the domain. There is a point, in the proof of Caratheodory's Theorem 17, where one has to have a slightly expanded notion of convergence on compact subsets. Assume U_n is open in $\mathbb{C}$ and $f_n : U_n \to \mathbb{C}$ is analytic for $n = 1, 2, 3, \ldots$. Assume also $U_n \subset U_{n+1}$ for $n = 1, 2, , \ldots$, Let $U = \bigcup_{n=1}^{\infty} U_n$. If K is a compact subset of U then there exists $N \in \mathbb{N}$ such that $K \subset U_N$, hence $K \subset U_n$ for $n \geq N$. It thus makes sense to say that the sequence $\{f_n\}$ converges uniformly over compact subsets to $f : U \to \mathbb{C}$ iff for each compact subset K of U, each $\epsilon > 0$, there exists $N_{\epsilon,K} \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N_{\epsilon,K}$, all $z \in K$. Implicitly one assumes $N_{\epsilon,K}$ is large enough so $K \subset U_n$ if $n \geq N_{\epsilon,K}$. Except at one point in the proof of Theorem 17, we will assume that $U_n = U$ for all $n \in \mathbb{N}$ when talking of uniform convergence over compact subsets. The following two results are quite standard; I add sketch of proofs only because of this slightly expanded notion of convergence.

Proposition 5 *Let U_n be open in $\mathbb{C}$ and let $f_n : U_n \to \mathbb{C}$ be analytic for $n = 1, 2, 3, \ldots$. Assume also $U_n \subset U_{n+1}$ for $n = 1, 2, , \ldots$ and assume that $\{f_n\}$ converges uniformly over compact subsets of U to f. Then f is analytic on U.*

Sketch of a proof. The usual proof, found in any complex variables textbook, applies, One can use Morera's Theorem, for example. If a continuous function f from an open set U to $\mathbb{C}$ satisfies that $\int_\gamma f\,dz = 0$ for each closed curve γ such that γ^* is contained in a disc contained[†] in U, then f is analytic in U. The limit function f will be continuous, since it is continuous on every compact subset of U, hence in a neighborhood of every point of U. If γ is a closed curve in U, contained in some open disc $W \subset U$ then, γ^* being compact, we will have that $\gamma^* \subset U_n$ for all $n \geq N$ for some $N \in \mathbb{N}$. By Cauchy's Theorem, $\int_\gamma f_n\,dz = 0$ for all $n \geq N$; since $f_n \to f$ uniformly on γ^*, we'll also have $\int_\gamma f\,dz = 0$. The result follows. ∎

Theorem 6 *(Montel's Theorem) Let U_n be open in $\mathbb{C}$, $U_n \subset U_{n+1}$, and let $f_n : U_n \to \mathbb{C}$ be analytic for $n = 1, 2, 3, \ldots$. For K a compact subset of U let $N_K \in \mathbb{N}$ be such that[‡] $K \subset U_n$ for $n \geq N_K$. Assume that for every compact subset K of U there exists a constant C_K such that $|f_n(z)| \leq C_K$ for all $z \in K$, all $n \geq N_K$. Then there exists a subsequence of $\{f_n\}$ converging uniformly on compact subsets of U to some function $f : U \to \mathbb{C}$. This limit is necessarily analytic.*

Sketch of a proof. This theorem is a fairly straightforward consequence of the Arzela-Ascoli Theorem. One shows that the restriction of the sequence elements to a compact subset of U is equicontinuous. This can be reduced to proving equicontinuity on closed discs contained in U. Let $W = D(w, r)$ be such that $\bar{W} \subset U$, let $R > r$ be such that $\overline{D(w, R)}$ is still contained in U. Then

$$f_n'(z) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta$$

for all $z \in \bar{W}$; if C is a bound for $|f_n(\zeta)|$ for all ζ in the compact set $\{\zeta \in \mathbb{C} : |\zeta - w| = R\}$ then one gets form the formula for f_n' that

$$|f_n'(z)| \leq \frac{CR}{(R - r)^2}$$

for all $z \in \bar{W}$. I am implicitly assuming here that $U_n = U$ for all n; however, the same argument holds in the slightly more general case, one has to

[†] There exist $z_0 \in U$, $r > 0$ such that $\gamma^* \subset D(z_0, r) \subset U$. It actually suffices to have this condition for γ the boundary of a triangle.

[‡] $N_K = 1$ for all compact K if $U_n = U$ for all n.

add simply: "for n large enough." Equicontinuity is a consequence of having uniformly bounded derivatives. For every compact subset one then can extract a subsequence converging uniformly on that subset. To get a single sequence one needs an additional diagonal argument. One can express U (as one can every open subset of $\mathbb{C}$ as a union $U = \bigcup_{m=1}^{\infty} K_m$ of a sequence $\{K_m\}$ of compact subsets that have the property that $K_m \subset K_{m+1}^\circ$ for each $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ we have $N_m \in \mathbb{N}$ such that $K_m \subset U_n$ for $n \geq N_m$. We may and will assume that $N_1 \leq N_2 \leq \cdots$. There is a subsequence $\{f_{n_k}\}$ of $\{f_n\}_{n \geq N_1}$ converging uniformly on K_1 to some continuous function f_1. Ignoring the terms (if any) of $\{f_{n_k}\}$ with $n_k < N_2$, there is a subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ converging uniformly to a continuous function f_2 on K_2. Necessarily $f_2|_{K_1} = f_1$. The notation can get quite messy here, we cannot keep on using subindices of subindices for subsequences, so in a detailed proof, one switches to double indices. The point is that for each $m \in \mathbb{N}$ one has a subsequence converging uniformly on K_m and this subsequence has in turn a subsequence converging uniformly on K_{m+1}. The diagonal sequence then converges uniformly on each K_m. Because of $K_m \subset K_{m+1}^\circ$, every compact subset of U is included in K_m fr some $m \in \mathbb{N}$. ■

Proposition 7 *For $n \in \mathbb{N}$ let U_n be open in $\mathbb{C}$, $U_n \subset U_{n+1}$, and let $f_n : U_n \to \mathbb{C}$ be analytic and univalent. Assume $\{f_n\}$ converges uniformly to f on compact subsets of $U = \bigcup_{n=1}^{\infty} U_n$ and assume U is connected. Then either f is univalent or f is constant.*

Proof. By Cauchy's formula was used above to see that f is analytic on D; one also uses it to prove that $\{f_n'\}$ will converge uniformly to f' on compact subsets of U. Assume for a contradiction that f is not univalent nor constant, and let $z_1, z_2 \in U$ be such that $f(z_1) = f(z_2)$. One can now find an open connected subset V of U such that the closure $\bar{V}$ is compact and such that $z_1, z_2 \in V \subset \bar{V} \subset U$. At this point we can replace U by V and assume that $\{f_n\}$ converges uniformly to f on V. The rest of the proof becomes standard, see for example Proposition 3.5 in Chapter 8 of Stein=Shakarchi *Complex Analysis.* They introduce the sequence $\{g_n\}$, where $g_n(z) = f_n(z) - f_n(z_1)$ for $n \in \mathbb{N}$. This sequence converges uniformly to $g = f - f(z_1)$ on V. Now $g(z_2) = 0$ and, since g is not identically 0 (otherwise f is constant), z_2 is an isolated zero of g. There is thus a radius $r > 0$ such that $g(z) \neq 0$ if $0 < |z - z_2| \leq r$. In particular, $r < |z_2 - z_1|$. By the argument principle,

$$\frac{1}{2\pi i} \int_{|z - z_2| = r} \frac{g'(z)}{g(z)}\, dz = \#\text{ of zeros of } g \text{ in } D(z_2, r) \geq 1.$$

(Stein-Shakarchi say equal to 1 where I have ≥ 1, but zeros are counted by their multiplicity, and hypothetically z_2 could be a zero of order > 1 of g). Since $|g(z)| > 0$ on the compact circle $C = \{z \in \mathbb{C} : |z - z_2| = r\}$, it is bounded away from 0, the same applies to elements g_n with n large enough, so that $1/g_n \to 1/g$ uniformly on C and

$$\frac{1}{2\pi i}\int_{|z-z_2|=r}\frac{g_n'(z)}{g_n(z)}\,dz \to \frac{1}{2\pi i}\int_{|z-z_2|=r}\frac{g'(z)}{g(z)}\,dz$$

as $n\to\infty$. But

$$\frac{1}{2\pi i}\int_{|z-z_2|=r}\frac{g_n'(z)}{g_n(z)}\,dz = 0$$

for all n, since each g_n is univalent and already 0 at z_1, thus never 0 in $D(z_2,r)$. A contradiction has been reached, and we are done. ■

4 More Specific Preliminary Results

Lemma 8 *Let $f\in S$. There exists a unique $F\in S$ such that $f(z^2)=F(z)^2$ for all $z\in D$. The function F is odd. Moreover, if $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, then*

$$F(z)=z+\frac{1}{2}a_2z^3+\frac{1}{2}(a_3-\frac{1}{4}a_2^2)z^5+\cdots\ .$$

Proof. Since $f(0)=0$, the function $h:z\mapsto f(z)/z$ is analytic in D. Then $h(0)=f'(0)=1$ and h it is never 0 since f is univalent and thus zero only at zero. It follows that $h(D)$ is a simply connected subset of $\mathbb{C}$, $0\notin h(D)$, $1\in h(D)$. Let log denote the analytic determination of the logarithm satisfying $\log 1=0$. Set

$$F(z)=ze^{\frac{1}{2}\log h(z^2)}=z\sqrt{h(z^2)}.$$

This is clearly an odd function and $F(z)^2=z^2h(z^2)=f(z^2)$. It is univalent; if $F(z)=F(w)$ then $f(z^2)=f(w^2)$, hence $z^2=w^2$. this implies $\sqrt{h(z^2)}=\sqrt{h(w^2)}$ and since h is never 0, we now get $z=w$ from

$$z\sqrt{h(z^2)}=F(z)=F(w)=w\sqrt{h(w^2)}.$$

That $F(0)+0$ is obvious. Moreover, $F'(z)=\left(1+z\frac{h'(z^2)}{h(z^2)}\right)e^{\frac{1}{2}\log h(z^2)}$; it follows that $F'(0)=1$ since $\log 1=0$. Thus $F\in S$, Uniqueness is clear since $F(z)^2=f(z^2)$ determines F up to sign. But $F'(0)=1$ determines the sign.

