GAP PROBABILITIES AND BETTI NUMBERS OF A RANDOM INTERSECTION OF QUADRICS

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Abstract. We consider the Betti numbers of an intersection of k random quadrics in $\mathbb{R}\mathrm{P}^n$. Sampling the quadrics independently from the Kostlan ensemble, as $n \to \infty$ we show that for each $i \geq 0$ the expected ith Betti number satisfies:

$$\mathbb{E}b_i(X) = 1 + O(n^{-M}) \quad \text{for all } M > 0.$$

In other words, each *fixed* Betti number of X is asymptotically expected to be one; in fact as long as $i = i(n)$ is sufficiently bounded away from $n/2$ the above rate of convergence is uniform (and in this range Betti numbers concentrate to their expected value).

For the special case $k = 2$ we study the expectation of the sum of *all* Betti numbers of X. It was recently shown [27] that this expected sum equals $n + o(n)$; here we sharpen this asymptotic, showing that:

$$\sum_{j=0}^{n} \mathbb{E}b_j(X) = n + \frac{2}{\sqrt{\pi}} n^{1/2} + O(n^c) \quad \text{for any } c > 0. \tag{1}$$

(the term $\frac{2}{\sqrt{\pi}} n^{1/2}$ comes from contributions of *middle* Betti numbers).

The proofs are based on a combination of techniques from random matrix theory and spectral sequences. In particular (1) is based on a reduction that requires an average count of the number of singular quadrics in a random pencil; this count turns out to be related to the derivative at zero of the gap probability $f_{\beta,n}$ in finite Gaussian β-ensembles ($\beta = 1, 2, 4$). We provide also new computations for this quantity and as n goes to infinity:

$$f'_{\beta,n}(0) \sim -\frac{2\sqrt{2}}{\pi} n^{1/2}.$$

1. Introduction

Random algebraic geometry. Quadratic equations are "universal" in algebraic geometry; every algebraic set (real or complex) can be described using quadratic equations, while possibly increasing the number of equations and of variables.

In contrast with complex algebraic geometry, which draws much of its power from the availability of *generic statements*, real algebraic geometry is by necessity more algorithmic. For instance, a basic problem that is difficult to study even case by case concerns the topology of an intersection of quadrics. In search of asymptotic results while being motivated by problems with many degrees of freedom (typical in applications), it is natural to fix the number of equations and let the number of variables increase.

A. Barvinok [5] considered this asymptotic while investigating the homological complexity[1] $b(X)$ of an intersection X of k quadrics in $\mathbb{R}\mathrm{P}^n$. Barvinok showed that $b(X)$ is bounded by a polynomial of degree $O(k)$ in n. The same problem was revisited in subsequent studies [3, 6, 10, 29].

In this paper, following [27], we take a probabilistic approach and consider the average Betti numbers of an intersection of random quadrics. Drawing on random matrix theory, we will see

[1]Here for a topological space X the number $b(X)$ denotes the sum of its Betti numbers.

that rather precise information can be extracted from the tables of spectral sequences, developed in [3, 29], when they are studied on average.

To state our main problem more precisely, let us denote by $X \subset \mathbb{R}\mathrm{P}^{n-1}$ the common zero locus of k quadratic forms:

$$X = \{[x] \in \mathbb{R}\mathrm{P}^{n-1} \,|\, q_1(x) = \cdots = q_k(x) = 0\}.$$

Here we choose the defining polynomials $q_1, \ldots, q_k$ to be independent random quadratic polynomials from the Kostlan ensemble (see [9, 12]). Equivalently, the corresponding symmetric matrices[2] $Q_1, \ldots, Q_k$ are sampled independently from the *Gaussian orthogonal ensemble* (the $\mathrm{GOE}(n)$ ensemble). The probability distribution on $\mathrm{Sym}(n, \mathbb{R})$ giving rise to this ensemble is obtained by defining for every open set $U \subset \mathrm{Sym}(n, \mathbb{R})$:

$$\text{probability that } Q \text{ belongs to } U = \frac{1}{2^{n/2}\pi^{n(n+1)/4}} \int_U e^{-\frac{1}{2}\mathrm{tr}(Q^2)} dQ$$

(here dQ stands for the Lebesgue element on the space of symmetric matrices).

Using the results from [9], one can compute the average Euler characteristic of X. Assuming $\dim(X)$ is even:

$$\mathbb{E}\chi(X) = a_0 + a_2 + \cdots + a_{\dim(X)},$$

where the a_{2j} are the coefficients of the power series $\sum_{j\geq 0} a_{2j} t^{2j} = (\frac{2}{1+t^2})^{k/2}$. This result is based on metric (as opposed to topological) properties of X and gives limited information on individual Betti numbers $b_i(X)$. The question we are interested in is:

(2) "What is the expected value of $b_i(X)$?"

In addition to computing the expectation of a single Betti number $b_i(X)$, we are also interested in understanding how they distribute in the range $i = 0, \ldots, \dim(X)$.

The deterministic part of this problem has been studied by the first author and A. A. Agrachev in [3] and by the first author in [28, 29]. The main ingredient is the use of *spectral sequences*, a powerful machinery from algebraic topology. The advantage of this technique is that for the case of intersection of *few* quadrics (compared to the number of variables) it gives very accurate approximations to the topology of X.

Addressing question (2), a probabilistic treatment of this spectral sequence was given by the first author in [27], where the topology of the intersection X of two independent random (Kostlan distributed) quadrics in $\mathbb{R}\mathrm{P}^{n-1}$ was studied. The author proved that as n goes to infinity:

$$\mathbb{E}b(X) = n + o(n) \tag{3}$$

where $b(X) = \sum_{i\geq 0} b_i(X; \mathbb{Z}_2)$ denotes the sum of all Betti numbers of X. Here we sharpen this result to two orders of precision (see Theorem 1 below).

More generally, we take an intersection of k quadrics and ask for the expectation of the Betti number $b_i(X)$; even allowing the index i to depend on n, as long as it stays sufficiently bounded away from $n/2$, the expectation converges very fast to 1.

Theorem (Intersection of k quadrics.)**.** *For every $k, i \in \mathbb{N}$, every $M > 0$ and every open set $J \subset [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ we have:*

$$\mathbb{E}b_i(X) = 1 + O(n^{-M}) \quad \textit{and} \quad \sum_{j \in nJ} \mathbb{E}b_i(X) = |\mathbb{N} \cap nJ| + O(n^{-M}).$$

[2]Fixing a scalar product on $\mathbb{R}^n$ we can associate to each quadratic form q a symmetric matrix Q by setting $q(x) = \langle x, Qx \rangle$ for all $x \in \mathbb{R}^n$.

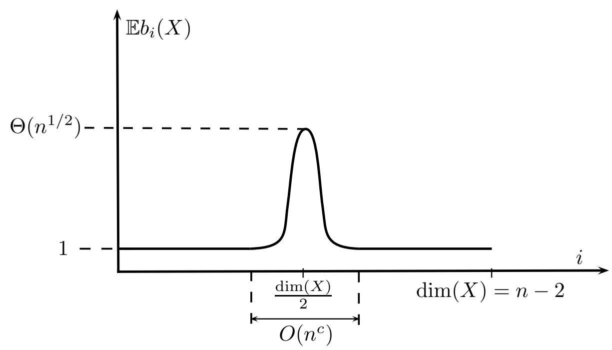

FIGURE 1. The limit distribution of the Betti numbers for the intersection of two random quadrics. The area below the depicted function (the sum of all Betti numbers) equals $n + \frac{2}{\sqrt{\pi}}n^{1/2} + O(n^c)$.

Thus, if we fix the complexity class (intersection of k quadrics) and the dimension of the homology, then as $n \to \infty$ the "pointwise" limit of the function $i \mapsto \mathbb{E}b_i(X)$ is one. In fact we prove that in the above theorem the convergence is uniform for Betti numbers b_i with $i = i(n)$ bounded away from $\frac{n}{2}$ by at least n^α, $0 < \alpha < 1$.

The proof of this result is based on a concentration of measure phenomenon from random matrix theory that forces the spectral sequence associated to X to have a limiting shape (with high probability) already at its second page. This leaves no room for higher differentials[3] and with high probability guarantees the existence of *exactly one* homology class of dimension i in X.

In the special case of $k = 2$ quadrics, the spectral sequence argument can be pushed further to study the distribution of *all* Betti numbers in the range $i = 0, \ldots, n$ (see Figure 1), and middle Betti numbers are responsible for the interesting contribution of $2n^{1/2}/\sqrt{\pi}$ appearing in the following result.

In general, we conjecture that for an intersection of k quadrics the middle Betti numbers give a contribution that is asymptotic to $c_k n^{(k-1)/2}$, for some constant c_k.

Theorem (Intersection of two quadrics)**.** *If X is the intersection of two random, independent Kostlan quadrics in $\mathbb{R}\mathrm{P}^n$:*

$$\mathbb{E}b(X) = n + \frac{2}{\sqrt{\pi}}n^{1/2} + O(n^c) \quad \text{for any } c > 0. \tag{4}$$

Let us give a brief explanation of the new term appearing in the above sum. A random application of the spectral sequence argument requires an average count of the number of singular lines in the span of the two quadrics defining X; using the kinematic formula from integral geometry, this average count can be reduced to the computation of the intrinsic volume of the set Σ of singular symmetric matrices of Frobenius norm one. This volume can be studied using tubes: it equals (one half) the derivative at zero of the volume of an ϵ-tube around Σ. Using an extension of Eckart-Young Theorem (Theorem 9 below) we can describe the ϵ-tube in terms of the eigenvalues: it consists of all symmetric matrices with Frobenius norm one and at least one eigenvalue smaller than ϵ in modulus. Ultimately, this leads to the appearance of the derivative at zero of the gap probability $f_{\beta,n}$ for the β-ensemble of Gaussian matrices (see below, the GOE

[3] This terminology will be explained in detail below.

case corresponds to $\beta = 1$):

$$f_{1,n}(\epsilon) = \text{probability that a GOE}(n) \text{ matrix has no eigenvalues in the interval } (-\epsilon, \epsilon).$$

Proposition (An asymptotic)**.** *The following asymptotic holds for the derivative at zero of the gap probability:*

$$f'_{\beta,n}(0) \sim -\frac{2\sqrt{2}}{\pi} n^{1/2}.$$

Although we only perform the asymptotic analysis for $\beta = 1, 2, 4$, the exact formula (6) discussed below holds for arbitrary $\beta > 0$. Using this language, one can rewrite (4) as:

$$\mathbb{E}b(X) = n - \sqrt{\frac{\pi}{2}} f'_{1,n}(0) + O(n^c) \quad \text{for any } c > 0.$$

Gap probability at zero. We now turn to the basic problem of evaluating exactly the derivative at zero of the gap probability. We consider the classical *finite* β-ensembles $G_{\beta,n}$ of random matrices (n is the size of the matrix) and the function:

$$f_{\beta,n}(\varepsilon) = \text{probability that a matrix from } G_{\beta,n} \text{ has no eigenvalues in } (-\varepsilon, \varepsilon).$$

Here $G_{1,n} = \text{GOE}(n)$ (the *orthogonal* ensemble), $G_{2,n} = \text{GUE}(n)$ (the *unitary* ensemble) and $G_{4,n} = \text{GSE}(n)$ (the *symplectic* ensemble); these are ensembles of random Hermitian matrices with Gaussian entries (see [16, 31, 37] for more details and properties of statistics of these ensembles). With slight abuse of notation, we will still denote by $G_{\beta,n}$ the Euclidean space of such Hermitian matrices endowed with the *Frobenius* norm; in particular $G_{1,n}$ is the set of real symmetric, $G_{2,n}$ the set of Hermitian and $G_{4,n}$ the set of quaternionic Hermitian matrices. We denote by $N_\beta = n + \frac{1}{2}n(n-1)\beta$ the dimension of $G_{\beta,n}$ as a real vector space. The asymptotics of a rescaled version of $f_{\beta,n}$, tracing its behavior close to zero, but letting n go to infinity first is well studied (see [4, Ch. 3] and the references therein).

