Celestial Mechanics Notes Set 1: Introduction to the N-Body Problem

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Contents

1 Introduction:

These notes are are meant to collect and develop fundamental properties of the N-body problem which we will use again and again in subsequent work, including the equations of motion for the problem, the ten classical integrals of motion, and important invariance properties of the equations. We will develop the kinematic properties of the simplest case of the problem, the case of two bodies. In addition we will discuss numerical integration of the problem, and examine some numerical experiments to get a feel for the great variety of dynamical behavior possible in the N body problem for $N > 2$.

These notes were written during the Fall of 2006 when I took Dr. Caesar Ocampo's graduate Celestial Mechanics I course in the aerospace engineering department at the University of Texas Austin. They have grown out of the homework assignments for that course and out of my attempts to grasp the material.

While I have written these notes for my own benefit, it is my hope that others might enjoy them, or find something useful in them. That being said, if these notes have an intended audience it would be a reader with more background in mathematics than physics, who wants to learn something about the N-body problem from a something of a dynamical systems perspective, namely someone very much like myself.

They intend to be in a conversational style, but when something is proven the style of the demonstration will hopefully be very familiar to math students. There will be many numerical experiments in these note sets. These will be treated as building intuition, and suggesting conjectures, but never as being proofs. There are many numerical methods that can provide rigorous proofs of various kinds of dynamical conjectures, but these will not be explored in this set of notes (though some of them may be mentioned when appropriate). All the numerical methods used here are quite elementary, consisting mostly of variations of Runge-Kutta and methods from linear algebra, and perhaps the qualitative theory of dynamical systems.

2 Elementary Principles and Equations of Motion

The material in this section is available in many sources, and is included here for the sake of completeness. In addition, while most of the following material is fundamental to students of physics and engineering, it may not be so to some students of mathematics (myself included).

The entire study of N-body dynamics is built on the following fundamental observation or principle from physics; A body at point P_1 whose mass is m_1 exerts a force on a body at P_2 whose mass is m_2. This force acts along the line determined by P_1 and P_2, is oriented so as to accelerate P_2 toward P_1, and has magnitude which is proportional to the product of m_1 and m_2 and inverse proportional to the square of the distance between P_1 and P_2.

This fact is know as Newton's Law of Universal Gravitation . No attempt is made to derive it here. We take it as coming from empirical observations or as an axiom. (I understand it can be derived as a limiting case of Einstein's general relativistic equations, but such a derivation would be far beyond the scope of these notes).

Having accepted the principle one wants to write it in mathematical language. Then

$$|\mathbf{F}_{21}| \sim \frac{m_1 m_2}{|\mathbf{r}_{21}|^2}$$

where $\mathbf{F}_{ij}$ is the force on particle at P_i due to the particle at P_j, and $\mathbf{r}_{ij}$ is the displacement vector between points P_i and P_j. This is the vector which points from P_i to P_j and whose magnitude is the distance between P_i and P_j. If P_i and P_j are points in Euclidean space then $\mathbf{r}_{ij} = P_j - P_i$.

This force on P_2 due to P_1 points from P_2 toward P_1. A unit vector in this direction is given by

$$u_{r_{21}} = \frac{\mathbf{r}_{21}}{|\mathbf{r}_{21}|}$$

Let the constant of proportionality be G. Then

$$\mathbf{F}_{21} = G\frac{m_1 m_2}{|\mathbf{r}_{21}|^2} u_{r_{21}} = G\frac{m_1 m_2}{|\mathbf{r}_{21}|^2}\frac{\mathbf{r}_{21}}{|\mathbf{r}_{21}|} = G\frac{m_1 m_2}{|\mathbf{r}_{21}|^3}\mathbf{r}_{21}$$

The units of the constant must be such as to cause the units on the left hand side of the equality to agree with those on the right. The units on the left are force or $(mu)(du)/(tu)^2$ (where (du) is distance units, (mu) is mass units, and (tu) is time units). The units on the right are $(mu)^2/(du)^2$. Then G has units $(du)^3/(mu)(tu)^2$.

The value of the constant of proportionality depends on the units one chooses. For a fixed set of units it can be measured by a simple experiment. Take two masses with $m_1 = m_2 = 1mu$. Hold these masses at rest $1du$ apart and measure the force acting on either of the masses. Then the value of G is equal to the magnitude of the force. From the point of view of mathematics it is convenient to simply choose units so that the numerical value of G is one. This will often be done when there is no reason to do otherwise.

Now we want to write down the equations of motion for N-particles interacting under mutual gravitation. To do this we must make use of two more fundamental principles from physics. First, by Newton's Third Law "to every action there is an equal and opposite reaction", so that

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

and we can write the forces on both particles.