Concerning the Taylor coefficients of $F(z)$, assume $F(z)=\sum_{k=0}^{\infty}c_kz^{2k+1}$, where $c_0=1$. Then

$$F(z)^2=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}c_kc_{n-k-1}\right)z^{2n},$$

and comparing with $f(z)^2=z^2+\sum_{n=2}^{\infty}a_nz^{2n}$ we get

$$\sum_{k=0}^{n-1}c_kc_{n-k-1}=a_n \tag{7}$$

for $n = 2, 3, \ldots$. For $n = 2$ we get $c_0c_1 + c_1c_0 = a_2$, so $2c_1 = a_2$, hence $c_1 = \frac{1}{2}a_2$. For $n = 3$, using $c_0 = 1, c_1 = \frac{1}{2}a_2$, the equation works out to $2c_2 + \frac{1}{4}a_2^2 = a_3$, hence $a_3 = \frac{1}{2}(a_3 - \frac{1}{4}a_2^2)$.

■

Notation If $f \in S$ and $F \in S$ is given by Lemma 8, we write $F(z) = \sqrt{f(z^2)}$.

Definition 1 *Let $f \in S$. We define $\tilde{f} : \mathbb{C}\backslash\bar{D}$ by*

$$\tilde{f}(z) = \frac{1}{f(1/z)}.$$

Lemma 9 *Let $f \in S$. Then $\tilde{f} : \mathbb{C}\backslash\bar{D}$ is univalent. Moreover, if $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$, then*

$$\tilde{f}(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$$

where $b_0 = -a_2$, $b_1 = -a_3 + a_2^2$, and the rest of the coefficients are not important.

Proof. As already mentioned in the proof of Lemma 8, the function $h(z) = f(z)/z$ is analytic and never 0 in D. Thus $1/h(z)$ is analytic in D and we can write

$$\frac{z}{f(z)} = \frac{1}{h(z)} = \sum_{n=0}^{\infty} c_n z^n.$$

Replacing z by $1/z$ and multiplying by z, we get

$$\tilde{f}(z) = \frac{1}{f(1/z)} = \sum_{n-0}^{\infty} c_n z^{1-n} = c_0 z + \sum_{n=1}^{\infty} c_n z^{1-n} = c + 0z + \sum_{n=0}^{\infty} b_n z^{-n},$$

where $b_n = c_{n+1}$. The coefficients c_n are not hard to determine. In the first place $c_0 = 1/h(0) = 1$. From

$$\left(\sum_{n=0}^{\infty} c_n z^n\right)\left(z + \sum_{n=1}^{\infty} a_n z^n\right) = z,$$

we get for $n \geq 1$

$$c_{n-1} + \sum_{j=2}^{n} a_j c_{n-j} = \begin{cases} 1 & \text{if n=1,} \\ 0 & \text{otherwise.} \end{cases}$$

For $= 1, 2$ this works out to $c_1 + a_2 = 0$, $c_2 + a_2c_1 + a_3c_0 = 0$, respectively, from which $c_1 = -a_2$, $c_2 = a_2^2 - a_3$; i.e., $b_0 = -a_2$, $b_1 = a_2^2 - a_3$. ■

In proving Löwner's Theorem 20, we will need the following *subordination principle.*

Theorem 10 *Let f, g be analytic in D with f univalent, $g(0) = f(0)$, and $g(D) \subset f(D)$. Then $|g'(0)| \leq |f'(0)|$ and*

$$g(D(0,r)) \subset f(D(0,r))$$

for all $r \in (0,1)$. Moreover, if $g'(0) = f'(0)$, then $g = f$.

Proof. Since f^{-1} is analytic, univalent, and defined on the range of g (by $g(D) \subset f(D)$), we can consider the function $h = f^{-1} \circ g$. We have $h(0) = 0$ because $f(0) = g(0)$. Thus $z \mapsto h(z)/z$ is analytic in D. Moreover,

$$h(D) = f^{-1}(g(D)) \subset f^{-1}(f(D)) = D,$$

so $|h(z)| \leq 1$ for all $z \in D$. Let $0 < r < 1$. If $|z| = r$ we have $|h(z)/z| \leq 1/r$ thus, by the maximum principle $|h(z)/z| \leq 1/r$ for all $|z| < r$. Keeping z fixed and letting $r \to 1$ we get that $|h(z)/z| \leq 1$ for all $z \in D$. Thus

$$|h'(0)| = \lim_{z \to 0} \left| \frac{h(z)}{z} \right| \leq 1.$$

Now

$$h'(z) = \left(f^{-1}\right)'(g(z)) \cdot g'(z) = \frac{1}{f'(f^{-1}(z))} g'(z);$$

letting $z \to 0$ we get $h'(0) = g'(0)/f'(0)$; coupled with $|h'(0)| \leq 1$ we proved $|g'(0)| \leq |f'(0)|$. Assume now $f'(0) = g'(0)$. In this case the value, and the absolute value of the analytic function $h(z)/z$ at 0 is 1, which is also an upper bound of its values; by the maximum principle, $h(z)/z = c$ a constant; necessarily $c = 1$ so that

$$f^{-1}(g(z)) = h(z) = z$$

for all $z \in D$. Applying f we get $g(z) = f(z)$ for all $z \in D$.

Finally, let $r \in (0,1)$. In all of D we have $|h(z)| \leq 1$, so $|h(z)| \leq |z|$ in D. Restricting to $D(0,r)$ we get $|h(z)| \leq |z| \leq r$ in $D(0,r)$, so $h(z) \in D(0,r)$ and then $g(z) = f(h(z)) \in f(D(0,r))$. ■

The following theorem due to Bieberbach (1916) could be what suggested the conjecture.

Theorem 11 *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$. Then $|a_2| \leq 2$ with equality if and only if f is the Koebe function, or the Koebe function composed with rotations; i.e.,*

$$f(z) = e^{-i\alpha} k(e^{i\alpha} z)$$

for all $z \in D$, some $\alpha \in \mathbb{R}$.

The proof of this theorem is an easy consequence of the following theorem due to Gronwall (1914).

Theorem 12 *Let $f \in S$ and assume $\tilde{f}(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$. Then $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$, with equality if and only if $\mathbb{C}\backslash\tilde{f}(\Omega)$ is a null set, where $\Omega = \mathbb{C}\backslash\bar{D} = \{z \in \mathbb{C} : |z| > 1\}$.*

Proof. Let $g = \tilde{f}$ (to avoid always to put on a tilde; one could easily forget). Notice first that $\lim_{|z|\to\infty} |g(z)| = \infty$. This is easiest seen by the definition,

$$\lim_{|z|\to\infty} |g(z)| = \lim_{|z|\to\infty} \frac{1}{|f(1/z)|} = \lim_{|z|\to 0} \frac{1}{|f(z)|} = \infty$$

since $f(0) = 0$.

For $r > 1$ let $\Omega_r = \{z \in \mathbb{C} : |z| > r\}$. Then $g(\Omega_r)$ is an open subset of $\mathbb{C}$ (analytic maps are open) that is easily seen to be bounded by $\gamma_r^* = \{g(z) : |z| = r\}$. It is unbounded (since $\lim_{|z|\to\infty} |g(z)| = \infty$). Because g is univalent, γ_r^* is a Jordan curve parameterized, for example, by $\gamma_r(t) = g(re^{it})$, $0 \leq t \leq 2\pi$. A Jordan curve divides the plane into two connected components, an unbounded one, $g(\Omega_r)$ in this case, and a bounded one. Let E_r denote the bounded component, so $E_r = \mathbb{C}\backslash\overline{g(\Omega_r)}$. There are several ways of seeing that the given parametrization of the boundary is positive. To not interrupt the flow here, I leave one such computation to the end. By Green's Theorem it follows that, writing $\gamma_r(t) = x_r(t) + iy_r(t)$,

$$Area(E_r) = \frac{1}{2}\int_{\gamma_r} (-y\,dx + x\,dy) = \frac{1}{2}\int_0^{2\pi} (-y_r(t)x_r'(t) + x_r(t)y_r'(t))\,dt.$$

Now

$$\begin{aligned}
-y_r x_r' + x_r y_r' &= \operatorname{Im}\{(x_r - iy_r)(x_r' + iy_r')\} = \operatorname{Im}\left\{\overline{g(re^{it})}\frac{d}{dt}\left(g(re^{it})\right)\right\} \\
&= \operatorname{Im}\left\{\left(re^{-it} + \sum_{n=0}^{\infty} \bar{b}_n r^{-n} e^{int}\right)\frac{d}{dt}\left(re^{it} + \sum_{n=0}^{\infty} b_n r^{-n} e^{-int}\right)\right\} \\
&= \operatorname{Im}\left\{\left(re^{-it} + \sum_{n=0}^{\infty} \bar{b}_n r^{-n} e^{int}\right)\left(ire^{it} - i\sum_{n=0}^{\infty} nb_n r^{-n} e^{-int}\right)\right\} \\
&= \operatorname{Im}\left\{ir^2 + i\sum_{n=0}^{\infty} \bar{b}_n r^{1-n} e^{i(n+1)t} - i\sum_{n=0}^{\infty} nb_n r^{1-n} e^{-i(n+1)t}\right. \\
&\quad \left. - i\sum_{n,m=0}^{\infty} m\bar{b}_n b_m r^{-(n+m)} e^{i(n-m)t}\right\}
\end{aligned}$$

Thus

$$\begin{aligned} Area(E_r) &= \frac{1}{2}\int_0^{2\pi} \operatorname{Im}\left\{ ir^2 + i\sum_{n=0}^{\infty}\bar{b}_n r^{1-n}e^{i(n+1)t} - i\sum_{n=0}^{\infty} nb_n r^{1-n}e^{-i(n+1)t} \right. \\ &\quad \left. - i\sum_{n,m=0}^{\infty} m\bar{b}_n b_m r^{-(n+m)}e^{i(n-m)t}\right\} dt \\ &= \frac{1}{2}\operatorname{Im}\left\{ ir^2\int_0^{2\pi} dt + i\sum_{n=0}^{\infty}\bar{b}_n r^{1-n}\int_0^{2\pi} e^{i(n+1)t}\,dt \right. \\ &\quad \left. -i\sum_{n=0}^{\infty} nb_n r^{1-n}\int_0^{2\pi} e^{-i(n+1)t}\,dt - i\sum_{n,m=0}^{\infty} m\bar{b}_n b_m r^{-(n+m)}\int_0^{2\pi} e^{i(n-m)t}\,dt\right\}. \end{aligned}$$

Since $\int_0^2 \pi e^{ikt}\,dt = 0$ if $k \in \mathbb{Z}$, $k \neq 0$, and it equals 2π if $k = 0$. It now follows that

$$Area(E_r) = \pi\left(r^2 - \sum_{n=0}^{\infty} n|b_n|^2 r^{-2n}\right).$$

The first term in the summation above is 0, so it doesn't need to be mentioned. Since $Area(E_r) \geq 0$ see that $\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \leq r^2$. As $r \downarrow 1$, E_r increases to $\mathbb{C}\backslash g(\Omega)$ while

$$r^2 - \sum_{n=0}^{\infty} n|b_n|^2 r^{-2n} \to 1 - \sum_{n=1}^{\infty} n|b_n|^2.$$

It follows that

$$Area(\mathbb{C}\backslash g(\Omega)) = 1 - \sum_{n=1}^{\infty} n|b_n|^2$$

proving the Theorem. That is $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$ because areas are non-negative, while equality is achieved if and only $Area(\mathbb{C}\backslash g(\Omega)) = 0$.