In our setting, we are actually interested in the gap probability for fixed n (in particular its derivative at zero). For *finite* ensembles the study goes back to the pioneering work of M. Gaudin [18] and later M. Jimbo, T. Miwa, Y. Môri and M. Sato [26]. Forrester and Witte [15], drawing on [26], have evaluated $f_{\beta,n}$ ($\beta = 1, 2$) using methods from integrable systems. For example, if $\beta = 1$ and n is even:[4]

$$f_{1,n}(\varepsilon) = \tau_{\sigma_V}(\varepsilon^2),$$

where τ_{σ_V} is a function satisfying:

$$\sigma_V(t) = t\frac{d}{dt}\log \tau_{\sigma_V}(t) \quad \text{and} \quad \lim_{t\to 0^+} \sigma_V(t) t^{-1/2} = -\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} = -c_n.$$

Using the two above relations, one can evaluate the derivative at zero of $f_{1,n}$ under the assumption that n is even (see [27]):

$$f'_{1,n}(0) = -2c_n.$$

In order to study the more general case $\beta > 0$ and arbitrary parity of n, we use the *joint density* of the eigenvalues $\lambda_1, \ldots, \lambda_n$ for $Q \in G_{\beta,n}$. We will denote such joint density by $F_{\beta,n}$; its explicit expression is given in [31]:

$$F_{\beta,n}(\lambda) = C_\beta(n) \exp\left(-\frac{\beta}{2}\sum_{j=1}^n \lambda_j^2\right) \prod_{1\le j<k\le n} |\lambda_k - \lambda_j|^\beta, \tag{5}$$

[4]Here we use the same notation as in [15] to help the reader comparing with this reference. The subscript of σ_V is due to the connection with the Painlevé fifth equation.

where $C_\beta(n)$ is a normalization constant (its explicit value is written in equation (15) below). In particular one can write $f_{\beta,n}(\varepsilon)$ as an integral of the joint density for the eigenvalues over the region $((-\infty,-\varepsilon)\cup(\varepsilon,\infty))^n$; from this we derive the following exact formula. Note that the following theorem holds for arbitrary $\beta>0$, but in our applications we are interested in $\beta=1,2,4$; for these three cases we develop more explicit formulas (Lemmas 14, 15 and 16) and the asymptotic stated in Corollary 18.

Theorem (The derivative of the gap probability at zero)**.**

$$f'_{\beta,n}(0) = -2n\frac{C_\beta(n)}{C_\beta(n-1)}\mathbb{E}_{Q\in G_{\beta,n-1}}\left\{|\det(Q)|^\beta\right\} \tag{6}$$

In fact all the quantities in the above equation are classically known (the expectation of the modulus of the determinant is computed in [31] using the Mellin transform; notice that it is for an ensemble of dimension one less than the original one). The asymptotic analysis of these quantities, as needed for Proposition 1, is still delicate and is part of the results of this paper.

Structure of the paper. We prove the result on an intersection of k quadrics in Sections 2 where we first recall what we need from Algebraic Topology, giving a short account of spectral sequences. The case of two quadrics is proved in Section 3. Supporting results in linear algebra and random matrix theory are proved aftwerward in Section 4 and the Appendix. In Section 4.1 we prove the generalization of the Eckart-Young Theorem for our set of β-Hermitian matrices. Section 4.2 is devoted to the explicit computation of the derivative of the gap probability at zero (Theorem 10); we apply this in Section 4.3 to study the volume of $\Sigma_{\beta,n}$ (Theorem 11). We provide asymptotic versions of Theorems 11 and 10 (respectively Corollaries 17 and 18) in the Appendix.

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1.1. **Notes.** P. Bürgisser [9] and S. S. Podkorytov [35] computed the expectation of the Euler characteristic of random algebraic sets defined by independent, centered random polynomials whose distribution is invariant by an orthogonal change of variables. Here we briefly review recent results on individual Betti numbers of random algebraic sets.

The first result in this direction was made by F. L. Nazarov and M. L. Sodin: in [33] the authors prove that a random spherical harmonic of degree d on S^2 has on average $c\cdot d^2$ nodal domains. Extending their technique, the current authors (motivated by P. Sarnak's letter [36]) studied the expectation of the number of connected components b_0 of a random hypersurface Y of degree d in $\mathbb{R}\mathrm{P}^n$ (here a homogeneous polynomial of degree d in $n+1$ is sampled at random uniformly from the unit sphere in the $L^2_{S^n}$-norm). In [30] they proved that there exist constants $C_n, c_n>0$ such that for large d:

$$c_n d^n \leq \mathbb{E}b_0(Y) \leq C_n d^n.$$

The novel techniques introduced in [33] can be extended to show that in fact $\mathbb{E}b_0(Y)/d^n$ has a limit as d goes to infinity. However the method is not explicit and yields little more than the non-vanishing of this limit. In the case $n=2$, where $\mathbb{E}b_0(Y)$ is asymptotic to $c\cdot d^2$ for $c>0$, M. Nastasescu [36, 32] computed this Nazarov-Sodin constant c numerically and found that it is approximately 0.0195.

In a sequence of papers [19, 20, 21] D. Gayet and J-Y. Welschinger proved that for a Kostlan hypersurface Y of degree d in $\mathbb{R}\mathrm{P}^n$ (i.e. whose defining polynomial has a distribution which is invariant by *unitary* change of coordinates), for every Betti number b_i there are two constants $a_{i,n}, A_{i,n} > 0$ such that:

$$a_{i,n} d^{n/2} \leq \mathbb{E} b_i(Y) \leq A_{i,n} d^{n/2}$$

Again, the problem of determining the sharp constants is far from trivial: the reason is that for the large degree limit no technique is available for exact computations (the upper bound is obtained using Morse inequalities and the lower bound using the barrier method from [30, 33]: neither of these methods produces equalities).

With Y. Fyodorov, the current authors in [17] studied the connected components of a more general family of orthogonally invariant ensembles and provided estimates on the constants.

1.2. **Related problems.** We discuss in this section some interesting related problems.

To start with, the theorem on the intersection of k quadrics answers question (2) above only for Betti numbers away from the middle ones. What is the behavior of these middle Betti numbers remains an open problem (for $k > 2$). Here we motivate the above conjecture that their contribution is of the order $n^{(k-1)/2}$ as $n \to \infty$.

Let us denote by $\Sigma_{\beta,n}$ the set of norm-one, singular matrices in the β-ensemble. For a generic choice of a k-dimensional space ($k \leq 3$) $\Sigma_{1,n} \cap W$ is smooth, but for larger k singularities are unavoidable. Let us set $\Sigma_W^{(1)} = \Sigma_{1,n} \cap W$ and denote by $\Sigma_W^{(r)}$ the set of singular points of $\Sigma_W^{(r-1)}$. In [29] the first author proves that for a generic choice of $W = \mathrm{span}\{q_1, \ldots, q_k\}$ the following inequality holds between the Betti numbers of $X = \{q_1 = \cdots = q_k = 0\}$ and those of $\Sigma_W^{(r)}$:

$$b(X) - n \leq \frac{1}{2} \sum_{r \geq 1} b(\Sigma_W^{(r)}).$$

Thus, the homological complexity of X is bounded in terms of the sum of the complexities of the $\Sigma_W^{(r)}$. For example in the case $k = 2$ we see that $b(X) - n$ is bounded by the number of singular lines in $W \cap \Sigma_{1,n}$ (actually, in this case the above formula is almost exact and is key for the theorem on the intersection of two random quadrics). Since they are defined by a homogeneous equation of degree n, the maximum number of singular lines is at most n; on the other hand, when we look at the average number we see asymptotically a constant times $n^{1/2}$.

In the case $k = 3$ the situation is fairly more complicated; here we have to compute the average number of components of the random curve:

$$\Sigma_W = \{(\omega_1, \omega_2, \omega_3) \in S^2 \,|\, \det(\omega_1 Q_1 + \omega_2 Q_2 + \omega_3 Q_3) = 0\}, \quad Q_1, Q_2, Q_3 \in \mathrm{GOE}(n).$$

A theorem of V. Vinnikov [39] states that *every*[5] real algebraic curve arises in this way; thus the above construction gives another possible model for random curves (already introduced in [27, 30]): curves arising as intersection of $\Sigma_{1,n}$ with a random three-plane.

Gayet and Welschinger's result on random algebraic manifolds states that fixing n and letting the degree go to infinity one gets on average a homological complexity which is of order the square root of the complete intersection in complex projective space [22]. Here for X we are performing the opposite limit (k and $d = 2$ are fixed and $n \to \infty$) but by analogy we guess that we should get on average the square root of the homological complexity of a complete intersection of k quadrics in complex projective space, which is $O(n^{k-1})$. In particular, because of the theorem on k quadrics, we see that this homological complexity should come from middle Betti numbers, which in turn are bounded by the r.h.s. of the above inequality.

Closely related to this is a random version of a problem studied by J. Adams, P. Lax and R. Phillips. In [1] the authors studied (topological) restrictions on the dimension of a subspace W of $G_{\beta,n}$ missing $\Sigma_{\beta,n}$ (i.e. W hits the set of singular matrices only at the origin). It turns out

[5]Except empty curves of degree n when $n \equiv 2 \bmod 4$.

that the maximal dimension $\rho_\beta(n)$ of such W is related to Radon-Hurwitz numbers (see [1]). In particular if $k \le \rho_\beta(n)$, it is natural to ask the question:

"What is the probability that a random k-dimensional space in $G_{\beta,n}$ misses $\Sigma_{\beta,n}$?"

Notice that in the case n is odd we must hit $\Sigma_{\beta,n}$ at a nonzero point (in fact the determinant is a real polynomial of *odd* degree and must vanish somewhere on W). For this reason we introduce the following event:

$$L_\beta(k,n) = \left\{ W \simeq \mathbb{R}^k \subset G_{\beta,n} \text{ such that } \max_{Q\in W} \mathrm{i}^+(Q) = \left\lfloor \frac{n+1}{2} \right\rfloor \right\}$$

where $\mathrm{i}^+(Q)$ denotes the number of positive eigenvalues of the matrix Q from $G_{\beta,n}$. It is not difficult to show that the probability of missing $\Sigma_{\beta,n}$ for even n equals the probability of $L_\beta(k,n)$. Thus we can ask more generally for the probability of $L_\beta(k,n)$ and its asymptotic.

This problem relates to determinantal curves (and consequently also to the above question on the topology of an intersection of three random quadrics) as follows:

$$\mathbb{P}\{L_1(3,n)\} = \text{probability that a random determinantal curve of degree } n \text{ is empty.}$$

Remark 1. The uniform probability distribution on the Grassmannian of k planes in $G_{\beta,n} \simeq \mathbb{R}^N$ is characterized as the *unique* probability distribution which is invariant under the action of the orthogonal group $O(N)$ (the Euclidean structure is the one given by the Frobenius norm). Sampling $Q_1, \dots, Q_k$ independenlty from the β-ensemble and considering their linear span produces a random k-plane whose probability distribution is invariant by the $O(N)$ action (by construction if $M : G_{\beta,n} \to G_{\beta,n}$ is an orthogonal linear map, then $M(Q_i)$ is distributed as Q_i); thus this k-plane is uniformly distributed in the Grassmannian in the above sense.