The second physical principle we will need is Newton's First Law. This is often stated as saying that an object in motion tends to move in a straight line at a constant velocity, unless acted on by an outside force, and an object at rest tends to stay at rest. This is perhaps a little too vague. It should be added that this statement holds in an inertial reference frame. Or better yet, that an inertial reference frame is defined to be one where this statement holds. As I understand it, if this later approach is taken, then the real content of Newton's First Law is that "Inertial Reference Frames Exist".

With this in mind we can clarify what it is we are trying to do. Namely; write down the equations of motion for an N-body system *in inertial coordinates.* The important thing for us is that it can be shown that in an inertial reference frame force is a vector. This implies that the total force on a particle can be found by adding the individual forces acting on the particle, and each of these can be computed by considering pairwise combinations as if each pair were in the simple two particle system like the one described above.

An excellent discussion of the logical pitfalls associated with even getting off the ground in the study of classical mechanics (i.e. with Newton's Laws and the associated definitions) is in [JS]. In particular, the authors prove the claim about forces in inertial coordinates. The exposition in the present notes is certainly not as logically sound as that give there, but is only intended to sketch how one arrives at the equations of motion.

To write down these equations we appeal to one final physical principle. Namely, Newton's Second Law of Motion, which in words is that the vector sum of the forces acting on a body is equal to the product of it's mass times it's vector acceleration. Again, this can be taken as the definition of force.

Then consider a system of N-particles interacting only under mutual gravitation. Putting the pieces of the above discussion together gives

$$\begin{aligned}
\mathbf{F}_1 &= \mathbf{F}_{12} + \mathbf{F}_{13} + \cdots + \mathbf{F}_{1N} = m_1\mathbf{a}_1 \\
\mathbf{F}_2 &= \mathbf{F}_{21} + \mathbf{F}_{23} + \cdots + \mathbf{F}_{2N} = m_2\mathbf{a}_2 \\
\vdots &= \vdots \\
\mathbf{F}_i &= \mathbf{F}_{i1} + \mathbf{F}_{i2} + \cdots + \mathbf{F}_{iN} = m_i\mathbf{a}_i \\
\vdots &= \vdots \\
\mathbf{F}_N &= \mathbf{F}_{N1} + \mathbf{F}_{N1} + \cdots + \mathbf{F}_{N(N-1)} = m_N\mathbf{a}_N
\end{aligned}$$

Isolating, say the i^{th} equation, and using both the definition of acceleration, and the formula for $\mathbf{F}_{ij}$ developed above gives

$$m_i\frac{d^2}{dt^2}\mathbf{r}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_i m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij}$$

for $1 \leq i \leq N$. Divide both sides by m_i and you have

$$\frac{d^2}{dt^2}\mathbf{r}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} \tag{1}$$

(we are assuming that the mass of each particle is constant). Eqn (1) holds for $1 \leq i \leq N$ so is a system of N second order autonomous vector differential equations.

It's often useful for both theoretical reasons, and for numerically integrating the system, to rewrite these as a system of first order differential equations. Then define the state vector $\mathbf{x} \in \mathbb{R}^{6N}$ by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{3N} \\ x_{3N+1} \\ \vdots \\ x_{6N} \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{3N} \\ \dot{x}_1 \\ \vdots \\ \dot{x}_{3N} \end{bmatrix} = \begin{bmatrix} r_1^1 \\ \vdots \\ r_N^3 \\ \dot{r}_1^1 \\ \vdots \\ \dot{r}_N^3 \end{bmatrix}$$

where r_i^l is the l^{th} component of the i^{th} vector (so $1 \leq l \leq 3$ and $1 \leq i \leq N$) and let

$$g_i(x_1, \ldots, x_{3N}) = g_i(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \sum_{\substack{j = 1 \\ j \neq i}}^{N} \frac{m_j}{|\mathbf{r}_{ij}|^3} \mathbf{r}_{ij}$$

Then

$$\dot{\mathbf{x}} \quad = \quad \begin{pmatrix} x_{3N+1} \\ \vdots \\ x_{6N} \\ g_1(x_1, \ldots, x_{3N}) \\ \vdots \\ g_N(x_1, \ldots, x_{3N}) \end{pmatrix}$$

is the vector field for the N-Body problem.

3 10 Classical Integrals of Motion

The N-body problem admits ten well know constants, or integrals of motion. These are expressions that are constant along solution trajectories. Then for a given set of initial conditions, one can compute the ten constants, and know that they will be the same at all later times. This can be very helpful when integrating the system; keeping track of the value of the constants is a good indicator of how accurate the numerical solution is. If the current value of any of the constant drifts too far from it's initial value, they the numerical results are no longer valid.

From a dynamical systems point of view, the constants of motion give global information about the solutions of the system. If we fix a value of a constant, we obtain a codimension one sub manifold of the phase space. Then an initial condition with that integral constant must stay in that integral manifold for all time.