Here is one way of seeing that γ_r is positively oriented. There might be simpler ways. By the argument principle, if λ_r is the positively oriented circle of radius $1/r$ $(r > 1)$, then

$$\frac{1}{2\pi i}\int_{\lambda_r}\frac{f'(z)}{f(z)}\,dz = \#(\text{zeros of } f \text{ in } |z| < 1/r) = 1.$$

Parameterizing the circle by $\lambda_r(t) = \frac{1}{r}e^{it}$, $0 \leq t \leq 2\pi$, this works out to

$$\frac{1}{2\pi r}\int_0^{2\pi}\frac{f'(\frac{1}{r}e^{it})}{f(\frac{1}{r}e^{it})}e^{it}\,dt = 1. \tag{8}$$

The curve γ_r has 0 in its interior; it will be positively oriented if the index of 0 with respect to this curve (winding number) is 1. We have

$$Ind_{\gamma_r}(0) = \frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z}\,dz = \frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{g(re^{it})}\frac{d}{dt}g(re^{it})\,dt = \frac{r}{2\pi}\int_0^{2\pi}\frac{g'(re^{it})}{g(re^{it})}e^{it}\,dt.$$

Since $g(z) = f(1/z)^{-1}$, one gets that $g'(z)/g(z) = f'(1/z)/[z^2 f(1/z)]$. Thus

$$Ind_{\gamma_r}(0) = \frac{1}{2\pi r}\int_0^{2\pi}\int_0^{2\pi} \frac{f'(\frac{1}{r}e^{-it})}{f(\frac{1}{r}e^{-it})} e^{-it}\, dt.$$

Changing variables by $s = 2\pi - t$, $dt = -ds$, $e^{-it} = e^{is}$, we get

$$Ind_{\gamma_r}(0) = -\frac{1}{2\pi r}\int_{2\pi}^{0} \frac{f'(\frac{1}{r}e^{is})}{f(\frac{1}{r}e^{is})} e^{is}\, ds = \frac{1}{2\pi r}\int_0^{2\pi} \frac{f'(\frac{1}{r}e^{is})}{f(\frac{1}{r}e^{is})} e^{is}\, ds = 1,$$

by (8). ■

We can now prove Bieberbach's Theorem 11. Assume $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in S$. We will apply Theorem 12 to $F \in S$, where $F(z) = \sqrt{f(z^2)} = \sum_{k=0}^{\infty} c_k z^{2k+1}$ is given by Lemma 8. Let $\tilde{F}(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$; by Theorem 12 we have $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$. In particular, $|b_1|^2 \leq 1$. By Lemma 9, the coefficient b_1 of $\tilde{F}$ is the square of the coefficient of z^2 of F (which is 0) minus one half of the coefficient of z^3, which is $\frac{1}{2}a_2$. Thus $b_1 = -\frac{1}{2}a_2$, hence $1 \geq \frac{1}{4}|a-2|^2$, proving $|a_2| \leq 2$. Assuming now that$|a_2| = 2$, then $|b_1| = 1$ and Theorem 12 implies $b_n = 0$ for $n \geq 2$, hence $\tilde{F}(z) = z - \frac{1}{2}a_2 z^{-1}$. Moreover, $\frac{1}{2}a_2 = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$. Then

$$F(z) = \frac{1}{\tilde{F}(1/z)} = \frac{z}{1-e^{i\alpha}z^2}$$

and $f(z^2) = F(z)^2 = \dfrac{z^2}{(1-e^{i\alpha}z^2)^2}$. Since $z \mapsto z^2$ is onto D from D, we have

$$f(z) = \frac{z}{(1-e^{i\alpha}z)^2} = e^{-i\alpha}\frac{e^{i\alpha}z}{(1-e^{i\alpha}z)^2} = e^{-i\alpha}k(e^{i\alpha}z).$$

■

The next big preliminary theorem is a particular case of a theorem of Caratheodory, which will be essential in proving that a certain family of elements in S are dense in S in the topology of convergence over compact subsets of D. It will then suffice to settle the Bieberbach conjecture for functions in this family. It is also an essential tool in Löwner's construction. To prove it we need a result of Koebe and a growth theorem. The growth theorem is also essential in getting a first bound on the Taylor coefficients of elements in S.

Theorem 13 *(Koebe 1/4 Theorem; Koebe 1907, Bieberbach 1916) Let $f \in S$. Then $f(D) \supset \{w \in \mathbb{C} : |w| < 1/4\}$.*

Proof. Let $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$. Assume $w \notin f(D)$. Then $g : D \to \mathbb{C}$ defined by $g(z) = wf(z)/(w - f(z))$ is easily seen to be in S and have Taylor expansion $g(z) = z + \left(a_2 + \frac{1}{w}\right) z^2 + \cdots$. By Theorem 11, both a_2 and $a_2 + \frac{1}{w}$ are less than or equal 2 in absolute value, thus

$$2 \geq \left|a_2 + \frac{1}{w}\right| \geq \frac{1}{|w|} - |a_2| ge \frac{1}{|w|} - 2.$$

It follows that $1/|w| \leq 4$; i.e., $|w| \geq 1/4$. ■

The growth theorem is more complicated than one would expect. It may help to get some silly computations out of the way. Assume in all this that $g : D \to \mathbb{C}$ is analytic and $g(z) \neq 0$ for $z \in D$. Let $\omega \in \mathbb{C}$, $|\omega| = 1$ be fixed and consider $r \mapsto g(r\omega)$. We have

$$\frac{\partial g(r\omega)}{\partial r} = g'(r\omega)\omega.$$

This is fairly obvious. Putting a bar on it;i.e., taking complex conjugates, one gets

$$\frac{\partial \overline{g(r\omega)}}{\partial r} = \overline{g'(r\omega)}\bar{\omega}.$$

Next

$$\frac{\partial |g(r\omega)|^2}{\partial r} = \frac{\partial}{\partial r}\left(g(r\omega)\overline{g(r\omega)}\right) = g'(r\omega)\omega\overline{g(r\omega)} + g(r\omega)\overline{g'(r\omega)}\bar{\omega} = 2\text{Re }\left(\omega g'(r\omega)\overline{g(r\omega)}\right).$$

Up to here we had no need to assume $g(z) \neq 0$ in D. But because of it $r \mapsto \log|g(r\omega)|$ is differentiable for $r \in (0,1)$ and

$$\begin{aligned} \frac{\partial}{\partial r}\log|g(r\omega)| &= \frac{1}{2}\frac{\partial}{\partial r}\log\left(|g(r\omega)|^2\right) = \frac{1}{2}\frac{1}{|g(r\omega)|^2}2\text{Re }\left(\omega g'(r\omega)\overline{g(r\omega)}\right) \\ &= \frac{1}{|g(r\omega)|^2}\text{Re }\left(\omega g'(r\omega)\overline{g(r\omega)}\right) = \text{Re }\left(\frac{\omega g'(r\omega)}{g(r\omega)}\right). \end{aligned}$$

We will use this with $g = f'$, where $f' \in S$. Since f is univalent, $f'(z) \neq 0$ for all $z \in D$ and our formula becomes

$$\frac{\partial}{\partial r}\log|f'(r\omega)| = \text{Re }\left(\frac{\omega f''(r\omega)}{f'(r\omega)}\right). \tag{9}$$

for all $r \in (0,1)$, $\omega \in \mathbb{C}$, $|\omega| = 1$.

Another thing to know for this theorem is that if $z \in D$, then the map $\psi : D \to \mathbb{C}$ defined by

$$\psi(\zeta) = \frac{\zeta + z}{1 + \bar{z}\zeta} \tag{10}$$

is the one an only univalent map mapping D onto itself, 0 to z and such that $\psi'(0) > 0$. In fact $\psi'(0) = 1 - |z|^2$. Let us tackle now the growth theorem.

Theorem 14 *Let $f \in S$. Then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2} \tag{11}$$

for all $z \in D$, with equality occurring only if $z = 0$ or if f is a rotation of the Koebe function; i.e., $f(z)e^{-i\alpha}k(ze^{i\alpha})$ for all $z \in D$, some $\alpha \in \mathbb{R}$.

Proof. Fix for a while $z \in D$ and define $g : D \to \mathbb{C}$ by

$$g(\zeta) = \frac{f \circ \psi(\zeta) - f(z)}{(1 - |z|^2) f'(z)},$$

where ψ is given by (10). This function of ζ is univalent (since f, ψ are univalent), $g(0) = 0$ and

$$g'(\zeta) = \frac{1}{(1 - |z|^2) f'(z)} f' \circ \psi(\zeta) \psi'(\zeta)$$

so that $g'(0) = 1$. In other words, $g \in S$. Finding the coefficient of ζ^2 in the Taylor expansion of g is not too hard, we have

$$g''(\zeta) = \frac{1}{(1 - |z|^2) f'(z)} \left(f'' \circ \psi(\zeta) (\psi'(\zeta))^2 + f' \circ \psi(\zeta) \psi''(\zeta) \right).$$

One sees that

$$\psi'(\zeta) = \frac{1 - |z|^2}{(1 + \bar{z}\zeta)^2}, \tag{12}$$

$$\psi''(\zeta) = -\frac{2\bar{z}(1 - |z|^2)}{(1 + \bar{z}\zeta)^3}, \tag{13}$$

so that $\psi'(0) = 1 - |z|^2$, $\psi''(0) = -2\bar{z}(1 - |z|^2)$. Thus, if $g(\zeta) = \zeta + A_2 \zeta^2 + \cdots$, then

$$\begin{aligned} A_2 &= \frac{1}{2} g''(0) = \frac{1}{(1 - |z|^2) f'(z)} \left(f''(z)(1 - |z|^2)^2 - 2 f'(z) \bar{z} (1 - |z|^2) \right) \\ &= \frac{1}{2} \left((1 - |z|^2) \frac{f''(z)}{f'(z)} - 2\bar{z} \right). \end{aligned}$$

By Theorem 11, $|A_2| \leq 2$. Multiplying this inequality by $2|z|/(1 - |z|^2)$ (and using that $|z||w| = |zw|$ for complex numbers z, w, we get

$$\left| \frac{z f''(z)}{f'(z)} - \frac{2|z|^2}{1 - |z|^2} \right| \leq \frac{4|z|}{1 - |z|^2}, \tag{14}$$

valid for all $z \in D$.