2. Betti numbers of intersections of random quadrics

Consider quadratic forms $q_1, \dots, q_k$ on $\mathbb{R}^n$ (i.e. homogeneous polynomials of degree two in n variables). Each q_i defines a (possibly empty) quadric hypersurface in the projective space $\mathbb{R}\mathrm{P}^{n-1}$ and we consider the set X obtained by intersecting together these hypersurfaces:

$$X = \{[x] \in \mathbb{R}\mathrm{P}^{n-1} \,|\, q_1(x) = \cdots = q_k(x) = 0\}.$$

Notice that, once a scalar product on $\mathbb{R}^n$ has been fixed, a quadratic form q determines a unique symmetric matrix Q given by the equation:

$$q(x) = \langle x, Qx \rangle \quad \text{for all } x \in \mathbb{R}^n.$$

In the random setting, it is assumed that the quadratic forms q_i are independent and *Kostlan* distributed (see [27]); this is equivalent to require that:

$$Q_i \in \mathrm{GOE}(n), \quad i = 1, \dots, k.$$

The first result we will prove is the following, which is valid for $i \in \{0, \dots, \dim(X)\}$ sufficiently bounded away from $\dim(X)/2$ (the "middle" Betti numbers).

Theorem 1. *Let X be a random intersection of k quadrics in $\mathbb{R}P^{n-1}$. For every $0 < \alpha < 1$ and $M > 0$, if $|i - \frac{n}{2}| \ge n^\alpha$. then:*

$$\mathbb{E}b_i(X) = 1 + O(n^{-M}).$$

Note that the convergence in the above range will be uniform. In particular we get the following corollary: since the error term in the above statement is of the order $O(n^{-M})$ *for all* $M > 0$, then an accumulation of at most n such error terms has the same property.

Corollary 2. *For every $M>0$ and every open set $J\subset[0,\frac{1}{2})\cup(1/2,1]$ we have:*

$$\sum_{j\in nJ}\mathbb{E}b_i(X)=|\mathbb{N}\cap nJ|+O(n^{-M}).$$

Before proving the theorems, we need a way for computing Betti numbers of X in a *deterministic* setting.

2.1. **Betti numbers of intersections of quadrics: a deterministic approach.** The cohomology of intersections of real quadrics is studied in [2, 3, 28, 29]. The main ingredient is a *cohomology spectral sequence* converging to the homology of X; since we will only need part of this machinery, here we present a simplification of the theory adapted to our needs.

Given the quadratic forms $q_1,\dots,q_k$ and the corresponding symmetric matrices $Q_1,\dots,Q_k$, we consider their span:

$$W=\operatorname{span}\{Q_1,\dots,Q_k\}\subset\operatorname{Sym}(n,\mathbb{R}).$$

For every symmetric matrix Q we denote by $\mathrm{i}^+(Q)$ the number of its positive eigenvalues (usually called the *index* of Q); we set

$$\mu=\max_{Q\in W}\mathrm{i}^+(Q)\quad\text{and}\quad\nu=\min_{Q\in W\setminus\{0\}}\mathrm{i}^+(Q).$$

Notice that for the generic choice of $q_1,\dots,q_k$ the span W is k-dimensional.

In order to study the topology of X we will need to consider also, for every $j\geq 0$ the following set:

$$\Omega^j=\{\omega Q\ \text{ s.t. }\ \omega\in S^{k-1}\ \text{ and }\ i^+(\omega Q)\geq j\}$$

where the notation ωQ simply means $\omega_1Q_1+\cdots+\omega_kQ_k$. By slightly abusing of notation we will simply think of Ω^j as a subset of S^{k-1}.

The Betti numbers[6] of the sets $\Omega^1\supset\Omega^2\supset\cdots\supset\Omega^n$ together with μ are the ingredients we need and we collect them into a table $E=(e_{i,j})$ defined for $i=0,\dots,k$ and $j=0,\dots,n-1$ by:

$$e_{i,j}=b_i(W,\Omega^{j+1}).$$

The zero-th and the k-th columns look like (there are μ zeros in the first column and $n-\nu$ in the second one):

$$e_{0,*}=\begin{array}{|c|}\hline 1\\\hline \vdots\\\hline 1\\\hline 0\\\hline \vdots\\\hline 0\\\hline\end{array}\qquad e_{k,*}=\begin{array}{|c|}\hline 0\\\hline \vdots\\\hline 0\\\hline 1\\\hline \vdots\\\hline 1\\\hline\end{array}$$

For the zero-th column: if $j\geq\mu$ then the set Ω^{j+1} is empty and $e_{0,j}=b_0(W,\Omega^{j+1})=1$; on the other hand if $j\leq\mu-1$ then $\Omega^{j+1}\neq\emptyset$ and $e_{0,j}=b_0(W,\Omega^{j+1})=0$. For the k-th column: if $j\leq\nu-1$ then $\Omega^{j+1}=S^{k-1}$ (*every* point ω in S^{k-1} has $\mathrm{i}^+(\omega Q)\geq\nu$); hence $e_{k,j}=b_k(W,S^{k-1})=b_{k-1}(S^{k-1})=1$. On the opposite if $j\geq\mu$ then there is at least a point on the sphere not in Ω^{j+1} (say a point where the index is minimum); in other words Ω^{j+1} is a proper open subset of the sphere S^{k-1} and $e_{k,j}=b_k(W,\Omega^j)=b_{k-1}(\Omega^j)=0$ (proper open subsets of S^{k-1} do not have homology in dimension $k-1$).

[6]Hereafter all homology and cohomology groups will be with $\mathbb{Z}_2$ coefficients.

Thus the table E can be viewed as a partitioned table (blank spots are zeros):

$$E = \begin{array}{|c|c|c|}
\hline
1 & & 0 \\
\vdots & & \vdots \\
1 & & 0 \\
\hline
0 & & 0 \\
\vdots & S & \vdots \\
0 & & 0 \\
\hline
0 & & 1 \\
\vdots & & \vdots \\
0 & & 1 \\
\hline
\end{array}$$

and the table S is the following:

$$S = \begin{array}{|c|c|c|c|}
\hline
b_0(\Omega^\mu) - 1 & b_1(\Omega^\mu) & \cdots & b_{k-2}(\Omega^\mu) \\
\hline
\vdots & \vdots & & \vdots \\
\hline
b_0(\Omega^{\nu+1}) - 1 & b_1(\Omega^{\nu+1}) & \cdots & b_{k-2}(\Omega^{\nu+1}) \\
\hline
\end{array}$$

(the -1 appear only in the first column).

Given the table E, for every $0 \leq i \leq n-1$ we define $b_i(E)$ by the sum of the elements on its $(n-1-i)$-th diagonal[7], i.e.:

$$b_i(E) = e_{0,n-1-i} + e_{1,n-i-2} + \cdots + e_{n-i-2,1} + e_{n-1-i,0}.$$

The following theorem is a reformulation of [3, Thm. A] and [28, Thm. 8].

Theorem 3. *The entries of E are related to the Betti numbers of the common zero locus $X \subset \mathbb{R}P^{n-1}$ of the generic quadrics $q_1, \ldots, q_k$ by:*

$$b_i(X) \leq b_i(E) \quad \forall i \geq 0 \quad \textit{and} \quad \chi(X) = \sum_{j=0}^{n-1} (-1)^j b_j(E)$$

Moreover assume for every nonzero *entry $e_{j,n-1-i-j}$ of the $(n-1-i)$-th diagonal we have:*

$$\sum_{t>j+2} e_{t,n-t-i} = 0 \quad \textit{and} \quad \sum_{t<j-2} e_{t,n-t-i} = 0 \tag{7}$$

then:

$$b_i(X) = b_i(E).$$

In the case $k = 2$ we also have:

$$|b(X) - b(E)| \leq 2. \tag{8}$$

Proof. We sketch the steps to derive the current statement from the cited ones.

First, [3, Thm. A] asserts that there exists a spectral sequence whose E_2 terms coincides with the above table E and that converges to the homology of X. Since this spectral sequence is a (finite in this case) sequence of groups with endomorphisms $(E_2, d_2), (E_3, d_3), \ldots, (E_\infty, d_\infty \equiv 0)$ with $E_\infty = H^*(X)$, and each term is the homology of the predecessor, then the first part of the statement follows (ranks can only decrease and the Euler characteristic is preserved).

Property (7) is a consequence of the fact that the endomorphisms of the spectral sequence have a bigrading and they can only influence the next term of the spectral sequence "locally". Let us explain this with a visual example. Let us pick an element $e_{i,j}$ (which belongs to the

[7]The strange but standard indexing is due to Alexander-Pontryiagin duality.

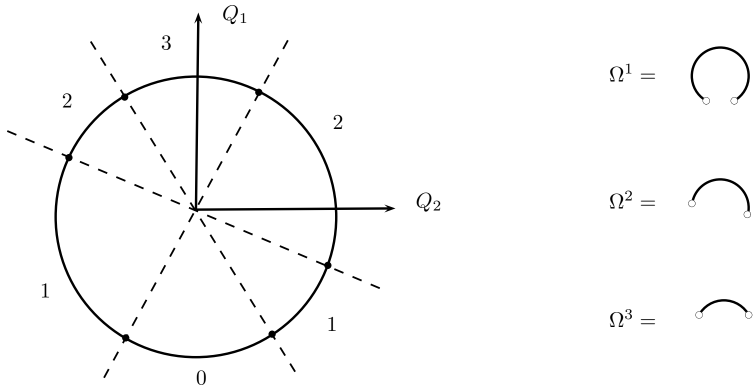


FIGURE 2. The index function for the empty intersection of two quadrics.

$(i+j)$-th diagonal and will give information on the $(n-1-i-j)$-th Betti number) and look at the left side of the diagonal below it and the right side of the one above it:

$e_{i-3,j+2}$						
	$e_{i-2,j+1}$	$*$	$*$			
		$*$	$e_{i,j}$	$*$		
			$*$	$*$	$e_{i+2,j-1}$	
						$e_{i+3,j-2}$

If the sum of all these elements is zero we say that $e_{i,j}$ *survives*. If every nonzero element in a diagonal survives, then the the sum of all of them equals the corresponding Betti number of X. In this example if every nonzero element in the diagonal containing $e_{i,j}$ survives, then $b_{n-1-i-j}(X) = b_{n-1-i-j}(E)$.

Finally, [28, Thm. 8] asserts that in the case $k=2$ one has:

$$b_k(X) = \tilde{e}_{0,n-k} + \tilde{e}_{1,n-k-1} + \tilde{e}_{2,n-k-2}$$

where the elements $\tilde{e}_{i,j}$ come from a table that *equals* E except for $e_{0,\mu}$ and $e_{2,\mu-1}$, but for which in any case $|e_{0,\mu} - \tilde{e}_{0,\mu}| \leq 1$ and $|e_{2,\mu-1} - \tilde{e}_{2,\mu-1}| \leq 1$. □

Example 1 (Empty intersection of two quadrics in $\mathbb{R}\mathrm{P}^2$). Consider the quadratic forms:

$$q_1(x) = x_0^2 + x_1^2 + x_2^2 \quad \text{and} \quad q_2(x) = 2x_0x_1 + 2x_2x_1 + 2x_0x_2 + x_1^2$$

with corresponding symmetric matrices:

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

In this case the index function over the circle $S^1 \subset \mathrm{span}\{Q_1, Q_2\}$ looks as in Figure 2. Consequently $\mu = 3$ and the table E consists of all zeros; this is confirmed by the fact that the common zero locus of q_1 and q_2 is empty (q_1 is positive definite).