These integral manifolds can give us information about the global behavior of trajectories. If the integral manifold has a compact component then we know there are are bounded solutions. If two initial conditions are in different path components of the integral surface, then we know they will be bounded away from each other for all time.

Then, by fixing all ten constants, we obtain ten energy surfaces, and we know that a trajectory whose initial conditions begin with those ten constants must evolve in the intersection of the ten manifolds for all time. This can greatly reduce the degrees of freedom of the problem.

The best possible situation is that we are considering a version of an N-body problem where the intersection of the ten integrals is one dimensional. Then the intersection of the integral manifolds is the actual trajectory of the system. Such a system is said to be completely integrable as we can, at leat implicitly, write down the solution curves.

But even when this does not happen the reduction can greatly simplify the analysis. It can happen that the system at hand is a small perturbation of a completely integrable system, and that some of our understanding of the integrable problem translates into information we can use in the perturbed system.

Having discussed how helpful these quantities can be, we turn to finding them.

3.1 Conservation of Linear Momentum

We define the instantaneous center of or mass for an N body system to be

$$\mathbf{r}_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i(t)$$

where $M = \sum_{i=1}^{N} m_i$. We will prove

Theorem 1

$$\ddot{\mathbf{r}}_{cm} = 0$$

Proof: We compute

$$\mathbf{r}_{cm}(t) = \frac{1}{M}\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i(t) = \frac{1}{M}\sum_{i=1}^{N} m_i \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} \tag{2}$$

We show by induction on N that the right hand side of Eqn 2 is zero.

For the base case, let $N = 2$. Then

$$\frac{1}{M}\sum_{i=1}^{2} m_i \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} = \frac{1}{M}\left(G\frac{m_2}{|\mathbf{r}_{12}|^3}\mathbf{r}_{12} + m_2 G\frac{m_1}{|\mathbf{r}_{21}|^3}\mathbf{r}_{21}\right)$$

$$= \frac{Gm_1m_2}{M}\left(\frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} - \frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|^3}\right)$$

$$= \frac{Gm_1m_2}{M}(0)$$

as

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i = -(\mathbf{r}_i - \mathbf{r}_j) = -\mathbf{r}_{ji}$$

(which is a fact we will use again and again throughout these notes). This establishes the base case.

For the inductive step, assume that the claim is true for some fixed number $K > 2$ of bodies. From this assumption we must show that the claim holds for $K + 1$ bodies.

Explicitly, assume that

$$\ddot{\mathbf{r}}_{cm_k}(t) = \frac{1}{M}\sum_{i=1}^{K} m_i \sum_{\substack{j=1 \\ j\neq i}}^{K} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} = 0$$

Now add one more body to this system and compute

$$\begin{aligned}
\ddot{\mathbf{r}}_{cm_{K+1}}(t) &= \frac{1}{M}\sum_{i=1}^{K+1} m_i \sum_{\substack{j=1 \\ j\neq i}}^{K+1} \frac{Gm_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} \\
&= \frac{1}{M}\sum_{i=1}^{K} m_i \sum_{\substack{j=1 \\ j\neq i}}^{K+1} \frac{Gm_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} \\
&+ \frac{1}{M} m_{K+1} \sum_{\substack{j=1 \\ j\neq K+1}}^{K+1} \frac{Gm_j}{|\mathbf{r}_{(K+1)j}|^3}\mathbf{r}_{(K+1)j}
\end{aligned}$$

We manipulate just the first term;

$$\frac{1}{M}\sum_{i=1}^{K} m_i \sum_{\substack{j=1\\ j\neq i}}^{K+1} \frac{Gm_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} = \frac{1}{M}\sum_{i=1}^{K} m_i \left[\sum_{\substack{j=1\\ j\neq i}}^{K} \frac{Gm_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} + \frac{Gm_{N+1}}{|\mathbf{r}_{i(K+1)}|^3}\mathbf{r}_{i(K+1)} \right]$$

$$\begin{aligned}
&= \frac{1}{M}\sum_{i=1}^{K} m_i \sum_{\substack{j=1\\ j\neq i}}^{K} \frac{Gm_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} + \frac{1}{M}\sum_{i=1}^{K} m_i \frac{Gm_{N+1}}{|\mathbf{r}_{i(K+1)}|^3}\mathbf{r}_{i(K+1)} \\
&= 0 + \frac{1}{M}\sum_{i=1}^{K} m_i \frac{Gm_{N+1}}{|\mathbf{r}_{i(K+1)}|^3}\mathbf{r}_{i(K+1)}
\end{aligned}$$

where the zero term comes from invoking the induction hypotheses. Then plugging this back into the computation where we left off gives