We let again $z \in D$, write $z = r\omega$, where $r \in (0, 1)$ and $|\omega| = 1$ and now keep ω fixed for a while. Using that the real part of a complex number is bounded by its absolute value, we get from (14) that

$$\left| \mathrm{Re} \left(\frac{r\omega f''(r\omega)}{f'(r\omega)} - \frac{2r\omega}{1 - r^2} \right) \right| \leq \frac{4r}{1 - r^2}.$$

We can cancel a factor of r from all numerators and then write this inequality in the form

$$\frac{-4 + 2r}{1 - r^2} \leq \mathrm{Re} \, \frac{\omega f''(r\omega)}{f'(r\omega)} \leq \frac{4 + 2r}{1 - r^2}.$$

In view of (9), we proved

$$\frac{-4+2r}{1-r^2} \leq \frac{\partial}{\partial r} \log |f'(r\omega)| \leq \frac{4+2r}{1-r^2}.$$

If we now integrate with respect to r, from 0 to r we get ($\log |f'(0)| = \log 1 = 0$)

$$\log \frac{1-r}{(1+r)^3} = \int_0^r \frac{-4+2\rho}{1-\rho^2}\, d\rho \leq \log |f'(r\omega)| \leq \int_0^r \frac{4+2\rho}{1-\rho^2}\, d\rho = \log \frac{1+r}{(1-r)^3}.$$

Setting $z = r\omega$, removing the logarithms, we proved

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}. \tag{15}$$

At this stage we can finally get to the main estimate. The upper estimate in (11) is almost immediate. We have if $z = r\omega$. $|\omega| = 1$, $0 < r < 1$,

$$f(z) = f(r\omega) = \omega \int_0^r f'(\rho\omega)\, d\rho;$$

Taking absolute values and applying the second inequality in (15),

$$|f(z)| \leq \int_0^r \frac{1+\rho}{(1-\rho)^3}\, d\rho = \frac{r}{(1-r)^2} = \frac{|z|}{(1-|z|)^2},$$

proving the second inequality in (11). The lower estimate seems to need more work. We can divide D into two disjoint sets A, B, where

$$A = \{z \in D : |f(z)| \geq 1/4\}, \qquad B = \{z \in D : |f(z)| < 1/4\}.$$

Since the function $r \mapsto r/(1+r)^2$ is increasing in the interval $0 \leq r \leq 1$; its maximum value is assumed for $r = 1$ and this maximum equals 1/4; that is, $|z|/(1+|z|)^2 \leq 1/4$ for all $z \in D$. Thus the lower estimate of(11), namely $|f(z)| \geq |z|/(1+|z|)^2$, holds for all $z \in A$. The lower estimate is also clear if $z = 0$, so it suffices to see it also holds for all $z \in B\backslash\{0\}$, so let $z \in B\backslash\{0\}$. Then $|f(z)| < \frac{1}{4}$, hence also $|tf(z)| < 1/4$ for $0 \leq t \leq 1$ so that by the Koebe 1/4 Theorem 13, $tf(z) \in f(D)$ for all $t \in [0,1]$ and we can define a differentiable arc γ by $\gamma(t) = f^{-1}(tf(z))$. This is an arc in the disc from $0 = \gamma(0) = f^{-1}(0)$ to $z = \gamma(1) = f^{-1}(f(z))$. Suppose we write $\gamma(t) = x(t) + iy(t)$, where x, y are real valued functions of t $[0,1]$. Notice that $\gamma(t) \neq 0$ except if $t = 0$; in fact, since f, f^{-1} are univalent, $\gamma(z) = 0$ implies $tf(z) = 0$, hence either $t = 0$ or $z = 0$. Due to that, the function $t \mapsto |\gamma(t)| = \sqrt{x(t)^2 + y(t)^2}$ is differentiable for $0 < t < 1$, and continuous for $0 \leq t \leq 1$. We have

$$\frac{d}{dt}|\gamma(t)| = \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}} = \frac{\mathrm{Re}\,(\gamma(t)\gamma'(t))}{|\gamma(t)|} \leq \frac{|\gamma(t)\gamma'(t)|}{|\gamma(t)|} = |\gamma'(t)|.$$

This last thing is true for any complex valued differentiable function; the derivative of its absolute value is dominated by the absolute value of the derivative. But taking into account the definition of γ we also have

$$\gamma'(t) = \frac{\partial}{\partial t}\left(f^{-1}(tf(z))\right) = \frac{1}{f'(f^{-1}(tf(z)))} f(z) = \frac{f(z)}{f'(\gamma(t))},$$

so that $f'(\gamma(t))\gamma'(t) = f(z)$ for all $t \in [0,1]$. We have to proceed now with a little bit of care to make sure the inequalities are correctly treated. By what we just saw, $|f(z)| = |f'(\gamma(t))|\,|\gamma'(t)|$ for all $t \in [0,1]$. Thus, using the first inequality in (15),

$$|f(z)| = \int_0^1 |f(z)|\,dt = \int_0^1 |f'(\gamma(t))|\,|\gamma'(t)|\,dt \geq \int_0^1 \frac{1-|\gamma(t)|}{(1-|\gamma(t)|)^3}|\gamma'(t)|\,dt.$$

Since $(1-|\gamma(t)|)/[(1-|gm(t)|)^3]$ is always positive we can use the estimate above of the derivative of the absolute value dominated by the absolute value of the derivative to get

$$|f(z)| \geq \int_0^1 \frac{1-|\gamma(t)|}{(1-|\gamma(t)|)^3}\frac{d}{dt}|\gamma(t)|\,dt.$$

We can now change variables by $r = |\gamma(t)|$ to get

$$|f(z)| \geq \int_0^{|z|} \frac{1-r}{(1+r)^3}\,dt = \int_0^{|z|}\left(-\frac{1}{(1+r)^2}+\frac{2}{(1+r)^3}\right)dr = \frac{|z|}{(1+|z|)^2}.$$

This proves the lower estimate in (11).

It remains to be proved that if equality occurs for either estimate and $z \neq 0$, then f is a rotation of the Koebe function. The key is inequality (14); if this inequality is sharp, then all further inequalities will be sharp. So equality in (11) happens for some $z \neq 0$ only if (14) is an equality for that value of z. One sees at once this implies that the coefficient A_2 of the function g defined so much earlier in this proof will satisfy $|A_2| = 2$, hence g is a rotation of the Koebe function by Theorem 11. One now has to see that this implies that f is also such a rotation. This is another set of calculations; since I don't think we'll need this part I'll omit them. ■

An easy but necessary corollary to the growth theorem is the fact that for each $n \in \mathbb{N}$, the Taylor coefficients of functions on S can be uniformly bounded by a constant depending only on n.

Corollary 15 *There exist constants $A_n \in \mathbb{R}$ for $n \in \mathbb{N}$ such that if $f(z) = z + \sum_{n=1}^{\infty} a_n z^n \in S$, then $|a_n| \leq A_n$ for $n \in \mathbb{N}$.*

Proof. Let $f(z) = z + \sum_{n=1}^{\infty} a_n z^n \in S$. Then

$$a_n = \frac{1}{2\pi i}\int_{|z|=1/2} \frac{f(z)}{z^{n+1}}\,dz.$$

(The circle of radius $1/2$ can be replaced by any circle of radius $r \in (0,1)$). If $|z| = 1/2$ we have by the upper estimate in (11) that

$$|f(z)| \leq \frac{1/2}{(1/2)^2} = 2,$$

so that

$$|a_n| \leq \frac{2^{n+2}}{2\pi} \int_{|z|=1/2} dz = 2^{n+2}.$$

■

The Bieberbach conjecture could be phrased as stating that one can take $A_n = n$. Actually, for the De Branges' proof the following corollary seems to be more relevant. The idea of the proof is the same as for the previous corollary.

Corollary 16 *There exist constants $C_k \in \mathbb{R}$ for $k \in \mathbb{N}$ such that if $f \in S$ and $\log(f(z)/z) = \sum_{n=1}^{\infty} c_k z^k$, then $|c_k| \leq C_k$ for $k \in \mathbb{N}$.*

Proof. $h : z \mapsto f(z)/z$ takes the unit disc D onto a simply connected region containing 1 (the value that makes $f(z)/z$ analytic at 0), but not 0. In that region there is a unique analytic determination of the logarithm such that $\log 1 = 0$. One has of course for $w \in h(D)$ that $\log w = \log|w| + i \arg w$ and, while there is a bit of indeterminacy in what $\arg w$ could be[§], one will have $|\arg w| \leq 2\pi$. Thus $|\log w| \leq |\log|w|| + 2\pi$. Using the growth theorem, the function $\log f(z)/z$ will thus satisfy

$$\left|\log \frac{f(z)}{z}\right| \leq 2 \log \frac{1}{1-|z|^2} + 2\pi;$$

for $|z| = 1/2$ we get the estimate

$$\left|\log \frac{f(z)}{z}\right| \leq 2 \log 4 + 2\pi.$$

Using

$$c_k = \frac{1}{2\pi i} \int_{|z|=1/2} \frac{\log(f(z)/z)}{z^{k+1}}\, dz,$$

the result follows. ■

The following is the particular case of the theorem by Caratheodory.

Theorem 17 *Assume $\{U_n\}$ is a decreasing or increasing sequence of simply connected open subsets of $\mathbb{C}$ and for each $n \in \mathbb{N}$ assume $0 \in U_n$ and let f_n map D conformally onto U_n and satisfy $f_n(0) = 0$, $f_n'(0) > 0$. If the sequence is decreasing, assume $0 \in \left(\bigcap_{n=1}^{\infty} U_n\right)^{\circ}$ and let U be the connected component of 0 in $\left(\bigcap_{n=1}^{\infty} U_n\right)^{\circ}$. If the sequence is increasing, assume $U = \bigcup_{n=1}^{\infty} U_n \neq \mathbb{C}$. Then $\{f_n\}$ converges uniformly to a function f on compact subsets of D; moreover, f is a conformal mapping of D onto U, and the sequence $\{f_n^{-1}\}$ converges uniformly to f^{-1} on compact subsets of U.*

Proof. This is really a strong theorem, with a long proof. There is the awkwardness of the two cases, the increasing and decreasing domains cases. I'll try to break up the proof into segments, making it perhaps easier to assimilate.

[§] One may be able to take $-\pi \leq \arg w \leq \pi$, or one may be forced to select, $-\pi/2 \leq \arg w \leq 3\pi/2$; etc.