2.2. **The expectation of the maximum of the index.** Fix k, and let $W = \mathrm{span}\{q_1, q_2, .., q_k\}$ be the span of k (Kostlan distributed) random quadrics. This translates to a span of random matrices $\{Q_1, Q_2, .., Q_k\}$ in GOE. By homogeneity, we may assume the matrices have been rescaled by $\frac{1}{\sqrt{n}}$. Let μ be as above the maximum of the index function $\mathrm{i}^+(\omega Q)$ over $\omega = (\omega_1, \ldots, \omega_k) \in S^{k-1}$; here and below we use the notation $\omega Q = \omega_1 Q_1 + \cdots + \omega_k Q_k$; notice that if $\omega \in S^{k-1}$ then $\omega Q \in \mathrm{GOE}$.

Lemma 4. *For every $\alpha, M > 0$*

$$\mathbb{P}\left\{\mu \geq \frac{n}{2} + n^{\alpha}\right\} = O(n^{-M}),$$

for every $M > 0$.

Proof of Lemma. Fix $\alpha > 0$, and let E_n denote the event

$$E_n := \left\{\mu \geq \frac{n}{2} + n^{\alpha}\right\},$$

and

$$A_n := \left\{\|Q_i\| \leq 4\sqrt{n}, i = 1, 2, .., k\right\}.$$

Using the bound $\|Q_i\| \leq \sqrt{n}\|Q_i\|_{\mathrm{op}}$ along with the fact that $\mathbb{P}\{\|Q_i\|_{\mathrm{op}} \geq 4\}$ is exponentially small [37][Ch. 2], we have:

$$\mathbb{P}\{E_n\} = \mathbb{P}\{E_n \cap A_n\} + O(c_1^n), \tag{9}$$

for some $0 < c_1 < 1$.

Let $\mathcal{N}_n$ be a maximal ε_n-net of points on the sphere S^{k-1} ($\varepsilon_n > 0$ is specified later), i.e. $\mathcal{N}_n$ is a finite set of points $\omega_i \in S^{k-1}$ that are separated from each other by a distance of at least ε_n, and $\mathcal{N}_n$ is maximal with respect to set inclusion.

Claim 1: Let

$$E_{n,j} := \left\{\mathrm{i}^+(\omega_j Q) \geq \frac{n}{2} + n^{\alpha}/2\right\}.$$

and

$$R_{n,j} := \{\omega_j Q \text{ has } \geq n^{\alpha}/2 \text{ eigenvalues in } (-\varepsilon_n 4\sqrt{nk}, \varepsilon_n 4\sqrt{nk})\}.$$

Then:

$$E_n \cap A_n \subset \cup_{\omega_j \in \mathcal{N}_n}\{E_{n,j} \cup R_{n,j}\}.$$

Claim 2: For each $M > 0$, we have:

$$\mathbb{P}\{E_{n,j}\} = O(n^{-M}).$$

If $\varepsilon_n < n^{-3/2}$ then we also have:

$$\mathbb{P}\{R_{n,j}\} = O(n^{-M}).$$

Choosing, for instance, $\varepsilon_n = 1/n^2$ and applying both Claims we have

$$\mathbb{P}\{E_n \cup A_n\} < \sum_j (\mathbb{P}E_{n,j} + \mathbb{P}R_{n,j}) = O(n^{-M+2k}),$$

since the number of terms in the sum is $|\mathcal{N}_n| = O(1/\varepsilon_n^{k-1}) = O(1/n^{2k-2})$. This proves the result in light of (9).

It remains to prove the Claims.

To see Claim 1, assume the events E_n and A_n occur and note that for some $\omega \in S^{k-1}$ we have $\mu = \mathrm{i}^+(\omega Q)$ and this ω is at most ε_n away from one of the points $\omega_j \in \mathcal{N}_n$. The change in index $|\mathrm{i}^+(\omega Q) - \mathrm{i}^+(\omega_j Q)|$ counts the number of changes in sign of the eigenvalues. If the ith eigenvalue

λ_i changes sign, then $|\lambda_i(\omega Q) - \lambda_i(\omega_j Q)| > |\lambda_i(\omega_j Q)|$. The Weilandt-Hoffman estimate [37][Ch. 1] states:

$$\sum_{i=1}^{n} |\lambda_i(\omega Q) - \lambda_i(\omega_j Q)|^2 < \|\omega Q - \omega_j Q\|^2.$$

In particular, for an eigenvalue that changes sign:

$$|\lambda_i(\omega_j Q)| < |\lambda_i(\omega Q) - \lambda_i(\omega_j Q)| < \|\omega Q - \omega_j Q\| \leq \varepsilon_n 4\sqrt{nk},$$

where the last inequality is implied by the event A_n.

This implies that the change in the index $|\mathrm{i}^+(\omega Q) - \mathrm{i}^+(\omega_j Q)|$ is at most the number of eigenvalues of $\omega_j Q$ with absolute value less than $\varepsilon_n 4\sqrt{nk}$. Thus, either

$$|\mathrm{i}^+(\omega Q) - \mathrm{i}^+(\omega_j Q)| \leq n^\alpha/2,$$

or $\omega_j Q$ has at least $n^\alpha/2$ eigenvalues in $(-\varepsilon_n 4\sqrt{nk}, \varepsilon_n 4\sqrt{nk})$. These two possibilities imply the events $E_{n,j}$ and $R_{n,j}$ respectively, so this proves Claim 1.

In order to prove Claim 2, we state a result providing a uniform convergence estimate for Wigner's semi-circle law. We state just a special case of the result from [14]:

Theorem 5 (L. Erdös, H-T. Yau, J. Yin, 2012)**.** *Let X be a (rescaled) random matrix in GOE(n), and assume $|b_n| < 5$. There exist positive constants $A > 1$, C, c, and $\phi < 1$, such that*

$$\mathbb{P}\left\{|N_X(-\infty, b_n) - n \cdot m(-\infty, b_n)| \geq (\log n)^L\right\} < C \exp\{-c(\log n)^{\phi L}\},$$

where $L = A \log\log n$, $N_X(-\infty, b_n)$ counts the number of eigenvalues of X in the interval $(-\infty, b_n)$, and m is the semi-circle measure.

Let us first apply this to event $E_{n,j}$. We have:

$$\mathbb{P}\{E_{n,j}\} = \mathbb{P}\{|N_X(-\infty, 0) - n/2| \geq n^\alpha/2\} < \mathbb{P}\{|N_X(-\infty, 0) - n/2| \geq (\log n)^L\},$$

for all large enough n. So, $\mathbb{P}\{E_{n,j}\} < C \exp\{-c(\log n)^{\phi L}\}$. For any $M > 0$, eventually (whatever the values of the constants):

$$c(\log n)^{\phi L - 1} > M.$$

Thus,

$$C \exp\{-c(\log n)^{\phi L}\} = O(\exp\{-M(\log n)\}) = O(n^{-M}).$$

Next we apply the same theorem to the event $R_{n,j}$. Let $b_n = \varepsilon_n 4\sqrt{nk}$, and $X = \omega_j Q$ (which is a matrix in GOE). First, we have:

$$R_{n,j} = \{|N_X(-\infty, b_n) - N_X(-\infty, -b_n)| \geq n^\alpha/2\}.$$

By the triangle inequality,

$$|N_X(-\infty, b_n) - N_X(-\infty, -b_n)| \leq |N_X(-\infty, b_n) - m(-\infty, b_n)| + |N_X(-\infty, -b_n) - m(-\infty, -b_n)|.$$

Accordingly, $R_{n,j} \subset \hat{R}_{n,j} \cup \tilde{R}_{n,j}$, where

$$\hat{R}_{n,j} := \{|N_X(-\infty, b_n) - m(-\infty, b_n)| \geq n^\alpha/4\},$$

$$\tilde{R}_{n,j} := \{|N_X(-\infty, -b_n) - m(-\infty, -b_n)| \geq n^\alpha/4\}.$$

Applying Theorem 5 to $\mathbb{P}\{\hat{R}_{n,j}\}$ and $\mathbb{P}\{\tilde{R}_{n,j}\}$, we have:

$$\mathbb{P}\{R_{n,j}\} \leq \mathbb{P}\{\hat{R}_{n,j}\} + \mathbb{P}\{\tilde{R}_{n,j}\} = O(n^{-M}).$$

□

As a corollary we derive the following proposition, which computes the expectation of μ.

Proposition 6. *For every k-dimensional $W \subset \mathrm{GOE}(n)$ as above and every $0 < \alpha < 1$ we have:*

$$\mathbb{E}\mu = \frac{n}{2} + O(n^{\alpha})$$

Proof. From one hand we have $\mu \geq n/2$, which gives $\mathbb{E}\mu \geq n/2$. On the other hand:

$$\begin{aligned}\mathbb{E}\mu &= \sum_{j \geq 0} \left(\frac{n+j}{2}\right) \mathbb{P}\left\{\mu = \frac{n+j}{2}\right\} \\ &\leq \left(\frac{n}{2} + n^{\alpha}\right) \mathbb{P}\left\{\mu \leq \frac{n}{2} + n^{\alpha}\right\} + n\mathbb{P}\left\{\mu > \frac{n}{2} + n^{\alpha}\right\} \\ &\leq \frac{n}{2} + n^{\alpha} + O(n^{1-M})\end{aligned}$$

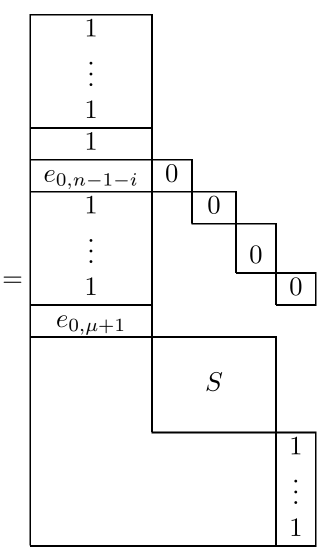
where in the last inequality we have used the assertion of Lemma 4 (if $(n+j)/2$ is not an integer, then the corresponding event is empty). □

2.3. **Proof of Theorem 1.** Fix k and let $i = i(n)$ be as in the statement. Then as n goes to infinity the table E has a fixed number of columns $(k+1)$ and the number of rows is increasing; moreover Betti numbers of X can be studied (deterministically) using Theorem 3.