$$\begin{aligned}
\ddot{\mathbf{r}}_{cm_{K+1}}(t) &= \frac{1}{M}\sum_{i=1}^{K} m_i \frac{Gm_{N+1}}{|\mathbf{r}_{i(K+1)}|^3}\mathbf{r}_{i(K+1)} \\
&+ \frac{1}{M} m_{K+1} \sum_{\substack{j=1\\ j\neq K+1}}^{K+1} \frac{Gm_j}{|\mathbf{r}_{(K+1)j}|^3}\mathbf{r}_{(K+1)j} \\
&= \frac{Gm_{K+1}}{M}\sum_{i=1}^{K}\left(\frac{m_i}{|\mathbf{r}_{i(K+1)}|^3}\mathbf{r}_{i(K+1)} + \frac{m_i}{|\mathbf{r}_{(K+1)i}|^3}\mathbf{r}_{(K+1)i} \right) \\
&= \frac{Gm_{K+1}}{M}\sum_{i=1}^{K} \frac{m_i}{|\mathbf{r}_{i(K+1)}|^3}(\mathbf{r}_{i(K+1)} - \mathbf{r}_{i(K+1)}) \\
&= 0
\end{aligned}$$

which was to be shown. Then the claim holds for all $N \geq 2$ by induction.

□

It follows that

$$\dot{\mathbf{r}}_{cm}(t) = \mathbf{v}_{cm}(t) = c_1$$

and

$$\mathbf{r}_{cm}(t) = c_1 t + c_2$$

Evaluating these gives and using the initial conditions gives

$$c_1 = \mathbf{v}_{cm}(0) = \frac{1}{M}\sum_{i=1}^{N}\mathbf{v}_i(0)$$

and

$$\mathbf{v}_{cm}(0) \times 0 + c_2 = \mathbf{r}_{cm}(0) = \frac{1}{M}\sum_{i=1}^{N}\mathbf{r}_i(0)$$

The first integration gives three constants of motion, one for each component. Explicitly;

$$\frac{1}{M}\sum_{i=1}^{N} m_i v_i^1(t) = \frac{1}{M}\sum_{i=1}^{N} m_i v_i^1(0)$$

$$\frac{1}{M}\sum_{i=1}^{N} m_i v_i^2(t) = \frac{1}{M}\sum_{i=1}^{N} m_i v_i^2(0)$$

$$\frac{1}{M}\sum_{i=1}^{N} m_i v_i^3(t) = \frac{1}{M}\sum_{i=1}^{N} m_i v_i^3(0)$$

The center of mass moves, but with constant velocity. From the second integration we had that know

$$\mathbf{r}_{cm}(t) = c_1 t + c_2$$

So the integral of motion is

$$\mathbf{r}_{cm}(t) - c_1 t = c_2$$

But we have evaluated these constants and know that

$$\mathbf{r}_{cm}(t) - \mathbf{v}_{cm}(0)t = \frac{1}{M}\sum_{i=1}^{N}\mathbf{r}_i(0)$$

Then the components of this gives three more constants of motion;

$$\frac{1}{M}\left[\sum_{i=1}^{N} m_i r_i^1(t) - t\sum_{i=1}^{N} m_i v_i^1(0)\right] = \frac{1}{M}\sum_{i=1}^{N} r_i^1(0)$$

$$\frac{1}{M}\left[\sum_{i=1}^{N} m_i r_i^2(t) - t\sum_{i=1}^{N} m_i v_i^2(0)\right] = \frac{1}{M}\sum_{i=1}^{N} r_i^2(0)$$

$$\frac{1}{M}\left[\sum_{i=1}^{N} m_i r_i^3(t) - t\sum_{i=1}^{N} m_i v_i^3(0)\right] = \frac{1}{M}\sum_{i=1}^{N} r_i^3(0)$$

Bringing us to six integrals.

3.2 Conservation of Angular Momentum

Define the angular momentum of an N body system to be

$$h(t) = \sum_{i=1}^{N} h_i(t) = \sum_{i=1}^{N} m_i(\mathbf{r}_i(t) \times \mathbf{v}_i(t)) = \sum_{i=1}^{N} m_i(\mathbf{r}_i(t) \times \dot{\mathbf{r}}_i(t))$$

We will prove

Theorem 2

$$\dot{h}(t) = 0$$

Proof: First we compute a little;

$$\begin{aligned}
\dot{h}(t) &= \frac{d}{dt} \sum_{i=1}^{N} m_i(\mathbf{r}_i(t) \times \dot{\mathbf{r}}_i(t)) \\
&= \sum_{i=1}^{N} m_i[\dot{\mathbf{r}}_i(t) \times \dot{\mathbf{r}}_i(t) + \mathbf{r}_i(t) \times \ddot{\mathbf{r}}_i(t)]
\end{aligned}$$