1. Assume already proved that the sequence $\{f_n\}$ converges uniformly on compact subsets of D to a function f. In this step we prove:f is a univalent mapping of D to $\mathbb{C}$. In fact, Proposition 7 allows only two possibilities because each f_n is univalent, f is either univalent or constant. In our case that it is constant means it is 0 (since $f_n(0) = 0$ for all n). If the sequence of domains $\{U_n\}$ is increasing, let $r > 0$ be such that $D(0,r) \subset U_1 \subset U$. If it is decreasing, we are assuming that $0 \in \left(\bigcap_{n=1}^{\infty} U_n\right)^{\circ}$, thus there is $r > 0$ such that $D(0,2r) \subset \bigcap_{n=1}^{\infty} U_n$, hence also $D(0,r) \subset U$, the connected component of 0 of the interior of the intersection of all the domains. Consider $f_n^{-1} : D(0,r) \to D$. Since $f_n^{-1}(0) = 0$, we have that $z \mapsto f_n^{-1}(z)/z$ is analytic in $D(0,r)$; on the boundary $\{z \in \mathbb{C} : |z| = r\}$ it satisfies¶ $|f_n^{-1}(z)/z| \leq 1/r$, thus this estimate holds in all of $D(0,r)$ by the maximum principle. Letting $z \to 0$ in $|f_n^{-1}(z)/z| \leq 1/r$ we get

$$1/r \geq \left| \left(f_n^{-1}\right)' (0) \right| = \frac{1}{|f_n'(0)|}$$

so that $|f_n'(0)| \geq r$ for all n. Since f_n' converges uniformly to f' on compact subsets of D, this implies $|f'(0)| \geq r > 0$, thus f' is not identically 0 and hence neither is f. It follows that f is univalent.

2. Still assuming proved that the sequence $\{f_n\}$ converges uniformly on compact subsets of D to a function f, we prove next that $f(D) \subset U$. Let $w_0 \in f(D)$; $w_0 = f(z_0)$ for some $z_0 \in D$. Non constant analytic functions are open mappings, so $f(D)$ is an open set and f is a conformal mapping of D onto $f(D)$. It follows that there exists $r > 0$ such that $D(w_0, r) \subset f(D)$; consequently $W = f^{-1}(D(w_0, r))$ is an open subset of D containing z_0. We now select $\rho > 0$ so that the closed disc of radius ρ centered at z_0 is included in W. Let $C = \{z \in \mathbb{C} : |z - z_0| = r\}$ be the boundary of this disc. We will prove that the image of half of this disc, namely $f(D(z_0, \rho/2))$, is included in U_n for n large enough. After that we have to use different arguments for the case of an increasing or a decreasing sequence of domains.

So let $w_1 \in f(D(z_0, \rho/2))$. Then $w_1 = f(z_1)$ for some z_1 in the open disc of radius $\rho/2$ centered at z_0, hence, f being univalent, $f(z) \neq w_1$ for all $z \in C$. Since C is compact, there is $\delta > 0$ such that $|f(z) - w_1| \geq \delta$ for all $z \in C$. Because $\{f_n\}$ converges uniformly to f on the compact subset C of D there is $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \delta$ for all $n \geq N, z \in C$. We can apply now Rouché's Theorem to the functions $f(z) - w_1$ and $f_n(z) - f(z)$. Since

$$|f_n(z) - f(z)| < \delta \leq |f(z) - w_1|$$

for $z \in C$, Rouché's Theorem states that $f(z) - w_1$ and $(f(z) - w_1) + (f_n(z) - f(z)) = f_n(z) - w_1$ have the same number of zeros in the disc bounded by C; that is, in $D(z_0, \rho)$. Now $f(z) - w_1$ has a unique such 0 in that disc (and anywhere else); it is 0 only for $z = z_1$. It follows that for $n \geq N$, there exists a unique ζ_n in the disc bounded by C such that $f_n(\zeta_n) = w_1$. Thus $w_1 \in U_n$ for $n \geq N$. Here is where we need to split the argument.

¶$|f_n^{-1}(z)| \leq 1$ for all $z \in D(0,r)$ since we map into D.

Suppose first the sequence $\{U_n\}$ decreases. Then $w_1 \in U_n$ for large values of n implies $w_1 \in U_n$ for all n and, since $w_1 1$ was an arbitrary point in $f(D(z-0,\rho/2))$ we conclude that $f(D(z_0,\rho/2)) \subset \bigcap_{n=1}^{\infty} U_n$. Now $f(D(z_0,\rho/2))$ is open and contains $w_0 = f(z_0)$, thus $w_0 \in \left(\bigcap_{n=1}^{\infty} U_n\right)^{\circ}$. Since w_0 was arbitrary in $f(D)$ this proves that $f(D) \subset \left(\bigcap_{n=1}^{\infty} U_n\right)^{\circ}$. But $0 = f(0) \in f(D)$ and $f(D)$ is connected, thus $f(D) \subset U$, the connected component of 0 of the interior of $\bigcap_{n=1}^{\infty} U_n^{\circ}$.

Next assume the sequence $\{U_n\}$ is increasing. Then the situation is somewhat simpler, we take $w_1 = w_0$ to get that $w_0 \in U_n \subset U$ for n large enough; since w_0 was arbitrary in $f(D)$ we proved $f(D) \subset U$.

3. Once again with the same assumptions (and hence conclusions) of **1.**; that is, that $\{f_n\}$ converges uniformly to f on compact subsets of D, we prove that $U \subset f(D)$. Together with **2.**, this will prove $f(D) = U$.

Consider the sequence of univalent analytic functions $\{f_n^{-1}\}$; for $n \in \mathbb{N}$, f_n^{-1} is analytic on $U_n \supset U$. In the case of a decreasing sequence $\{U_n\}$ of domains we restrict this sequence of inverses to U, the connected component of 0 in the interior of the intersection of all U_n's, so that $f_n^{-1} : U \to D$ is analytic for all n. The sequence is obviously uniformly bounded (by 1, since it takes values in the unit disc), hence by Montel's Theorem 6 there is a subsequence $\{f_{n_k}^{-1}\}$ converging uniformly on compact subsets of U. The same result is true in the case of an increasing sequence of domains; the sequence $\{f_n^{-1}\}$ is bounded uniformly since all its elements take values in D so Montel's Theorem 6 also applies to give a subsequence $\{f_{n_k}^{-1}\}$ converging uniformly on compact subsets of $U = \bigcup_{n=1}^{\infty} U_n$. Set $g_k = f_{n_k}^{-1}$ and let $g = \lim_{k\to\infty} g_k$. By Proposition 7 either g is univalent or identically 0. We do have $g_k'(0) = 1/f_{n_k}'(0) \to 1/f'(0) \neq 0$, so that $g'(0) = \lim_{k\to\infty} g_k'(0) \neq 0$, hence g is not constant, thus univalent.

Let $w_0 \in U$ and let $z_0 = g(w_0) \in D$. We want to see that $f(z_0) = w_0$, proving $w_0 \in f(D)$. Set $w_k = f_{n_k}(z_0)$, then $w_{n_k} \to f(z_0)$ as $k \to \infty$. Because g_k converges uniformly to g, one will have that $g_k(w_k)$ converges to $g(f(z_0))$. But $g_k(w_k) = g_k(f_{n_k}(z_0)) = z_0$ for all $k \in \mathbb{N}$, thus $g(f(z_0)) = z_0 = g(w_0)$. Since g is univalent, $w_0 = f(z_0)$. This completes the proof that $U \subset f(D)$, hence of $U = f(D)$.

4. We are still under the assumptions of **1.**, thus can assume everything proved so far under these assumptions. In **3.** we proved that $\{f_n^{-1}\}$ has a subsequence converging to a univalent function g uniformly over compact subsets of U, and also proved that if $w_0 \in U = f(D)$, then $f(g(w_0)) = w_0$. This implies that $g = f^{-1}$. If we replace $\{f_n\}$ with any subsequence, the same argument would prove that this subsequence has a subsequence converging uniformly on compact subsets of U to f^{-1}. If every subsequence of a sequence in a metric space has a subsequence converging to the same limit, then the whole sequence has to converge to that limit. It follows that $\{f_n^{-1}\}$ converges uniformly over compact subsets of U to f^{-1}.

5. Everything has been proved so far, except that the sequence actually converges uniformly on compact subsets of D. We now stop assuming this and prove it. It will suffice to prove that $\{f_n\}$ is uniformly bounded on compact subsets of D. In fact, then by Montel's Theorem 6 there will be a subsequence converging uniformly on compact subsets of D. To this subsequence we can apply all that we proved so far; in particular its limit f will be a conformal map of D onto U satisfying $f(0) = 0$ and $f'(0) > 0$. There being only one such map, all converging subsequences must have the same limit, thus the whole sequence converges to that limit.

To prove that the sequence is uniformly bounded on compact subsets of D we notice first that all that f_n lacks for being in S is that $f_n'(0)$ might not be 1; that means that $f_n/f_n'(0) \in S$ for all $n \in \mathbb{N}$, hence by the growth Theorem 14, specifically the upper estimate in (11),

$$|f_n(z)| \leq \frac{|z||f_n'(0)|}{(1-|z|)^2}$$

for all $n \in \mathbb{N}, z \in D$. We need to see that the sequence $\{|f_n'(0)|\}$ is bounded. Using again the fact that $f_n/f_n'(0) \in S$, the Koebe 1/4 Theorem 13 implies that

$$\{z \in \mathbb{C} : |z| < 1/4\} \subset \frac{1}{|f_n'(0)|} f_n(D); \qquad \text{i.e.} D(0, \frac{1}{4}|f_n'(0)|) \subset f_n(D) = U_n.$$

If there is a subsequence with $|f_{n_k}'(0)| \to \infty$, then there is a subsequence U_{nk} *increasing* to $\mathbb{C}$. This is only possible in the case of a decreasing sequence of sets if $U_n = \mathbb{C}$ for all n; but U_n is conformally equivalent to D, and $\mathbb{C}$ is not. On the other hand, if $\{U_n\}$ increases then we explicitly assume that $U \neq \mathbb{C}$, so again this possibility is excluded. It follows that $\{|f_n'(0)|\}$ is a bounded sequence, completing the proof that $\{f_n\}$ is uniformly bounded on compact subsets of D and of the theorem. ∎

The main application of this theorem of Caratheodory to the matter at hand is to prove that a certain type of functions, the so called *slit maps*, are dense in S (in the topology of uniform convergence on compact sets). This reduces proving the conjectures for slit maps. If f is a slit map, Charles Löwner (1923) associates with it a sort of homotopy taking it to the identity function;specifically a map $F : D \times [0, \infty) \to \mathbb{C}$ such that $(z, t) \mapsto F(z, t)$ is differentiable in t, analytic in z, $F(z, 0) = z$ for all $z \in D$, and $\lim_{t\to\infty} e^t F(t, z) = f(z)$ on uniformly on compact subsets of D. He shows that such an F can be constructed satisfying a certain differential equation. With this Löwner was able to prove $|a_3| \leq 3$ for slit maps and, because slit maps are dense in S, also for all $f \in S$. I believe he also proved $|a_4| \leq 4$. De Branges' proof relies heavily on Löwner's construction. The details are developed in the next sections. We conclude this section with the definition of slit maps and the density theorem.