Let us start by proving the following property:

$$b_i(X) = 1 \quad \text{for all } i < n - \mu - k - 2 \tag{10}$$

(in fact for the generic X, because of Poincaré duality the same will hold for $i > \mu + 2$). Let us consider the table E and focus on the diagonal above the one containing $e_{0,n-1-i}$:

$E =$ 

Since $n - \mu - i > k + 2$, then this diagonal consists of *all* zeros (except $e_{0,n-i}$): in fact since the number of ones below $e_{0,n-1-i}$ is bigger than the number of columns of S, the diagonal $(e_{0,n-i}, e_{1,n-i-1}, \ldots, e_{n-i,0})$ does not hit S, nor the ones in the last columns. This implies:

$$\sum_{t>2} e_{t,n-t-i} = 0 \quad \text{and} \quad \sum_{t<-2} e_{t,n-t-2} = 0$$

(the second condition is automatically satisfied because for negative indices i, j the numbers $e_{i,j}$ are zeros). In particular for such an i the second part of Theorem 3 implies:

$$b_i(X) = b_i(E) = 1.$$

Let now $0<\alpha<1$ and $i=i(n)\leq \lfloor \frac{n}{2}-n^\alpha\rfloor$; recall that by Lemma 4 we have:

$$\mathbb{P}\left\{\mu\leq \frac{n}{2}+n^\alpha\right\}\geq 1-O(n^{-M})\quad \text{for all } M>0.$$

Interpreting b_i as a nonnegative function on $\mathrm{Sym}(n,\mathbb{R})^k$ endowed with the probability distribution $d\gamma_k$ arising from $\mathrm{GOE}(n)^k$, this enables to compute for $\alpha'<\alpha$:

$$\begin{aligned}\mathbb{E}b_i &= \int_{\mathrm{Sym}(n,\mathbb{R})^k} b_i d\gamma_k = \int_{\{\mu>\frac{n}{2}+n^{\alpha'}\}} b_i d\gamma_k + \int_{\{\mu\leq\frac{n}{2}+n^{\alpha'}\}} b_i d\gamma_k \\ &\geq \int_{\{\mu\leq\frac{n}{2}+n^{\alpha'}\}} d\gamma_k \geq 1+O(n^{-M'}).\end{aligned}$$

In the previous chain of inequalities we have used the fact that $\mu\leq\frac{n}{2}+n^{\alpha'}$ combined with $i=i(n)\leq\frac{n}{2}-n^\alpha$ gives (for large enough n):

$$n-\mu-k-2\geq n-\frac{n}{2}-n^{\alpha'}-k-2=\frac{n}{2}-n^{\alpha'}-k-2\geq\frac{n}{2}-n^\alpha\geq i(n)$$

which in turn implies, because of (10), that for each $j\leq i(n)$ on $\{\mu\leq\frac{n}{2}+n^{\alpha'}\}$ one has $b_j\equiv 1$.

This proves that for every $0<\alpha<1$ and $M>0$ if $i=i(n)\leq\lfloor\frac{n}{2}-n^\alpha\rfloor$ we have $\mathbb{E}b_i(X)\geq 1+O(n^{-M})$.

For the opposite inequality we need to know the following uniform bound from real algebraic geometry [29] on the *sum* of the Betti numbers of an intersection X of k quadrics in $\mathbb{R}\mathrm{P}^n$:

$$b_i(X)\leq b(X)\leq O(n^{k-1}). \tag{11}$$

Reasoning as above we obtain:

$$\begin{aligned}\mathbb{E}b_i &= \int_{\mathrm{Sym}(n,\mathbb{R})^k} b_i d\gamma_k = \int_{\{\mu>\frac{n}{2}+n^{\alpha}\}} b_i d\gamma_k + \int_{\{\mu\leq\frac{n}{2}+n^{\alpha}\}} b_i d\gamma_k \\ &\leq O(n)^{k-1}\mathbb{P}\left\{\mu>\frac{n}{2}+n^\alpha\right\}+\mathbb{P}\left\{\mu\leq\frac{n}{2}+n^\alpha\right\} \\ &\leq O(n^{-M+k-1})+1\quad\text{for all } M>0.\end{aligned}$$

In particular we have proved that for every $0<\alpha<1$ and $M>0$ if we take $i<\lfloor\frac{n}{2}-n^\alpha\rfloor$ we have:

$$\mathbb{E}b_i(X)=1+O(n^{-M}).$$

Poincaré duality implies that the same holds true for $i>\lfloor\frac{n}{2}+n^\alpha\rfloor$.

3. The case of two quadrics

3.1. **Deterministic result for the case of two quadrics.** Continuing the discussion of Section 2.1, let us denote by $b(S)$ the sum of all the numbers in the table S (and similarly by $b(E)$ the sum of all the numbers in the table E). Let us also define the set of singular, norm-one matrices:

$$\Sigma_{1,n}=\{Q\in\mathrm{Sym}(n,\mathbb{R})\,|\,\det(Q)=0\quad\text{and}\quad\|Q\|_F^2=1\}.$$

The following properties hold for S.

Proposition 7. *For the generic choice of $q_1,\dots q_k$ we have $\nu=n-\mu$; moreover if $k=2$*

$$b(S)=n-2\mu+\frac{1}{2}\mathrm{Card}(W\cap\Sigma_{1,n})$$

Proof. For the generic choice of $q_1, \dots, q_k$ the set $\Sigma_{1,n} \cap W$ is a proper algebraic set (it is by definition the set of points in $S^{k-1} \subset W$ where the determinant is zero) and each set Ω^j is an open set; thus the minimum and the maximum of the index are attained at points where the determinant doesn't vanish; call these points Q_M and Q_m respectively. Then, since $\det(Q_M) \neq 0 \neq \det(Q_m)$, we have:

$$n - \mu = n - \mathrm{i}^+(Q_M) \leq n - \mathrm{i}^+(-Q_m) = \nu \leq \mathrm{i}^+(-Q_M) = n - \mu.$$

Let us move to the second part of the statement. In the case $k = 2$, for the generic choice of q_1, q_2 we have $\nu = n - \mu$ and the table S is given by one single column

$$S = \begin{array}{|c|} \hline b_0(\Omega^\mu) - 1 \\ \hline \vdots \\ \hline b_0(\Omega^{n-\mu+1}) - 1 \\ \hline \end{array}$$

Each set Ω^j is a disjoint union of open intervals of S^1 (it is a proper subset of the circle) and $b_0(\Omega^j) = \frac{1}{2} b_0(\partial\Omega^j)$. In particular:

$$b(S) = n - 2\mu + \frac{1}{2} \sum_{j=n-\mu+1}^{\mu} b_0(\partial\Omega^j).$$

Consider now the sum $\sum_{j=n-\mu+1}^{\mu} b_0(\partial\Omega^j)$: for the generic choice of q_1, q_2, it equals the number of points ω on the circle S^1 where $\mathrm{i}^+(\omega Q)$ changes its value. This happens exactly at the points where the determinant vanishes, i.e. at the points in $W \cap \Sigma_{1,n}$. In other words:

$$\bigcup_{j=n-\mu+1}^{\mu} \partial\Omega^j = W \cap \Sigma_{1,n}.$$

□

3.2. **Random intersection of two quadrics.** We move now to the random case and prove the following.

Theorem 8. *For $k = 2$ the following formula holds:*

$$\mathbb{E} b(X) = n + \frac{2}{\sqrt{\pi}} n^{1/2} + O(n^c) \quad \textit{for any } c > 0.$$

Proof. The proof goes along the lines of the proof of Theorem 1 from [27]. In the case $k = 2$ we can use the last part of the statement of Theorem 3 and write:

$$b(X) = b(E) + O(1).$$

In this case the first column of E has $n - \mu$ ones and the last has ν many; but for the generic choice of q_1, q_2 Proposition 7 implies $\nu = n - \mu$; in particular:

$$b(E) = n - \mu + b(S) + n - \mu = 2n - 2\mu + b(S) = 3n - 4\mu + \frac{1}{2}\mathrm{Card}(W \cap \Sigma_{1,n}) \tag{12}$$

(here $W = \mathrm{span}\{q_1, q_2\}$). The expectation of μ is computed in Proposition 6 and provides:

$$\mathbb{E} 4\mu = 2n + O(n^\alpha) \quad \text{for all } \alpha > 0. \tag{13}$$

It remains to compute $\frac{1}{2}\mathbb{E}\mathrm{Card}(W \cap \Sigma_{1,n})$. We start by noticing that by assumption for every $g \in SO(N)$ the random quadratic forms q and gq have the same distribution (here $N =$

$\dim \mathrm{Sym}(n,\mathbb{R})$ and the action is not by change of variable, but directly on the space of the coefficients). Thus we have:

$$\mathbb{E}\mathrm{Card}(W\cap\Sigma_{1,n})=\frac{\int_{SO(N)}\mathbb{E}\mathrm{Card}(gW\cap\Sigma_{1,n})dg}{|SO(N)|}=\frac{\mathbb{E}\int_{SO(N)}\mathrm{Card}(gW\cap\Sigma_{1,n})dg}{|SO(N)|}=\frac{2|\Sigma_{1,n}|}{|S^{N-2}|}.$$

The first equality is because for every $g\in SO(N)$ we have $\mathbb{E}\mathrm{Card}(gW\cap\Sigma_{1,n})=\mathbb{E}\mathrm{Card}(W\cap\Sigma_{1,n})$; the second is just linearity of expectation, and the third one is the integral geometry formula [12] (there is no expected value because the integral is constant).

Thus it remains only to compute the term $\frac{|\Sigma_{1,n}|}{|S^{N-2}|}$ (the intrinisic volume of the set of singular, norm-one symmetric matrices). This is done in the next section in Theorem 11, and the asymptotic we need is provided in Corollary 17 in the Appendix:

$$\frac{1}{2}\mathbb{E}\mathrm{Card}(W\cap\Sigma_{1,n})=\frac{|\Sigma_{1,n}|}{|S^{N-2}|}=\frac{2}{\sqrt{\pi}}n^{1/2}+O(1). \tag{14}$$

Combining now (13) and (14) into the expectation of (12) we get the result.

□

4. Intrinsic volume of the set of singular matrices

In Section 4.2, we provide a formula for the derivative at zero of the gap probability, and in Section 4.3 we use this to compute the intrinsic volume of the set of singular matrices. First we establish in Section 4.1 the key ingredient for relating these two quantities; an adaptation of the Eckart-Young theorem.

4.1. Eckart-Young theorem.

In its classical statement, the theorem of Eckart and Young [11] provides the distance (in the Frobenius norm) between an invertible matrix Q and the set Z of matrices with determinant zero:

$$d(Q,Z)=\|Q^{-1}\|_{\mathrm{op}}^{-1}.$$

We will be interested in computing the above distance in the metric space $G_{\beta,n}$ (the distance is again induced by the Fobenius norm, but a priori it could be bigger than the above one; a statement for the case of real symmetric matrices already appeared in [24]). Notice that if $Q\in G_{\beta,n}$ is invertible then $\|Q^{-1}\|_{\mathrm{op}}^{-1}$ equals the least singular value $\sigma(Q)$ of Q.

Theorem 9. *Let $Q\in G_{\beta,n}$ be invertible and Z be the set of matrices in $G_{\beta,n}$ with determinant zero. Then:*

$$d_{G_{\beta,n}}(Q,Z)=\sigma(Q).$$

Proof. Given $Q\in G_{\beta,n}$ invertible we consider the function:

$$f_Q: X\mapsto \|Q-X\|^2$$

and we look for a minimum on Z. We first prove that one such minimum must have rank $n-1$. In fact let $\hat{X}$ such that:

$$\mathrm{rank}(\hat{X})\leq n-2 \quad \text{and} \quad \|Q-\hat{X}\|^2=\min_{X\in Z}\|Q-X\|^2.$$

For $\beta=1,2,4$ let V_β be respectively $\mathbb{R},\mathbb{C}$ and $\mathbb{H}$; then for every $v\in V_\beta^n$ of norm one and $\varepsilon\in\mathbb{R}$ we have $\mathrm{rank}\left(\hat{X}+\varepsilon v\overline{v}^T\right)\leq n-1$ (adding a rank-one matrix to a rank-two one can increase the rank by at most one) and:

$$\left\|Q-\hat{X}-\varepsilon v\overline{v}^T\right\|^2=\|Q-\hat{X}\|^2-2\varepsilon\left\langle Q-\hat{X},v\overline{v}^T\right\rangle+\varepsilon^2\geq\|Q-\hat{X}\|^2.$$

In particular for every $\varepsilon \in \mathbb{R}$ we have $\varepsilon^2 \geq 2\varepsilon \left\langle Q - \hat{X}, v\overline{v}^T \right\rangle = 2\varepsilon \mathrm{tr}\left((Q-\hat{X})\overline{v}v^T\right)$, which implies $\mathrm{tr}\left((Q-\hat{X})\overline{v}v^T\right) = 0$. Since this holds for every v of norm one, then $Q = \hat{X}$ which is impossible.