But since $\mathbf{x} \times \mathbf{x} = 0$ for any vector $\mathbf{x}$ this is

$$\begin{aligned}
\dot{h}(t) &= \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times \ddot{\mathbf{r}}_i(t) \\
&= \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times \frac{g_i(\mathbf{r}_1, \ldots, \mathbf{r}_N)}{m_i} \\
&= \sum_{i=1}^{N} m_i \mathbf{r}_i(t) \times G \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{|\mathbf{r}_{ij}|^3} \mathbf{r}_{ij}(t) \\
&= G \sum_{i=1}^{N} m_i \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{|\mathbf{r}_{ij}|^3} \mathbf{r}_i(t) \times \mathbf{r}_{ij}(t) \\
&= G \sum_{i=1}^{N} m_i \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{|\mathbf{r}_{ij}|^3} \mathbf{r}_i(t) \times (\mathbf{r}_j(t) - \mathbf{r}_i(t)) \\
&= G \sum_{i=1}^{N} m_i \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{|\mathbf{r}_{ij}|^3} \mathbf{r}_i(t) \times \mathbf{r}_j(t)
\end{aligned}$$

Then completion of the proof requires showing that the right hand side above is zero. Again we proceed by induction.

Take as the base case $N = 2$ bodies. In this case

$$
\begin{aligned}
G\sum_{i=1}^{2} m_i \sum_{\substack{j=1\\ j\neq i}}^{2} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_i(t)\times\mathbf{r}_j(t) &= G\frac{m_1 m_2}{|\mathbf{r}_{12}|^3}\mathbf{r}_1(t)\times\mathbf{r}_2(t) + \frac{m_2 m_1}{|\mathbf{r}_{21}|^3}\mathbf{r}_2(t)\times\mathbf{r}_1(t)\\
&= 0
\end{aligned}
$$

by the anti-symmetry of the cross product.

Now assume that for some fixed number $K > 2$ of bodies we have the claim. Then we compute the derivative term for the system with one additional mass. this gives

$$
G\sum_{i=1}^{K+1} m_i \sum_{\substack{j=1\\ j\neq i}}^{K+1} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_i(t)\times\mathbf{r}_j(t)
$$

$$
\begin{aligned}
&= G\sum_{i=1}^{K} m_i \sum_{\substack{j=1\\ j\neq i}}^{K+1} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_i(t)\times\mathbf{r}_j(t)\\
&+ Gm_{K+1} \sum_{\substack{j=1\\ j\neq K+1}}^{K+1} \frac{m_j}{|\mathbf{r}_{(K+1)j}|^3}\mathbf{r}_{K+1}(t)\times\mathbf{r}_j(t)
\end{aligned}
$$

Working with the first of these terms we have

$$
G\sum_{i=1}^{K} m_i \sum_{\substack{j=1\\ j\neq i}}^{K+1} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_i(t)\times\mathbf{r}_j(t)
$$

$$
= G\sum_{i=1}^{K} m_i \left(\sum_{\substack{j=1\\ j\neq i}}^{K} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_i(t)\times\mathbf{r}_j(t) + \frac{m_{K+1}}{|\mathbf{r}_{i(K+1)}|^3}\mathbf{r}_i(t)\times\mathbf{r}_{K+1}(t) \right)
$$

$$= G\left[\sum_{i=1}^{K} m_i \sum_{\substack{j=1\\ j\neq i}}^{K} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_i(t)\times\mathbf{r}_j(t) + \sum_{i=1}^{K} m_i \frac{m_{K+1}}{|\mathbf{r}_{i(K+1)}|^3}\mathbf{r}_i(t)\times\mathbf{r}_{K+1}(t)\right]$$

$$= 0 + G\sum_{i=1}^{K} m_i \frac{m_{K+1}}{|\mathbf{r}_{i(K+1)}|^3}\mathbf{r}_i(t)\times\mathbf{r}_{K+1}(t)$$

where the term vanished due to the induction hypotheses. Substituting this into the original computation gives

$$G\sum_{i=1}^{K+1} m_i \sum_{\substack{j=1\\ j\neq i}}^{K+1} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_i(t)\times\mathbf{r}_j(t)$$

$$\begin{aligned}
&= G\sum_{i=1}^{K} m_i \frac{m_{K+1}}{|\mathbf{r}_{i(K+1)}|^3}\mathbf{r}_i(t)\times\mathbf{r}_{K+1}(t)\\
&+ Gm_{K+1} \sum_{\substack{j=1\\ j\neq K+1}}^{K+1} \frac{m_j}{|\mathbf{r}_{(K+1)j}|^3}\mathbf{r}_{K+1}(t)\times\mathbf{r}_j(t)\\
&= G\left(\sum_{j=1}^{K} \frac{m_j m_{K+1}}{|\mathbf{r}_{j(K+1)}|^3}\mathbf{r}_j(t)\times\mathbf{r}_{K+1}(t) - \sum_{j=1}^{K} \frac{m_j m_{K+1}}{|\mathbf{r}_{(K+1)j}|^3}\mathbf{r}_j(t)\times\mathbf{r}_{K+1}(t)\right)\\
&= 0
\end{aligned}$$

which gives the theorem.