Definition 2 *A slit map is a function $f \in S$ such that $f(D) = \mathbb{C}\backslash\Gamma^*$ where Γ is a Jordan arc going to infinity; i.e. $\Gamma : [0, \infty) \to \mathbb{C}$, Γ is continuous and injective, $\lim_{t\to\infty} |\Gamma(t)| = \infty$.*

Lemma 18 *The Koebe function is a slit map. In fact,*

$$k(D) = \mathbb{C}\backslash\{z \in \mathbb{C} : z \in \mathbb{R}, z \leq 1/4\}.$$

Proof. A proof using nothing from complex analysis could be quite messy. The way to proceed is to ask oneself what k does to the boundary of the circle. So we consider

$$k(e^{it}) = \frac{e^{it}}{(1-e^{it})^2} \qquad \text{for} \quad 0 \leq t \leq 2\pi.$$

If we multiply numerator and denominator by the conjugate of the denominator; i.e., by $(1-e^{-it})^2$, the numerator becomes

$$e^{it}(1-e^{-it})^2 = e^{it} - 2 + e^{-it} = 2(cost - 1) = -4\sin(t/2),$$

the denominator becomes

$$((1-e^{it})(1-e^{-it}))^2 = (1 - e^{it} - e^{-it} + 1)^2 = 4(1-\cos t)^2 = 16\sin^2(t/2)$$

so that

$$k(e^{it}) = -\frac{1}{4\sin(t/2)}.$$

As t ranges from 0 to π, the values of $k(e^{it})$ ar negative and go from ∞ along the negative real axis to $-1/4$. As t ranges from π to 2π, the values of $k(e^it)$ go back along the negative real axis to ∞. One concludes that $k(D)$ is the complement in $\mathbb{C}$ of the closed interval $(-\infty, 1/4]$ of the real axis. ■

Note. Even though I defined the Jordan arc missed by the range of a slit map as the image of an injective map from $[0, \infty)$ to $\mathbb{C}$, it will also be the image of the boundary of the disc by the slit map (extended to the boundary of the disc). As such it will be a Jordan arc traversed twice, from infinity to its beginning and back to infinity.

Theorem 19 *Let $f \in S$. There exists a sequence $\{f_n\}$ of slit maps converging uniformly to f on compact subsets of D.*

Proof. We may assume that f is defined and univalent in a disc $D(0, R)$ with $R > 1$. In fact, for $r \in (0, 1)$ define f_r by $f_r(z) = f(rz)$; this is clearly a univalent map defined in $D(0, 1/r)$ and it is easy to see that $\lim_{r\to 1-} f_r = f$

uniformly over compact subsets of D. Assuming now f univalent in $D(0,R)$ consider the curve $\gamma:[0,2\pi]\to\mathbb{C}$ given by $\gamma(t)=f(e^{it})$. This is a closed Jordan curve and there are a number of ways of seeing that the single bounded component of $\mathbb{C}\backslash\gamma^*$ is $f(D)$. Let $\Gamma_0:[0,\infty)\to\mathbb{C}$ be a differentiable Jordan arc (i.e., injective path) connecting $\gamma(0)=\gamma(1)=f(1)$ with ∞ intersecting γ only at $\gamma(0)$ and define the path Γ_n for $n\in\mathbb{N}$ as being the portion of the curve γ from $t=1/n$ to $t=2\pi$ followed by Γ_0; for example by

$$\Gamma_n(t)=\begin{cases}\gamma(t), & \frac{1}{n}\le t\le 2\pi,\\ \Gamma_0(t-2\pi), & t>2\pi.\end{cases}$$

Let $U_n=\mathbb{C}\backslash\Gamma_n^*$ for $n\in\mathbb{N}$. Then U_n is simply connected, $U_n\neq\mathbb{C}$, $0\in U_n$. By the Riemann mapping theorem, there exists a unique univalent f_n mapping D onto U_n in such a way that $f_n(0)=0$ and $f_n'(0)>0$. Moreover the sequence $\{U_n\}$ is decreasing and $\bigcap_{n=1}^{\infty}U_n=\mathbb{C}\backslash(\gamma^*\cup\Gamma^*)$. It is clear that the connected component of 0 in this intersection is precisely the region bounded by γ^*, namely $f(D)$. By Theorem 17, the particular case of a Theorem of Caratheodory, the sequence $\{f_n\}$ converges uniformly on compact sets to a conformal mapping of D onto $f(D)$ which sends 0 to 0 and has positive derivative at 0. But there is only one such map, and f is such a map, thus the sequence converges to f uniformly on compact subsets of D. ■

5 Löwner's Differential Equations

In this section we prove Löwner's Theorem, which was

Theorem 20 *Let $f\in S$ be a slit map; $f(z)=z+\sum_{n=1}^{\infty}a_nz^n$. There exists a continuous $g:D\times[0,\infty)\to\mathbb{C}$ such that $z\mapsto g(z,t):D\to\mathbb{C}$ is univalent for all $t\in\mathbb{R}$ and $t\mapsto g(z,t):(0,\infty)\to\mathbb{C}$ is differentiable for all $z\in D$. Moreover:*

1. *$g(z,t)=e^t\left(z+\sum_{n=2}^{\infty}a_n(t)z^n\right)$ for $z\in D$, $t\ge 0$, where $a_n:[0,\infty)\to\mathbb{C}$ are continuous, and differentiable for $t>0$, and $a_n(0)=a_n$ for each $n\in\mathbb{N}$. In particular, $g(\cdot,0)=f$.*
2. *g satisfies the following differential equation:*

$$\frac{\partial g}{\partial t}(z,t)=z\frac{1+\kappa(t)z}{1-\kappa(t)z}\frac{\partial g}{\partial z}(z,t) \tag{16}$$

where $\kappa:[0,\infty)\to\mathbb{C}$ is continuous and $|\kappa(t)|=1$ for all $t\in[0,\infty)$.

This is again a theorem with a pretty long proof. Not so much the construction of g; that is easily done as we shall see, but proving that it has the desired properties. I will prove it in a more or less narrative form, taking up the rest of this section.

Let $\Gamma : [0, \infty) \to \mathbb{C}$ be a continuous injective map, $\lim_{t\to\infty} |\Gamma(t)| = \infty$, such that Γ^* is the complement of $f(D)$. That is, $f(D) = \{z \in \mathbb{C} : z \neq \Gamma(t) \text{ for } 0 \leq t < \infty\}$. For $t \geq 0$ let $\Gamma_t : [t, \infty) \to \mathbb{C}$ be the restriction of Γ to $[t, \infty)$ and set

$$U_t = \mathbb{C} \backslash \Gamma_t^* = \{z \in \mathbb{C} : z \neq \Gamma(s) \text{ for } s \geq t\}.$$

The sets U_t are simply connected for all $t \geq 0$, $U_0 = f(D)$. Being simply connected, and not $\mathbb{C}$, there exists a unique univalent function $G_t : D \to \mathbb{C}$ such that $G_t(D) = U_t$, $g_t(0) = 0$ and $G_t'(0) > 0$. Notice that $f(0) = 0$ so $0 \in f(D) = U_0 \subset U_t$ for all $t \geq 0$. The Taylor expansion of G_t can be written in the form

$$G_t(z) = \beta(t) \left(z + \sum_{n=2}^{\infty} B_n(t) z^n \right).$$

(We just factor out the coefficient of z, which is positive). We will prove

Lemma 21 *The function β is continuous and strictly increasing from $[0, \infty)$ to $[1, \infty)$.*

We now define $g : D \times [0, \infty) \to \mathbb{C}$ by

$$g(z, t) = e^t \left(z + \sum_{n=2}^{\infty} B_n(\beta^{-1}(e^t) z) z^n \right)$$

for $z \in D, t \geq 0$. Notice that if $t \geq 0$ then $e^t \geq 1$, so e^t is in the domain of β^{-1}, and gets mapped back to $[0, \infty)$ by β^{-1}.

Before continuing it might be instructive to see how all of this looks for the case of the Koebe function. As seen above, for the Koebe function $f(D) = \mathbb{C} \backslash \{z \in \mathbb{C} : z \in \mathbb{R}, z \leq -1/4\}$. The easiest parametrization of the missing Jordan arc is $\Gamma(t) = -\frac{1}{4} - t$ for $t \geq 0$. One verifies that then

$$U_t = \mathbb{C} \backslash \Gamma_t^* = \mathbb{C} \backslash \{z \in \mathbb{C} : z \in \mathbb{R}, z \leq -1/4 - t\}$$

and one also sees easily that the univalent map mapping D onto G_t is simply $G_t = (4t + 1)k$. Now

$$G_t(z) = (4t + 1)k(z) = (4t + 1) \sum_{n=1}^{\infty} n z^n.$$

We have $\beta(t) = 4t + 1$, $B_n(t) = n$ for all n, t. Since B_n is constant for all n, we now get

$$g(z, t) = e^t \sum_{n=1}^{\infty} n z^n = e^t k(z) = \frac{e^t z}{(1 - z)^2}.$$

Now

$$\frac{\partial g}{\partial t}(z, t) = g(z, t),$$

while
$$\frac{\partial g}{\partial z}(x,t) = e^t \frac{1+z}{(1-z)^3} = \frac{1+z}{z(1-z)} g(z,t).$$
Thus
$$\frac{\partial g}{\partial t}(z,t) = z\frac{1-z}{1+z}\frac{\partial g}{\partial z}(x,t),$$
which is Löwner's differential equation with $\kappa(t) \equiv -1$.

Maybe it is time to prove something. The domains U_t are (as mentioned) simply connected for all $t \geq 0$ and $U_0 = f(D)$. We also have $U_s \subset U_t$ if $s < t$ (the amount of Jordan arc we take away gets smaller as t increases) and, in fact, the inclusion is proper: $U_s \neq U_t$ if $s < t$. We prove here:

Lemma 22 *The map $(z,t) \mapsto G_t(z) : D \times [0,\infty) \to \mathbb{C}$ is continuous.*

Proof. Let $(z_0, t_0) \in D \times [0,\infty)$ and assume the map is not continuous at (z_0, t_0), There exists then $\epsilon > 0$, and there exists a sequence $(z_k, t_k) \in D \times [0,\infty)$ converging to (t_0, z_0) such that $|G_{t_k}(z_k) - G_{t_0}(z_0)| \geq \epsilon$ for all k. We can divide the situation into the case in which $\{t_k\}$ approaches t_0 from the right, and the case it approaches it from the left; i.e., we can divide into the case in which $\{t_k\}$ decreases, and the case it increases. Assume first $\{t_k\}$ decreases to t_0. It is clear that then the domains $\{U_{t_k}\}$ is a decreasing sequence of arrays with intersection equal to U_{t_0}. The set U_{t_0} is a connected (in fact, simply connected) set containing 0, thus by Caratheodory's Theorem 17, $\{G_{t_k}\}$ converges uniformly over compact subsets to a conformal mapping of D onto U_{t_0}; by the uniqueness part of the Riemann mapping theorem, this mapping has to be G_{t_0}. Let N be a compact neighborhood of z_0 in D; for example $N = \{z \in \mathbb{C} : |z - z_0| \leq (1 - |z_0|)/2\}$. There is then $K \in \mathbb{N}$ such that $|G_{t_k}(z) = G_{t_0}(z)| < \epsilon/2$ for all $k \geq K$, $z \in N$. But G_{t_0} is analytic on D, there is thus a neighborhood W of z_0 in D such that $|G_{t_0}(z) - G_{t_0}(z_0)| < \epsilon/2$ if $z \in W$. Now $N \cap W$ is a neighborhood of z_0; since $z_k \to z_0$ as $k \to \infty$, there is $K' \in \mathbb{N}$ such that $k \geq K'|$ implies $z_k \in N \cap W$. This means that
$$|G_{t_k}(z_k) - G_{t_0}(z_0)| \leq |G_{t_k}(z_k) - G_{t_0}(z_k)| + |G_{t_0}(z_k) - G_{t_0}(z_0)| < \epsilon$$
for $k \geq \max(K, K')$; contradicting the choice of ϵ.