Thus we restrict to find minima of f_Q on the *smooth* stratum Z_1 of Z where the corank is one: since the stratum is smooth we can use Lagrange multipliers rule. Over this stratum we have $(\nabla \det)_X = \alpha(X)$ (the adjoint matrix of X) and $(\nabla f_Q)_X = 2(Q - X)$. Thus $X \in Z_1$ is a critical point of f_Q if and only if for some λ we have $\lambda(Q - X) = \alpha(X)$. In particular multiplying both sides of this equation by X and using the identity $\alpha(X)X = \det(X)\mathbb{1}$ we get:

$$X \in \mathrm{Crit}(f_Q|_{Z_1}) \quad \text{implies} \quad QX = X^2.$$

In particular since $\overline{X}^T = X$ we get:

$$QX = X^2 = \left(\overline{X}^T\right)^2 = \overline{QX}^T = \overline{X}^T\overline{Q}^T = XQ$$

which says Q and X can be simultaneously diagonalized by β-unitary operators; since conjugation by such operators is an isometry in $G_{\beta,n}$, then we can assume both Q and X are already diagonal andx satisfy $QX = X^2$; it remains to compute the norm of $Q - X$ to detect minima. If $Q = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $X = (x_1, \ldots, x_n)$, then for every $k = 1, \ldots, n$ we have $\lambda_k x_k = x_k^2$ and $\lambda_k, x_k \in \mathbb{R}$. Thus:

$$\|Q - X\|^2 = \sum_{k=1}^{n} \lambda_k^2 - x_k^2.$$

Now we already know that the matrix X has rank $n-1$ and exactly one of the x_k is zero, say x_s. In particular $\|Q - X\|^2 = x_s^2$ and the minimum of f_Q is attained when $x_s^2 = \sigma(Q)^2$. □

Remark 2. In fact the proof can be adapted to find critical points of f_Q over a stratum of Z where the corank is bigger, say r. Then a corresponding minimum is obtained by setting in the diagonal form of Q the first r singular values to zero. Also note that if Q has multiple eigenvalues we can find several minima.

4.2. **Gap probabilities.** In this section we derive the explicit formula for $f'_{\beta,n}(0) = \lim_{\varepsilon\to 0^+} f'_{\beta,n}(\varepsilon)$.

Theorem 10.

$$f'_{\beta,n}(0) = -2n\frac{C_\beta(n)}{C_\beta(n-1)}\mathbb{E}_{Q\in G_{\beta,n-1}}\left\{|\det(Q)|^\beta\right\}.$$

Proof. We make use of the important and well-known formula [31, Ch. 3] for the joint p.d.f. $F_{\beta,n}(\lambda)$ of the eigenvalues $\lambda = (\lambda_1, \lambda_2, .., \lambda_n)$ of a random matrix in the $G_{\beta,n}$ ensemble:

$$F_{\beta,n}(\lambda) = C_\beta(n)\exp\left(-\frac{\beta}{2}\sum_{j=1}^{n}\lambda_j^2\right)\prod_{j,k\in[1,n]}|\lambda_k - \lambda_j|^{\beta/2},$$

where

$$C_\beta(n) = (2\pi)^{-n/2}\beta^{n(n-1)\beta/4+n/2}\prod_{j=1}^{n}\frac{\Gamma(1+\beta/2)}{\Gamma(1+j\beta/2)}. \tag{15}$$

By definition, $f_{\beta,n}(\varepsilon)$ is an integral of $F_\beta(\lambda)$ over the set where all eigenvalues have absolute value at least ε, i.e.

$$f_{\beta,n}(\varepsilon) = \mathbb{P}\{\sigma(Q) \geq \varepsilon\} = \int_{((-\infty,\varepsilon)\cup(\varepsilon,+\infty))^n} F_{\beta,n}(\lambda)d\lambda, \tag{16}$$

Next differentiate both sides of this equation with respect to ε. Since ε only appears in the limits of integration (and not in the integrand), this simply produces boundary integrals; by symmetry under permutations of the variables, these n integrals are equal, so we just multiply the last one by n (here we have used $\hat{\lambda}$ to denote the first $n-1$ entries in λ):

$$f'_{\beta,n}(\varepsilon) = -n \int_{((-\infty,-\varepsilon)\cup(\varepsilon,+\infty))^{n-1}} \left(F_{\beta,n}(\hat{\lambda}, -\varepsilon) + F_{\beta,n}(\hat{\lambda}, \varepsilon) \right) d\hat{\lambda}. \tag{17}$$

For example, let us consider the case $n = 2$; we have:

$$\begin{aligned} f_{\beta,2}(\varepsilon) &= \left(\int_{-\infty}^{-\varepsilon}\int_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon}\int_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty}\int_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty}\int_{-\infty}^{-\varepsilon} \right) F_{\beta,2}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \\ &= \left(\int_{-\infty}^{-\varepsilon}\int_{-\infty}^{-\varepsilon} - \int_{-\infty}^{-\varepsilon}\int_{\infty}^{\varepsilon} + \int_{\infty}^{\varepsilon}\int_{\infty}^{\varepsilon} - \int_{\infty}^{\varepsilon}\int_{-\infty}^{-\varepsilon} \right) F_{\beta,2}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \end{aligned}$$

Thus taking derivative the fundamental Theorem of calculus implies:

$$\begin{aligned} f'_{\beta,2}(\varepsilon) = &- \int_{-\infty}^{-\varepsilon} F_{\beta,2}(-\varepsilon, \lambda_2) d\lambda_2 - \int_{-\infty}^{-\varepsilon} F_{\beta,2}(\lambda_1, -\varepsilon) d\lambda_1 + \int_{\infty}^{\varepsilon} F_{\beta,2}(-\varepsilon, \lambda_2) d\lambda_2 - \int_{-\infty}^{-\varepsilon} F_{\beta,2}(\lambda_1, \varepsilon) d\lambda_2 \\ &+ \int_{\infty}^{\varepsilon} F_{\beta,2}(\varepsilon, \lambda_2) d\lambda_2 + \int_{\infty}^{\varepsilon} F_{\beta,2}(\lambda_1, \varepsilon) d\lambda_1 - \int_{-\infty}^{-\varepsilon} F_{\beta,2}(\varepsilon, \lambda_2) \lambda_2 + \int_{\infty}^{\varepsilon} F_{\beta,2}(\lambda_1, -\varepsilon) d\lambda_1 \\ = &- \int_{-\infty}^{-\varepsilon} F_{\beta,2}(-\varepsilon, \lambda_2) d\lambda_2 - \int_{-\infty}^{-\varepsilon} F_{\beta,2}(\lambda_1, -\varepsilon) d\lambda_1 - \int_{\varepsilon}^{\infty} F_{\beta,2}(-\varepsilon, \lambda_2) d\lambda_2 - \int_{-\infty}^{-\varepsilon} F_{\beta,2}(\lambda_1, \varepsilon) d\lambda_2 \\ &- \int_{\varepsilon}^{\infty} F_{\beta,2}(\varepsilon, \lambda_2) d\lambda_2 - \int_{\varepsilon}^{\infty} F_{\beta,2}(\lambda_1, \varepsilon) d\lambda_1 - \int_{-\infty}^{-\varepsilon} F_{\beta,2}(\varepsilon, \lambda_2) \lambda_2 - \int_{\varepsilon}^{\infty} F_{\beta,2}(\lambda_1, -\varepsilon) d\lambda_1 = (*) \end{aligned}$$

Collecting together and using the invariance under permutations of the variables of $F_{\beta,2}$, we can rewrite the above expression as:

$$\begin{aligned} (*) = &\int_{(-\infty,-\varepsilon)\cup(\varepsilon,+\infty)} F_{\beta,2}(-\varepsilon, \lambda_2) + F_{\beta,2}(\varepsilon, \lambda_2) d\lambda_2 - \int_{(-\infty,-\varepsilon)\cup(\varepsilon,+\infty)} F_{\beta,2}(\lambda_1, -\varepsilon) + F_{\beta,2}(\lambda_1, \varepsilon) d\lambda_1 \\ = &-2 \int_{(-\infty,-\varepsilon)\cup(\varepsilon,+\infty)} F_{\beta,2}(-\varepsilon, \lambda_2) + F_{\beta,2}(\varepsilon, \lambda_2) d\lambda_2 \end{aligned}$$

Taking the limit $\varepsilon \to 0$ in the general case (17), the dominated convergence Theorem gives:

$$\lim_{\varepsilon \to 0} f'_{\beta,n}(\varepsilon) = -2n \int_{\mathbb{R}^{n-1}} F_{\beta,n}(\hat{\lambda}, 0) d\hat{\lambda}. \tag{18}$$

The integrand $F_{\beta,n}(\hat{\lambda}, 0)$ can be rearranged in an interesting way that expresses it in terms of the density $F_{\beta,n-1}(\hat{\lambda})$ and the determinant $|\lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1}|$:

$$\begin{aligned} F_{\beta,n}(\hat{\lambda}, 0) &= C_\beta(n) \exp\left(-\frac{\beta}{2} \sum_{j=1}^{n-1} \lambda_j^2 \right) \left\{ \prod_{j,k\in[1,n]} |\lambda_k - \lambda_j|^{\beta/2} \right\}_{\lambda_n = 0} \\ &= C_\beta(n) \exp\left(-\frac{\beta}{2} \sum_{j=1}^{n-1} \lambda_j^2 \right) \left\{ \prod_{j,k\in[1,n-1]} |\lambda_k - \lambda_j|^{\beta/2} \right\} |\lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1}|^\beta \\ &= \frac{C_\beta(n)}{C_\beta(n-1)} |\lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1}|^\beta F_{\beta,n-1}(\hat{\lambda}) \end{aligned}$$

Substituting this into (18) finally leads to an expression for $\lim_{\varepsilon\to 0} f'_{\beta,n}(\varepsilon)$ in terms of the Mellin transform:

$$\begin{aligned}\lim_{\varepsilon\to 0} f'_{\beta,n}(\varepsilon) &= -2n\frac{C_\beta(n)}{C_\beta(n-1)}\int_{\mathbb{R}^{n-1}} |\lambda_1\cdot\lambda_2\cdots\lambda_{n-1}|^\beta F_{\beta,n-1}(\hat\lambda)d\hat\lambda \\ &= -2n\frac{C_\beta(n)}{C_\beta(n-1)}\mathbb{E}_{Q\in G_{\beta,n-1}}\left\{|\det(Q)|^\beta\right\}\end{aligned}$$

□

4.3. **The intrinsic volume of $\Sigma_{\beta,n}$.** In this section we compute the intrinsic volume (induced by the Frobenius norm) of the set:

$$\Sigma_{\beta,n} = \{Q\in G_{\beta,n}\ :\|Q\|^2 = 1 \text{ and } \det(Q)=0\}.$$

Let us recall that the Mellin transform $\mathcal{M}^+_{n-1}(\beta,s)$ [31] provides the moments of the determinant of a random matrix from the β-ensemble. Namely,

$$\mathcal{M}^+_n(\beta,s) = \frac{1}{2}\mathbb{E}_{\beta,n}|\det|^{s-1} = \frac{1}{2}\mathbb{E}_{\beta,n}|\lambda_1\cdot\lambda_2\cdots\lambda_n|^{s-1}, \tag{19}$$

where we have used the notation $\mathbb{E}_{\beta,n}$ to emphasize that the expectation is taken using the probability density associated to $G_{\beta,n}$. In terms of this, we state a precise formula for the volume of $\Sigma_{\beta,n}$.