□

This shows that $h(t) = C$ and evaluating at the initial conditions gives $C = h(0)$. Then the three components of angular momentum are integrals of motion. Explicitly they are

$$\sum_{i=1}^{N} m_i[r_i^2(t)v_i^3(t) - r_i^3(t)v_i^2(t)] = h_1(0)$$

$$\sum_{i=1}^{N} m_i[r_i^3(t)v_i^1(t) - r_i^1(t)v_i^3(t)] = h_2(0)$$

$$\sum_{i=1}^{N} m_i[r_i^1(t)v_i^2(t) - r_i^2(t)v_i^1(t)] = h_3(0)$$

Then we have developed a total of nine integrals of motion. The last conserved quantity is is energy, and this is the topic of the next section.

3.3 Conservation of Energy

The tenth integral for the N-body problem is mechanical energy . As a student of mathematics with only modest background in physics, energy is both a more perplexing and romantic quaintly than the previous nine constants of motion, whose definitions are straight forward, and whose implications are modest. (Or if not modest, at least linear. Restraining a system to a constant value of angular, or linear momentum reduces the dimension of the phase space, but only by linear subspaces. One could never hope, for example to gain compactness by these restrictions). More attention is given in this section to mechanical energy than was given in the previous sections to the other nine constants of motion in order to come to happier terms with the idea of energy, and see what it means to someone whose lab is a chalkboard.

No attempt is made to justify the energy concept on intuitive physical grounds; in fact concepts like instantaneous work and the principle of least action are not discussed at all here. Instead we make some purely mathematical observations, and try to develop a connection between these.

For a moment then, consider the situation of "one particle dynamics", where we have a particle at $\mathbf{r} \in R^N$ under the influence of Newton's second law;

$$m\frac{d^2}{dt^2}\mathbf{r} = F(\mathbf{r})$$

whose vector field is a function only of the position of the particle.

Now suppose we were to take the path integral of the force along a solution curve. This is related to the concept of work, but we can think of it as just a possibly interesting dynamical observable. From a mathematical standpoint it is a reasonable observable, as we have two separate expressions for force (the equation of motion), and because path integrals are objects about which much is known (their connection with cohomology can lead to global information about the phase space).

Choosing arbitrary initial conditions $\mathbf{x}_0 = (\mathbf{r}_0, \dot{\mathbf{r}}_0)$, and integrating along the path (in configuration space) for a time $t = T$, we have

$$\int_{\mathbf{r}(0)}^{\mathbf{r}(T)} m\ddot{\mathbf{r}}\; d\mathbf{r} \quad \equiv \quad \int_0^T m\ddot{\mathbf{r}}^T\dot{\mathbf{r}}\, dt$$

The integrand of the later expression is an antiderivative;

$$
\begin{aligned}
\ddot{\mathbf{r}}^T\dot{\mathbf{r}} &= \frac{1}{2}[\ddot{\mathbf{r}}^T\dot{\mathbf{r}} + \ddot{\mathbf{r}}^T\dot{\mathbf{r}}] \\
&= \frac{1}{2}[\ddot{\mathbf{r}}^T\dot{\mathbf{r}} + \dot{\mathbf{r}}^T\ddot{\mathbf{r}}] \\
&= \frac{1}{2}\frac{d}{dt}(\dot{\mathbf{r}}^T\dot{\mathbf{r}}) \\
&= \frac{d}{dt}\frac{|\dot{\mathbf{r}}|^2}{2}
\end{aligned}
$$

Then using just the fundamental theorem of calculus and the definition of the path integral, we have that

$$
\begin{aligned}
\int_{\mathbf{r}(0)}^{\mathbf{r}(T)} m\ddot{\mathbf{r}}\ d\mathbf{r} &= \int_0^T m\ddot{\mathbf{r}}^T\dot{\mathbf{r}}\,dt \\
&= \int_0^T m\frac{d}{dt}\frac{|\dot{\mathbf{r}}|^2}{2}\ dt \\
&= m\frac{|\dot{\mathbf{r}}(T)|^2}{2} - m\frac{|\dot{\mathbf{r}}(0)|^2}{2} \\
&= m\frac{|\mathbf{v}(T)|^2}{2} - m\frac{|\mathbf{v}(0)|^2}{2}
\end{aligned}
\tag{4}
$$

So the value of the path integral along the path in configuration space depends only on the initial and final velocities.