The proof if the sequence $\{t_{n_k}\}$ increases to t_0 is almost identical, using the increasing domains part of Caratheodory's Theorem 17 ■

Let $\beta(t) = G_t'(0)$ so that the Taylor expansion of G_t begins as $G_t(z) = \beta(t)z + \cdots$. By construction $\beta(t) > 0$, so that we can divide all coefficients of the Taylor expansion of G_t by $\beta(t)$ and write the expansion in the form it was written above, namely
$$G_t(z) = \beta(t)\left(z + \sum_{n=2}^{\infty} B_n(t) z^n\right). \tag{17}$$

The coefficient of z^n is thus $\beta(t)B_n(t)$. Fix for a moment, r, $0 < r < 1$. By the Cauchy formula for the coefficients we have that

$$\beta(t) = \frac{1}{2\pi i}\int_{|z|=r}\frac{G_t(z)}{z^2}\,dz, \tag{18}$$

$$B_n(t) = \frac{1}{2\pi i\beta(t)}\int_{|z|=r}\frac{G_t(z)}{z^{n+1}}\,dz, \quad n = 2, 3, \ldots. \tag{19}$$

By the continuity of $(t, z) \mapsto G_t(z)$ (which translates to uniform continuity on compact sets), it follows from (18) that β is continuous; since it is never 0, one gets from (19) that $B_n : [0, \infty) \to \mathbb{C}$ is continuous for $n = 1, 2, 3, \ldots$.

For the next step we appeal to the subordination principle Theorem 10. If $s < t$ then $G_s(D) = U_s \subset U_t = G_t(D)$, also $G_s(0) = 0 = G_t(0)$, and both G_s, G_t are univalent. By the subordination principle

$$\beta(s) = G'_s(0) = |G'_s(0)| \le |G'_t(0)| = G'_t(0) = \beta(t);$$

in addition, $\beta(s) = \beta(t)$ would imply $G_s = G_t$, hence $U_t = U_s$. But $U_t \neq U_s$. Thus $\beta(s) < \beta(t)$; the function β is strictly increasing. Since $G_0 = f$, since f is the only univalent function mapping D onto $U_0 = f(D)$, 0 to 0, and such that $f'(0) > 0$, we also have that $\beta(0) = 1$. We proved:

Lemma 23 *The functions $\beta : [0, \infty) \to \mathbb{R}$, $B_n : [0.\infty) \to \mathbb{C}$ defined by (17), satisfying (18), (19), are continuous.*

We also proved most of Lemma 21. The only thing that remains to be proved is $\lim_{t\to\infty}\beta(t) = \infty$. Let $M \in \mathbb{R}$, $M > 0$. Considering the parametrization $t \mapsto \Gamma(t)$ of our Jordan arc, we have that $\lim_{t\to\infty}|\Gamma(t)| = \infty$, so there exists $t_0 > 0$ such that $|\Gamma(t)| > M$ if $t \ge t_0$. Another way of expressing this is to say that the closed disc $A = \{w \in \mathbb{C} : |w| \le M\}$ is a compact set disjoint from $\Gamma^*_{t_0}$, which is the complement of $G_{t_0}(D)$. Thus $G_{t_0}^{-1}(A)$ is a compact subset of D, hence there is ρ, $0 < \rho < 1$ such that $G_{t_0}^{-1}(A) subset D(0, \rho)$. It follows that if $\rho \ge |z| < 1$, then $|G_{t_0}(z)| > M$. As usual, because $G_{t_0}(0) = 0$ and (being univalent) has no other zeroes in the disc, the function $z \mapsto z/G_(t_0)(z)$ is analytic in D. For $|z| = \rho$ we have

$$\left|\frac{z}{G_{t_0}(z)}\right| \le \frac{\rho}{M} < \frac{1}{M};$$

by the maximum principle this inequality must hold now for all $z \in D(0, \rho)$. Letting $z \to 0$,

$$\frac{1}{M} \ge \lim_{z\to 0}\left|\frac{z}{G_{t_0}(z)}\right| = \left|\lim_{z\to 0}\frac{z}{G_{t_0}(z)}\right| = \frac{1}{|G'_{t_0}(0)|} = \frac{1}{\beta(t_0)};$$

that is, $\beta(t_0) > 1/M$. Since β is increasing, this proves $\beta(t) > M$ for $t \ge t_0$. Since $M > 0$ was arbitrary, we are done.

Lemma 21 is now proved in its entirety. ■

Depending how one parameterizes the Jordan arc Γ, one can get β to be almost any strictly increasing, continuous function mapping $[0, \infty)$ onto $[1, \infty)$. To fix β and get some uniqueness into the process we now define ϕ by $\phi(t) = \beta^{-1}(e^t)$ if $t \geq 0$, and then, for $z \in D$, $t \geq 0$, define $g(z, t)$ by

$$g(z,t) = G_{\phi(t)}(z) = G_{\beta^{-1}(e^t)}(z).$$

When convenient, we shall also write $g_t(z)$ for $g(z, t)$; that is, $g_t = G_{\phi(t)}$ is the unique conformal mapping from D onto $U_{\phi(t)}$ such that $g_t(0) = 0$ and $g_t'(0) > 0$¿

The map ϕ is strictly increasing and continuous, a homeomorphism of $[0, \infty)$ onto itself. It is thus an easy consequence of Lemma 22 and of (17) that:

Lemma 24 *The map $g : D \times [0, \infty) \to \mathbb{C}$ is continuous.*

It is also evident that the Taylor expansion at 0 of $(\cdot, t)$ has the form given in Theorem 20, namely

$$g(z,t) = e^t \left(z + \sum_{n=2}^{\infty} a_n(t) z^n \right), \tag{20}$$

where $a_n(t) = B_n(\beta^{-1}(e^t))$ is continuous from $[0, \infty)$ to $\mathbb{C}$; $a_n(0) = a_n$ for $n \geq 2$. From now on we will use the notation $\frac{\partial g}{\partial z}$ for the complex derivative of the function $z \mapsto g(z, \cdot)$; use $\frac{\partial g}{\partial t}$ (once differentiability has been established) of the real variable complex valued function $t \mapsto g(\cdot, t)$.

And now to the really tough part, the differentiability of g in the t variable, and the differential equation. There could be simpler proofs. This proof first establishes a differential equation for a sort of inverse function of g, then turns it around. I believe it is how Löwner proceeded. We have to begin with some facts and notation.

If $t \geq 0$, $w \in U_t$, we define $g^{-1}(w, t) \in D$ by $g(g^{-1}(w, t), t) = w$. In other words, $g^{-1}(\cdot, t) = g_t^{-1}$, where $g_t(z) = g(z, t)$. The map $w \mapsto g^{-1}(w, t)$ is a conformal mapping of U_t onto D, sending 0 to 0.

Suppose $\varphi : D \to \mathbb{C}$ is univalent and let $U = \varphi(D)$ so that φ is a conformal mapping of D onto U. If the boundary of U is a Jordan curve, then φ extends to a homeomorphism of $\bar{D}$ onto $\bar{U}$. This is proved in Stein-Shakarchi for the case of the boundary of U being a polygon (Chapter 8, Theorem 4.2), left as an exercise if the boundary of U is a piecewise smooth curve (Chapter 8, Exercise 18) and as a problem in the general case (Chapter 8, Problem 6). But a more genera; (the most general?) result can be found in Rudin's *Real and Complex Analysis.* To state it we need a definition. Let Ω be an open subset of $\mathbb{C}$ and let $z - 0 \in \partial\Omega$. We say z_0 is a *simple point* of the boundary of Ω iff the following property is satisfied: Whenever $\{z - n\}$ is a sequence of points in Ω converging to z_0, there exists an arc $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(t) \in \Omega$ for $0 \leq t < 1$, $\gamma(1) = z_0$ and there exist points $\{t_n\}$ in $[0, 1)$ such that $\gamma(t_n) = z_n$ for $n \in \mathbb{N}$.

To give an example, consider the slit disc $\Omega = \{z \in \mathbb{C} : |z| < 1,\}\backslash\{z \in \mathbb{R} : z \geq 0\}$. Then

$$\partial\Omega = \{z \in \mathbb{C} : |z| = 1\} \cup \{z \in \mathbb{R} : 0 \leq z \leq 1\}.$$

Then all points of $\{z \in \mathbb{C} : |z| = 1, z \neq 1\}$ are simple boundary points; all points of $\{z \in \mathbb{R} : 0 \leq z \leq 1\}$ are not simple.

There is then a further generalization in which a Jordan curve is a curve in the plane extended by ∞; i.e., on the Riemann sphere. For example the half plane $\{\mathrm{Im}\, z > 0\}$ is bounded by such a Jordan curve, namely the real line. One can then get the following result, where by Jordan curve one understands either a closed Jordan curve in the plane or a Jordan arc; i.e., an injective, continuous map $\gamma : \mathbb{R} \to \mathbb{C}$ such that $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$.

Theorem 25 *Assume U, V are open simply connected subsets of $\mathbb{C}$ both bounded by Jordan curves. If $f : U \to V$ is a conformal mapping of U onto V, then f extends to a homeomorphism from $\bar{U}$ to $\bar{V}$. The extension is necessarily unique.*

I won't prove this theorem. It is an easy consequence of the results mentioned above; one simply needs to factor $f = f_1 \circ f_2^{-1}$ where $f_1 : D \to V$, $f_2 : D \to U$ are conformal equivalences. One thing to notice is that if the Jordan curve bounding U is a regular Jordan curve, compactly embedded in $\mathbb{C}$, and V is bounded by a generalized Jordan curve, one going from ∞ to ∞, then there will be a point on the boundary of U mapped to ∞ by the extended f.