Theorem 11. *For $\beta = 1,2,4$:*

$$|\Sigma_{\beta,n}| = 2n\sqrt{2\pi}\frac{C_\beta(n)}{C_\beta(n-1)}\mathcal{M}^+_{n-1}(\beta,\beta+1)\cdot|S^{N_\beta-2}|,$$

where $N_{\beta,n} = \dim G_{\beta,n} = n+\frac{1}{2}n(n-1)\beta$, and $C_\beta(n)$ is defined in (15).

Remark 3. Plugging in the exact values for the normalization constants we have:

$$\frac{C_\beta(n)}{C_\beta(n-1)} = \frac{\beta^{(n-1)\beta/2+1/2}\Gamma(1+\beta/2)}{\sqrt{2\pi}\Gamma(1+\beta n/2)}$$

which in turn implies:

$$|\Sigma_{\beta,n}| = 2n\beta^{\frac{n\beta-\beta+1}{2}}\frac{\Gamma(1+\beta/2)}{\Gamma(1+\beta n/2)}\mathcal{M}^+_{n-1}(\beta,\beta+1)\cdot|S^{N_\beta-2}|.$$

We prove this using Theorem 9 along with several lemmas. We first reduce the problem to some Random Matrix Theory computations. To avoid cumbersome notation, let us suppress the dependence on β and n for the next two statements.

Proposition 12.

$$|\Sigma| = |S^{N-1}|\cdot\lim_{\varepsilon\to 0}\frac{\mathbb{P}\{\sigma(Q)\le\varepsilon\|Q\|\}}{2\varepsilon}$$

Proof. Since Σ is an algebraic subset of the sphere S^{N-1} of codimension one, then its intrinsic volume is computed by:

$$|\Sigma| = \lim_{\varepsilon\to 0}\frac{|\mathcal{U}_{S^{N-1}}(\Sigma,\varepsilon)|}{2\varepsilon},$$

where $\mathcal{U}_{S^{N-1}}(\Sigma,\varepsilon)$ is an ε-tube aorund Σ in S^{N-1} (i.e. the set of points in S^{N-1} at distance less than ε from Σ).

Consider now the functions $v(\varepsilon)=|\mathcal{U}_{S^{N-1}}(\Sigma,\varepsilon)|$ and $\hat{v}(\varepsilon)=|\mathcal{U}_G(Z,\varepsilon)\cap S^{N-1}|$ (where $\mathcal{U}_G(Z,\varepsilon)$ is an ε-tube of Z in G). We will prove that these functions have the same derivative at zero. First notice that if $d_{S^{N-1}}(X,\Sigma)\leq\varepsilon$ then also $d_G(X,Z)\leq\varepsilon$; hence:

$$\mathcal{U}_{S^{N-1}}(\Sigma,\varepsilon)\subset\mathcal{U}_G(Z,\varepsilon)\cap S^{N-1}. \tag{20}$$

Assume now that $d_M(X,Z)\leq\varepsilon$ for $X\in S^{N-1}$. Then the geodesic joining X to Σ is an "arc" on the sphere and by the triangle inequality $d_{S^{N-1}}(X,\Sigma)\leq d_G(X,Z)+1-\cos(d_G(X,Z))\leq\varepsilon+\varepsilon^2$ (the last inequality for ε small enough). In particular we get the inclusion:

$$\mathcal{U}_G(Z,\varepsilon)\cap S^{N-1}\subset\mathcal{U}_{S^{N-1}}(\Sigma,\varepsilon+\varepsilon^2). \tag{21}$$

Combining (20) and (21) gives $v(\varepsilon)\leq\hat{v}(\varepsilon)\leq v(\varepsilon+\varepsilon^2)$, which in turn implies $\lim_{\varepsilon\to 0}\frac{v(\varepsilon)}{2\varepsilon}=\lim_{\varepsilon\to 0}\frac{\hat{v}(\varepsilon)}{2\varepsilon}$. In particular this implies that $|\Sigma|$ is also computed by:

$$|\Sigma|=\lim_{\varepsilon\to 0}\frac{|\mathcal{U}_G(Z,\varepsilon)\cap S^{N-1}|}{2\varepsilon}. \tag{22}$$

We apply now Theorem 9 getting that:

$$\mathcal{U}_G(Z,\varepsilon)\cap S^{N-1}=\{\sigma(Q)\leq\varepsilon\}\cap S.$$

Since the probability distribution on the ensemble G is uniform on the unit sphere, then the volume of $\{\sigma(Q)\leq\varepsilon\}\cap S$ equals $|S^{N-1}|$ times the probability of the cone generated by $\{\sigma(Q)\leq\varepsilon\}\cap S^{N-1}$; in other words:

$$|\mathcal{U}_G(Z,\varepsilon)\cap S^{N-1}|=|S^{N-1}|\cdot\mathbb{P}\{\sigma(Q)\leq\varepsilon\|Q\|\}.$$

□

In the following, we have the least singular value in mind for the function $\sigma(Q)$, but we state the Lemma in more generality.

Lemma 13. *Fix N and for $Q\in\mathbb{R}^N$ suppose $\sigma(Q)$ is a continuous positive function that is homogeneous of degree one, so $\sigma(Q)=\|Q\|\sigma(Q/\|Q\|)$. Define:*

$$f(\varepsilon)=\mathbb{P}\{\sigma(Q)\geq\varepsilon\}\quad\textit{and}\quad g(\varepsilon)=\mathbb{P}\{\sigma(Q)\geq\varepsilon\|Q\|\}.$$

Then,

$$\lim_{\varepsilon\to 0}-g'(\varepsilon)=\left(\lim_{\varepsilon\to 0}-f'(\varepsilon)\right)\frac{\sqrt{2}\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})}.$$

Proof. First we establish the equation:

$$f(\varepsilon)=\frac{\mathrm{Vol}(S^{N-1})}{(2\pi)^{N/2}}\int_0^\infty g(\varepsilon/r)r^{N-1}e^{-\frac{r^2}{2}}\,dr. \tag{23}$$

Starting from the definition for f, we have:

$$\begin{aligned}
f(\varepsilon)&=\frac{1}{(2\pi)^{N/2}}\int_0^\infty\int_{S^{N-1}}\chi_{\{\sigma(Q)\geq\varepsilon\}}r^{N-1}e^{-\frac{r^2}{2}}\,d\theta dr\\
&=\frac{1}{(2\pi)^{N/2}}\int_0^\infty\underbrace{\int_{S^{N-1}}\chi_{\{\sigma(Q)\geq\varepsilon\}}\,d\theta}_{\mathrm{Vol}(S^{N-1})g(\varepsilon/r)}\,r^{N-1}e^{-\frac{r^2}{2}}\,dr\\
&=\frac{\mathrm{Vol}(S^{N-1})}{(2\pi)^{N/2}}\int_0^\infty g(\varepsilon/r)r^{N-1}e^{-\frac{r^2}{2}}\,dr.
\end{aligned}$$

This proves (23). Differentiating (23), we get:

$$f'(\varepsilon) = \frac{\mathrm{Vol}(S^{N-1})}{(2\pi)^{N/2}} \int_0^\infty g'(\varepsilon/r) r^{N-2} e^{-\frac{r^2}{2}} dr. \tag{24}$$

Next we take the limit $\varepsilon \to 0$ and apply the dominated convergence theorem. Note that for each ε, $g'(\varepsilon/r)$ as a function of r is continuous and has a limit as $\varepsilon \to 0$ and $\varepsilon \to +\infty$. This implies that $|g'(\varepsilon/r)|$ is uniformly bounded by say M. Thus, for all $\varepsilon > 0$, the integrands in (24) are all dominated by the integrable function $Mr^{N-2}e^{-\frac{r^2}{2}}$. This justifies the following:

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_0^\infty g'(\varepsilon/r) r^{N-2} e^{-\frac{r^2}{2}} dr &= \int_0^\infty \lim_{\varepsilon\to 0} g'(\varepsilon/r) r^{N-2} e^{-\frac{r^2}{2}} dr \\ &= \int_0^\infty \lim_{\varepsilon\to 0} g'(\varepsilon) r^{N-2} e^{-\frac{r^2}{2}} dr \\ &= \lim_{\varepsilon\to 0} g'(\varepsilon) \int_0^\infty r^{N-2} e^{-\frac{r^2}{2}} dr.\end{aligned}$$

Applying this while taking the limit $\varepsilon \to 0$ in (24) yields the statement in the Lemma:

$$\begin{aligned}\lim_{\varepsilon\to 0} f'(\varepsilon) &= \Big(\lim_{\varepsilon\to 0} g'(\varepsilon)\Big) \underbrace{\frac{\mathrm{Vol}(S^{N-1})}{(2\pi)^{N/2}} \int_0^\infty r^{N-2} e^{-\frac{r^2}{2}} dr}_{\frac{\Gamma(\frac{N-1}{2})}{\sqrt{2}\Gamma(\frac{N}{2})}} \\ &= \Big(\lim_{\varepsilon\to 0} g'(\varepsilon)\Big) \frac{\Gamma(\frac{N-1}{2})}{\sqrt{2}\Gamma(\frac{N}{2})}.\end{aligned}$$

□

We are finally in the position to give the proof of Theorem 11, which in fact is just given by the following chain of equalities:

$$\begin{aligned}|\Sigma_{\beta,n}| &= |S^{N_\beta-1}| \cdot \lim_{\varepsilon\to 0} \frac{1 - \mathbb{P}\{\sigma(Q) \geq \varepsilon \|Q\|\}}{2\varepsilon} = |S^{N_\beta-1}| \cdot \lim_{\varepsilon\to 0} \frac{1-g(\varepsilon)}{2\varepsilon} \quad \text{(by Proposition 12)} \\ &= |S^{N_\beta-1}| \cdot \lim_{\varepsilon\to 0} \frac{-g'(\varepsilon)}{2} = |S^{N_\beta-1}| \frac{\sqrt{2}\Gamma(\frac{N_\beta}{2})}{\Gamma(\frac{N_\beta-1}{2})} \cdot \frac{1}{2} \Big(\lim_{\varepsilon\to 0} -f'(\varepsilon)\Big) \quad \text{(by Lemma 13)} \\ &= \underbrace{2n|S^{N_\beta-1}| \frac{\sqrt{2}\Gamma(\frac{N_\beta}{2})}{\Gamma(\frac{N_\beta-1}{2})}}_{2n\sqrt{2\pi}|S^{N_\beta-2}|} \frac{C_\beta(n)}{C_\beta(n-1)} \mathcal{M}^+_{n-1}(\beta, \beta+1) \quad \text{(by Theorem 10 and (19))}.\end{aligned}$$

Appendix. Asymptotic analysis of Theorem 11

The following lemma is a combination of Prop. 7 and Cor. 3 from [20] and gives the exact formula for $\mathcal{M}^+_{n-1}(1,2)$ together with its asymptotic behavior.

Lemma 14.

$$\mathcal{M}^+_{n-1}(1,2) = \frac{1}{2} \begin{cases} \frac{2\sqrt{2}}{\pi}\Gamma\left(\frac{n+1}{2}\right) & \textit{for even } n, \\ (-1)^m \frac{(n-1)!}{m! 2^{n-1}} + (-1)^{m-1} \frac{4\sqrt{2}(n-1)!}{\sqrt{\pi} m! 2^{n-1}} \sum_{k=0}^{m-1} (-1)^k \frac{\Gamma(k+3/2)}{k!} & \textit{for odd } n = 2m+1 \end{cases}$$

Moreover as n goes to infinity (regardless its parity):

$$\mathcal{M}^+_{n-1}(1,2) \sim \frac{\sqrt{2}}{\pi}\Gamma\left(\frac{n+1}{2}\right).$$

As for the other two cases we have the following lemmas.