This relation is useful enough that the right hand side warrants it's own name. Define the functional $T : \mathbb{R}^N \to \mathbb{R}^+$ by

$$
T(\mathbf{v}) = m\frac{|\mathbf{v}|^2}{2}
$$

T is called the kinetic energy of the particle $\mathbf{r}$ as it depends only on velocity and can be evaluated for any vector in the tangent space. What has been shown here is that the path integral along a segment of a trajectory (work done moving a particle through a force field) is equal to the change in kinetic energy. Physicists call this the work/energy theorem.

In general this may not be so useful, as $\mathbf{v}(T)$ is an unknown. However as was mentioned above, in the context of the second order differential equation we have two expressions for $\ddot{\mathbf{r}}$ and so far we have taken advantage of only one of them. We also have that

$$
\int_{\mathbf{r}(0)}^{\mathbf{r}(T)} m\ddot{\mathbf{r}}\ d\mathbf{r} = \int_{\mathbf{r}(0)}^{\mathbf{r}(T)} F(\mathbf{r})\ d\mathbf{r}
$$

Is this expression of any use to us? Looking back at the development of the kinetic energy observable (functional) the key to the reduction of the path integral to an algebraic expression was the fact that the integrand was an anti-derivative. The fundamental theorem for line integrals (see for example [GP]) tells us that, if the integrand is an antiderivative (in this context the gradient of some scalar function f) then the integral reduces to the evaluation of the scalar function at the endpoints of the integration (the proof of this was essentially carried out in the development of the Kinetic energy above).

If all of these events conspire in our favor, then we call f a potential function (or the potential energy, as it is a functional/observial that depends only on the position of the particle) and have that

$$\begin{aligned} m\frac{|\mathbf{v}(T)|^2}{2} - m\frac{|\mathbf{v}(0)|^2}{2} &= \int_{\mathbf{r}(0)}^{\mathbf{r}(T)} m\ddot{\mathbf{r}}\ d\mathbf{r} \\ &= \int_{\mathbf{r}(0)}^{\mathbf{r}(T)} F(\mathbf{r})d\mathbf{r} \\ &= f(\mathbf{r}(T)) - f(\mathbf{r}(0)) \end{aligned}$$

This can be rearranged into

$$m\frac{|\mathbf{v}(0)|^2}{2} - f(\mathbf{r}(0)) = m\frac{|\mathbf{v}(T)|^2}{2} - f(\mathbf{r}(T))$$

the left hand side of which is independent of time. Then the right hand side is constant for all time as T was arbitrary. Define a new observable $E : \mathbb{R}^{2N} \to \mathbb{R}$ by

$$E(\mathbf{r}, \mathbf{v}) = \frac{|\mathbf{v}|^2}{2} - f(\mathbf{r})$$

E is the mechanical energy of the dynamical system. This shows that if a the force field of a system has a potential function, then mechanical energy is conserved along trajectories of the system.

(Note: the potential is sometimes defined so as to satisfy the differential equation $-\nabla f = F$ in which case the minus sings in our definition become pluses. The difference is purely cosmetic).

Extending these definitions to systems of N particles is straightforward. Let $\mathbf{r}_1, \ldots, \mathbf{r}_N \in \mathbb{R}^3$ (The restriction to three dimensions is not necessary from a mathematical standpoint, but conforms to the typical situation in mechanics). Suppose the system evolves under Newton's second law;

$$m_i \frac{d^2}{dt^2}\mathbf{r_i} = F_i(\mathbf{r}_1, \ldots, \mathbf{r}_N)$$

for $1 \leq i \leq N$. In analogy with what we did for a one particle system, we will take the path integrals of the forces from an arbitrary initial condition $\mathbf{r}_1(0), \ldots, \mathbf{r}_N(0)$ over a time interval of length T and add the results. This gives

$$\int_{\mathbf{r}_1(0)}^{\mathbf{r}_1(T)} m_1 \dot{\mathbf{r}}_1 \; d\mathbf{r}_1 + \ldots + \int_{\mathbf{r}_N(0)}^{\mathbf{r}_N(T)} m_N \dot{\mathbf{r}}_N \; d\mathbf{r}_N$$

$$\begin{aligned} &= m_1 \frac{|\mathbf{v}_1(T)|^2}{2} - m_1 \frac{|\mathbf{v}_1(0)|^2}{2} + \ldots + m_1 \frac{|\mathbf{v}_N(T)|^2}{2} - m_1 \frac{|\mathbf{v}_N(0)|^2}{2} \\ &= \sum_{i=1}^{N} T[\mathbf{v}_i]|_{\mathbf{v_i}(0)}^{\mathbf{v_i}(T)} \end{aligned}$$

As before this is a general result that (while it may or may not be of much use) holds for all mechanical systems. This leads to defining the kinetic energy of a system of particles to be the functional $T_{sys} : \mathbb{R}^{3N} \to \mathbb{R}^+$given by

$$T_{sys}(\mathbf{v}_1, \cdots, \mathbf{v}_N) = \sum_{i=1}^{N} m_i \frac{|\mathbf{v}_i|^2}{2}$$

(we will just write T for this when there is no possibility of confusion).