In our case the boundaries of the domains U_t onto which D maps conformally are obviously not Jordan curves but Jordan arcs. They go to ∞, but come from a finite point in the plane. However, the result can be modified as follows.

Proposition 26 *Assume $\varphi : D \to U$ is a conformal isomorphism, where $U = \mathbb{C}\backslash\Gamma^*$, $\Gamma : [a, \infty) \to \mathbb{C}$ a Jordan arc. Then*

1. *$\zeta_0 = \lim_{w\to\Gamma(a)} \varphi^{-1}(w)$ and $\zeta_\infty = \lim_{|w|\to\infty} \varphi^{-1}(w)$ exist; $\zeta_0 \neq \zeta_\infty$, $|\zeta_0| = |\zeta_\infty| = 1$.*

2. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\zeta_0 = e^{i\alpha}$, $\zeta_\infty = e^{i\beta}$. We can assume $\alpha < \beta < \alpha + 2\pi$. Then φ extends to a continuous map $\varphi : D\backslash\{\zeta_\infty$ to $\mathbb{C}$ such that $t \mapsto \varphi(e^{it}) : [\alpha, \beta) \to \mathbb{C}$ is injective ans a parametrization of the Jordan arc Γ^*. Similarly, $t \mapsto \varphi(t) : (\beta, \alpha] \to \mathbb{C}$ is injective and another parametrization of the Jordan arc Γ^* gone through in reverse direction (from ∞ to $\Gamma(a)$). This partitions the unit circle minus the point ζ_∞ into two arcs γ_1^*, γ_2^*,*

$$\gamma_1(t) = \varphi(e^{it}), \alpha \leq t < \beta, \gamma_2(t) = \varphi(e^{it}), \beta < t \leq \alpha$$

that have only the point ζ_0 in common.

Before proving this proposition, let us see how it works in the case of the Koebe function $k(z) = z/(1-z)^2$. I will avoid getting into too many actual calculations here. We saw already that $k(e^{it}) = -1/(4\sin(t/2))$. Going into a bit more detail, if

$$w = k(z) = \frac{z}{(1-z)^2},$$

one can solve to get, if $w \neq 0$,

$$z = \frac{2w+1 \pm \sqrt{4w+1}}{2w}.$$

One has to decide which determination of the square root works. Before deciding on it we notice that

$$\left(\frac{2w+1-\sqrt{4w+1}}{2w}\right)\left(\frac{2w+1+\sqrt{4w+1}}{2w}\right) = 1;$$

this can be verified either directly or as a consequence of the fact that both are roots of a quadratic equation with leading and constant coefficient equal to 1. It is a bit harder, but not horribly hard to verify that (for $w \neq 0$)

$$\left|\frac{2w+1-\sqrt{4w+1}}{2w}\right| = \left|\frac{2w+1+\sqrt{4w+1}}{2w}\right|$$

if and only if $w \in \mathbb{R}$ and $w \leq -1/4$; in other words, w is in the omitted Jordan arc $\Gamma^* = \{w \in \mathbb{R} : w \leq -1/4\}$. For all other values of w, one of $\dfrac{2w+1 \pm \sqrt{4w+1}}{2w}$ will be in the disc, the other one out of the disc. If $w \in \Gamma^*$, then $4w = 1 \leq 0$; we will take for our square root the one defined in the plane cut by the negative real axis such that the square root of positive numbers is positive. Consider then, just to pick a convenient value, $w = 2$. Then $4w + 1 = 9$, so

$$\frac{2w+1 \pm \sqrt{4w+1}}{2w} = \frac{5 \pm 3}{4},$$

which works out to 2 if we choose the + sign, 1/2 with the minus sign. So with this determination of the square root, we need the minus sign. I emphasize that if it works in one case it has to work in all because k has a well determined analytic inverse defined in $\mathbb{C}\backslash\Gamma^*$ and the inverse determines as much the choice of the square root as the other way around. Thus

$$k^{-1}(w) = \frac{2w+1-\sqrt{4w+1}}{2w}$$

for $w \neq 0$.

The limit for $w \to 0$ can be computed by Calculus 1 methods; it works out to 0 as it should; $\varphi^{-1}(0) = 0$. This time I will parameterize the Jordan arc by $\Gamma(t) = -t$, $1/4 \leq t < \infty$. We have

$$\zeta_0 = \lim_{w \to -1/4} k(w) = -1, \qquad \zeta_\infty = \lim_{|w| \to \infty} k^{-1}(w) = 1.$$

Thus we can take $\alpha = \pi$, $\beta = 2\pi$ and if we recall that $k(e^{it}) = -\dfrac{1}{4\sin(t/2)}$, we see that the arc γ_1 in this case is the lower semicircle from -1 to 1, omitting the end point 1; γ_2 is the upper semicircle from 1 to -1, omitting the initial point 1.

Sketch of the proof of Proposition 26. The first thing to see is, perhaps, that

$$\lim_{w\in\varphi(U), w\to\Gamma(a)} \varphi^{-1}(w)$$

exists. We can find a disc $W = D(w_1, \rho)$, $\rho > 0$, such that $\Gamma(a)$ is on the boundary of this disc and $\overline{W}\setminus\{\Gamma(a)\} \subset \varphi(D)$.

The idea is to have the result about conformal mappings quoted earlier do all the hard work. So We will extend Γ to an injective continuous map $\tilde{\Gamma} : (-\infty, \infty) \to \mathbb{C}$ such that $\lim_{t\to-\infty} |\tilde{\Gamma}(t)| = \infty$. Proving that this can be done is not entirely trivial; the original Jordan arc could do a lot of strange stuff, have spirals inside spirals and other weirdnesses, so a bit of care must be exercised. It can be done because the complement of Γ^* is an open connected set, hence path connected. The extended arc $\tilde{\Gamma}$ partitions the plane into two simply connected sets; that is, $\mathbb{C}\setminus\tilde{\Gamma}^* = U_1 \cup U_2$ where U_1, U_2 are two open disjoint simply connected sets. This is essentially the Jordan curve theorem on the Riemann sphere; i.e., $\mathbb{C} \cup \{\infty\}$. The standard form of the theorem has a bounded and an unbounded component; in principle the unbounded component is not simply connected, but it is so if we add ∞ to the picture; any curve in the unbounded component can be homotopically shrunk to ∞. If we restrict φ^{-1} to U_1 we get a conformal map of U_1 onto a subset $V_1 = \varphi^{-1}(U_1)$ of D. The set V_1 is a bounded simply connected open set; a standard result [**?**] proves its boundary is a Jordan curve. If we consider what the boundary can be, it has to be the image of the boundary of U_1. Part of this boundary, the part corresponding to the restriction of $\tilde{\Gamma}$ to $(-\infty, a)$ is in the range of φ, thus gets mapped onto a connected subset (an arc) in D. This arc begins at a point of ∂D and ends at another point of ∂D; namely, (with some abuse of notation) the point $\zeta_0 = \varphi^{-1}(\Gamma(a))$ and the point $\zeta_\infty = \varphi^{-1}(\infty)$. The image under this extended homeomorphism φ^{-1} of the original Jordan arc Γ^* has thus to be a connected subset of the circle ∂D; in other words, an arc that will go from ζ_0 to ζ_∞, closing the curve that is the image of $\tilde{\Gamma}^*$ under φ^{-1}. Similarly for U_2, which is mapped conformally onto $\varphi^{-1}(U_2) \subset D$. The image of Γ^* under the extension of φ^{-1} to a homeomorphism of the the closures of these sets has to be an arc of the boundary circle of D joining ζ_0 to ζ_∞. It is easy to see it cannot be the same arc as before, so it must be its complement.

The restrictions of φ to V_1 and to V_2 extend to continuous maps from$\bar{V}_1, \bar{V}_2$ to $\mathbb{C}$, respectively; these maps necessarily agree on $\left(\bar{V}_1 \cap \bar{V}_2\right) \setminus\{\zeta_\infty$, thus φ extends to a continuous map $\bar{D}\setminus\{\zeta_\infty\}$ to $\mathbb{C}$. We are basically done. ■

We return now to our situation, assuming again the hypothesis and notation of the statement of Theorem 20. If we apply this proposition to our map g_t, $t \geq 0$, we will write $\lambda(t)$ for the point called ζ_0 in the proposition, $\mu(t)$ for the point called ζ_∞. Then $|\lambda(t)| = |\mu(t)| = 1$. $\lambda(t) \neq \mu(t)$,

$$\lim_{w \to \Gamma(\phi(t))} g_t^{-1}(w) = \lambda(t), \quad \lim_{|w| \to \infty} g_t^{-1}(w) = \mu(t).$$

If we write $\lambda(t) = e^{i\alpha(t)}$, $\mu(t) = e^{i\beta(t)}$, $\alpha(t) < \beta(t) < \alpha(t) + 2\pi$, and if γ_1^*, γ_2^* are the two arcs

$$\gamma_1^* = \{e^{i\tau} \,:\, \alpha(t) \leq \tau < \beta(t)\}, \qquad \gamma_1^* = \{e^{i\tau} \,:\, \alpha(t) \leq \tau < \beta(t)\}$$

We fix now for a while, until further notice, $s, t \in \mathbb{R}$, $0 \leq s < t$. The function g_s is univalent from D onto $U_{\phi(s)} \subset U_{\phi(t)}$, g_t^{-1} is univalent from $U_{\phi(t)}$ onto D; it follows that the map h defined by

$$h(z) = g_t^{-1}(g_s(z))$$

is univalent from D into D. The arc segment $\{\Gamma(\phi(\sigma)) \,:\, s \leq \sigma < t\}$ is in $U_{\phi(t)}$ but not in $U_{\phi(s)}$. It's image under g_t^{-1} is an arc in D beginning at a point $g_t^{-1}(\Gamma(\phi(s))) \in D$, ending at $\lambda(t) \in \partial D$; $\lambda(t)$ as defined above. If we denote this arc by $J_{s,t}$, then h is a conformal mapping of D onto $D \backslash J_{s,t}$.

Consider now the extension of g_s to $\bar{D} \backslash \{\mu(s)\}$. Since it maps $\{e^{i\sigma} \,:\, \alpha(s) \leq \sigma < \beta(s)\}$ bijectively onto $\{\Gamma(\phi(\tau)) \,:\, s \leq \tau < \infty\}$, it maps a subarc of this arc onto $\{\Gamma(\phi(\tau)) \,:\, s \leq \tau \leq t\}$. Similarly, it maps a subarc of $\{e^{i\sigma} \,:\, \beta(s) < \sigma < \alpha(s) + 2\pi\}$ onto the sam arc $\{\Gamma(\phi(\tau)) \,:\, s \leq \tau \leq t\}$. These two subarcs of the boundary of the circle intersect at $\lambda(s)$.