Lemma 15.

$$\mathcal{M}^+_{n-1}(2,3) = \begin{cases} \frac{1}{\pi}\Gamma\left(\frac{n+1}{2}\right)^2 & \textit{for even } n, \\ \frac{n}{2\pi}\Gamma\left(\frac{n}{2}\right)^2 & \textit{for odd } n = 2m+1 \end{cases}$$

Moreover as n goes to infinity (regardless its parity):

$$\mathcal{M}^+_{n-1}(2,3) \sim \frac{1}{\pi}\Gamma\left(\frac{n+1}{2}\right)^2.$$

Proof. We start by recalling the following formula from [31]:

$$\mathcal{M}^+_{n-1}(2,3) = \frac{1}{2}\prod_{j=1}^{n-1}\frac{\Gamma(3/2+\lfloor j/2\rfloor)}{\Gamma(1/2+\lfloor j/2\rfloor)}.$$

Using the identity $\Gamma(z+1) = z\Gamma(z)$ in the above formula with $z = 1/2 + \lfloor j/2 \rfloor$, we can rewrite it as:

$$\begin{aligned}\mathcal{M}^+_{n-1}(2,3) &= \frac{1}{2}\prod_{j=1}^{n-1}\left(\frac{1}{2}+\left\lfloor\frac{j}{2}\right\rfloor\right) \\ &= \frac{1}{2}\left(\prod_{1\leq j\leq n-1,\ j\text{ even}}\frac{j+1}{2}\right)\cdot\left(\prod_{1\leq j\leq n-1,\ j\text{ odd}}\frac{j}{2}\right) \\ &= \frac{1}{2^n}\left(\prod_{2\leq k\leq n,\ k\text{ odd}}k\right)\cdot\left(\prod_{1\leq k\leq n,\ k\text{ odd}}k\right) = \frac{c_n}{2^n}\prod_{1\leq k\leq n,\ k\text{ odd}}k^2,\end{aligned}$$

where $c_n = 1$ for even n and n for odd ones. Thus if $n = 2m$ is even, we have:

$$\begin{aligned}\mathcal{M}^+_{n-1}(2,3) &= \frac{1}{2^n}(2m-1)!!^2 \\ &= \frac{2^{2m-n}}{\pi}\Gamma(m+1/2)^2 = \frac{1}{\pi}\Gamma\left(\frac{n+1}{2}\right)^2,\end{aligned}$$

where in the last line we have used the identity $\Gamma(m+1/2) = \sqrt{\pi}\frac{(2m-1)!!}{2^m}$. In the case $n = 2m+1$ is odd, recalling the value $c_{n\text{ odd}} = n$, have:

$$\begin{aligned}\mathcal{M}^+_{n-1}(2,3) &= n\frac{1}{2^n}(1\cdot 2\cdots(2m-1))^2 = n\frac{\Gamma(m+1/2)^2}{2\pi} \\ &= \frac{n}{2\pi}\Gamma\left(\frac{n}{2}\right)^2.\end{aligned}$$

The asymptotics are a simple application of Stirling's formula. □

Lemma 16.

$$\mathcal{M}^+_{n-1}(4,5) = \frac{4^{-n+1}}{\pi}\Gamma\left(n+\frac{1}{2}\right)^2 2H_n(-1),$$

where H_n is a hypergeometric function such that $2H_n(-1) = 1 + o(1)$ as $n \to \infty$. In particular as n goes to infinity:

$$\mathcal{M}_{n-1}^{+}(4,5) \sim \frac{1}{4^{n-1}\pi}\Gamma\left(n+\frac{1}{2}\right)^2.$$

Proof. We start by recalling equation (26.3.10) from [31]:

$$\begin{aligned}\mathcal{M}_{n-1}^{+}(4,5) &= \frac{1}{2^{2n-1}}\prod_{j=0}^{n-2}\frac{\Gamma(j+5/2)}{\Gamma(j+3/2)}\sum_{k=0}^{n-1}\binom{n-1}{k}(1)_k(3/2)_{n-1-k}\\ &= \frac{1}{2^{2n-1}}\left(\prod_{j=0}^{n-2}\frac{\Gamma(j+5/2)}{\Gamma(j+3/2)}\right)\cdot\frac{2}{\pi}\Gamma\left(n+1/2\right){}_2F_1\left(1,1-n,\frac{1}{2}-n,-1\right).\end{aligned}$$

In the above line ${}_2F_1$ denotes the hypergeometric function; let us set

$$H_n(-1) = {}_2F_1\left(1,1-n,\frac{1}{2}-n,-1\right).$$

With this notation we have:

$$\begin{aligned}\mathcal{M}_{n-1}^{+}(4,5) &= \frac{2H_n(-1)}{\sqrt{\pi}2^{2n-1}}\Gamma(n+1/2)\prod_{j=0}^{n-2}\frac{\Gamma(j+5/2)}{\Gamma(j+3/2)}\\ &= \frac{2H_n(-1)}{\sqrt{\pi}2^{2n-1}}\Gamma(n+1/2)\prod_{j=0}^{n-2}(j+3/2),\end{aligned}$$

where again the last line we have used $\Gamma(z+1) = z\Gamma(z)$ for $z = j+3/2$. Recalling also the identity $\prod_{j=0}^{n-2}(j+3/2) = \frac{2}{\pi}\Gamma(n+1/2)$, we finally get:

$$\mathcal{M}_{n-1}^{+}(4,5) = \frac{2H_n(-1)}{\pi 4^{n-1}}\Gamma\left(n+\frac{1}{2}\right)^2.$$

It remains to prove the limit $2H_n(-1) \to 1$. First we use the Pfaff transformation:

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\cdot{}_2F_1\left(a,b-c;c;\frac{z}{z-1}\right).$$

In our case:

$$2\cdot{}_2F_1\left(1,-n;-n-\frac{1}{2};-1\right) = {}_2F_1\left(1,-\frac{1}{2};-n-\frac{1}{2};\frac{1}{2}\right).$$

Now we use the series definition in terms of the Pockhammer symbol:

$${}_2F_1\left(1,-\frac{1}{2};-n-\frac{1}{2};\frac{1}{2}\right) = \sum_{k=0}^{\infty}\frac{(-1/2)_k}{(-n-1/2)_k}(1/2)^k.$$

We need to show that the right hand side $\to 1$. We have

$$\sum_{k=0}^{\infty}\frac{(-1/2)_k}{(-n-1/2)_k}(1/2)^k = 1 + \frac{1}{4(n+1/2)}\sum_{k=1}^{\infty}\frac{(1/2)_{k-1}}{(-n+1/2)_{k-1}}(1/2)^{k-1}. \tag{25}$$

We use the rough bound:

$$\left|\sum_{k=0}^{\infty}\frac{(1/2)_k}{(-n+1/2)_k}(1/2)^k\right| \le \sum_{k=0}^{n}(1/2)^k + \frac{1}{|(-n+1/2)_n|}\sum_{k=n+1}^{\infty}\frac{(1/2)_k}{(1/2)_{k-n}}(1/2)^k,$$

$$\le 2 + \frac{(1/2)^n}{|(-n+1/2)_n|}\sum_{k=n+1}^{\infty}\frac{k!}{(k-n)!}(1/2)^{k-n}$$

$$= 2 + \frac{n!}{|(-n+1/2)_n|}(1/2)^n\frac{F^{(n)}(1/2)}{n!},$$

where

$$F(z) = \frac{1}{1-z}.$$

We have

$$F^{(n)}(1/2)/n! = O(1)\cdot 2^n.$$

Applying this along with Stirling's approximation:

$$2 + \frac{n!}{|(-n+1/2)_n|}(1/2)^n\frac{F^{(n)}(1/2)}{n!} = 2 + \frac{n!}{|(-n+1/2)_n|}O(1) = o(n).$$

This shows that (25) equals $1 + \frac{1}{4(n+1/2)}o(n) = 1 + o(1)$, as desired. □

As a corollary we get the following asymptotic for the volume of $\Sigma_{\beta,n}$.

Corollary 17. *For each $\beta = 1,2,4$ we have:*

$$\frac{|\Sigma_{\beta,n}|}{|S^{N_\beta-2}|} \sim \frac{2}{\sqrt{\pi}}n^{\frac{1}{2}}.$$

Remark 4. In our main application of this asymptotic (Thm. 8) we will need, for $\beta = 1$, a more precise error bound:

$$\frac{|\Sigma_{1,n}|}{|S^{N_1-2}|} = \frac{2}{\sqrt{\pi}}n^{\frac{1}{2}} + O(1).$$

Proof. Recall from Theorem 11 (and the remark below it) that:

$$\frac{|\Sigma_{\beta,n}|}{|S^{N_\beta-2}|} = 2n\beta^{\frac{n\beta-\beta+1}{2}}\frac{\Gamma(1+\beta/2)}{\Gamma(1+\beta n/2)}\mathcal{M}^+_{n-1}(\beta,\beta+1).$$

The result follows applying Stirling's approximation to the asymptotic for $\mathcal{M}^+_{n-1}(\beta,\beta+1)$ given in Lemma 14, 15, 16.

The error bound stated in the Remark follows immediately from the error in Stirling's approximation for n even. For $n = 2m+1$ odd, reading the proof of Lemma 14 which was given in [20, Cor. 3], one can conclude that:

$$\mathcal{M}^+_{n-1}(1,2) = \frac{(n-1)!}{m!2^{n-1}}\left((-1)^m + \frac{4\sqrt{2}}{\sqrt{\pi}}S_m\right), \tag{26}$$

where

$$S_m = \frac{1}{2}\sum_{j=0}^{m/2-1}\frac{\Gamma(2j+2-1/2)}{\Gamma(2j+2)},$$

for m even, and

$$S_m = \frac{\Gamma(m+1/2)}{\Gamma(m)} - \frac{1}{2}\sum_{j=0}^{(m-1)/2-1}\frac{\Gamma(2j+2-1/2)}{\Gamma(2j+2)},$$

for m odd. Using the asymptotic [34]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}(1+O(1/z)),$$

along with an integral estimate for the sum we have (regardless of the parity of m):

$$S_m = \frac{\sqrt{m}}{2} + O(m^{-1/2}).$$

Applying this to (26) gives:

$$\mathcal{M}^+_{n-1}(1,2) = \frac{(n-1)!}{m!2^{n-1}}\left((-1)^m + \frac{4\sqrt{2}}{\sqrt{\pi}}S_m\right) = \frac{\sqrt{2}}{\pi}\Gamma\left(\frac{n+1}{2}\right)\left(1+O(n^{-1/2})\right).$$

Using Stirling's approximation for

$$\frac{C_1(n)}{C_1(n-1)} = \frac{\Gamma(1+1/2)}{\sqrt{2\pi}\Gamma(1+n/2)},$$

and plugging this into the exact formula gives:

$$\frac{|\Sigma_{1,n}|}{|S^{N_1-2}|} = \frac{2}{\sqrt{\pi}}n^{\frac{1}{2}}(1+O(n^{-1/2})).$$

The asymptotic of Theorem 10 follows again from Lemma 14, Lemma 15 and Lemma 16.

Corollary 18. *The following asymptotic holds for the derivative at zero of the gap probability:*

$$f'_{\beta,n}(0) \sim -\frac{2\sqrt{2}}{\pi}n^{1/2}.$$

□

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