Now, if there exists a functional $f : R^{3N} \to \mathbb{R}$ such that

$$D_{\mathbf{r}_1} f = F_i$$

then f is called a potential for the system and

$$\int_{\mathbf{r}_1(0)}^{\mathbf{r}_1(T)} m_1 \dot{\mathbf{r}}_1 \; d\mathbf{r}_1 + \ldots + \int_{\mathbf{r}_N(0)}^{\mathbf{r}_N(T)} m_N \dot{\mathbf{r}}_N \; d\mathbf{r}_N$$

$$= \int_{\mathbf{r}_1(0)}^{\mathbf{r}_1(T)} F_1 \; d\mathbf{r}_1 + \ldots + \int_{\mathbf{r}_N(0)}^{\mathbf{r}_N(T)} F_N \; d\mathbf{r}_N$$

$$= \int_0^T F_1 \dot{\mathbf{r}}_1 \; dt + \ldots + \int_0^T F_N \dot{\mathbf{r}}_N \; dt$$

$$= \int_0^T (F_1 \dot{\mathbf{r}}_1 + \ldots + F_N \dot{\mathbf{r}}_N) \, dt$$

$$= \int_0^T F \dot{\mathbf{r}} dt$$

$$\equiv \int_{\mathbf{r}(0)}^{\mathbf{r}(T)} F d\mathbf{r}$$

$$= \int_{\mathbf{r}(0)}^{\mathbf{r}(T)} \nabla f d\mathbf{r}$$

$$f(\mathbf{r}(T)) - f(\mathbf{r}(0))$$

where F is the vector $(F_1, \ldots, F_N)$ and $\mathbf{r}$ the vector $(\mathbf{r}_1, \ldots, \mathbf{r}_N)^T$, and ∇f is the $(1 \times 3N)$ vector of partial derivatives of f.

Equating the two expressions for the line integral and rearranging as in the one particle case gives

$$\sum_{i=1}^{N} m_i \frac{|\mathbf{v}_i(0)|^2}{2} - f(\mathbf{r}_1(0), \ldots, \mathbf{r}_N(0)) = \sum_{i=1}^{N} m_i \frac{|\mathbf{v}_i(T)|^2}{2} - f(\mathbf{r}_1(T), \ldots, \mathbf{r}_N(T))$$

Define the mechanical energy for the system of particles to be the functional $E_{sys} : \mathbb{R}^{6N} \to \mathbb{R}$ defined by

$$E(\mathbf{r}, \mathbf{v}) = T_{sys}(\mathbf{v}_1, \ldots, \mathbf{v}_N) - f(\mathbf{r}_1, \ldots, \mathbf{r}_N)$$

The agrement above shows that this quantity is conserved along trajectories (in the phase space) of the N particle system.

With these general considerations in hand we return to the the particular case of interest in celestial mechanics. We will show that the N-body problem has a potential function, by direct computation. Recall

$$\int_0^T (F_1 \dot{\mathbf{r}}_1 + \ldots + F_N \dot{\mathbf{r}}_N)\, dt$$

$$= \int_{\mathbf{r}_1(0)}^{\mathbf{r}_1(T)} F_1 \ d\mathbf{r}_1 + \ldots + \int_{\mathbf{r}_N(0)}^{\mathbf{r}_N(T)} F_N \ d\mathbf{r}_N$$

Considering these separately gives

$$\int_{\mathbf{r}_i(0)}^{\mathbf{r}_i(T)} F_i \ d\mathbf{r}_i = \int_{\mathbf{r}_i(0)}^{\mathbf{r}_i(T)} \left(G \sum_{j=1, j\neq i}^{N} \frac{m_i m_j}{|\mathbf{r}_{ij}|^3} \mathbf{r}_{ij} \right) \ d\mathbf{r}_i$$

$$= G \sum_{j=1, j\neq i}^{N} \int_{\mathbf{r}_i(0)}^{\mathbf{r}_i(T)} \frac{m_i m_j}{|\mathbf{r}_{ij}|^3} \mathbf{r}_{ij} \ d\mathbf{r}_i$$

$$= G\, m_i \sum_{j=1, j\neq i}^{N} \int_{\mathbf{r}_i(0)}^{\mathbf{r}_i(T)} m_j [(\mathbf{r}_j - \mathbf{r}_i)^T (\mathbf{r}_j - \mathbf{r}_i)]^{-3/2} \, (\mathbf{r}_j - \mathbf{r}_i)\, d\mathbf{r}_i$$

$$= \frac{G}{2} m_i \sum_{j=1, j\neq i}^{N} \frac{m_j}{|\mathbf{r}_{ij}|} \Big|_{\mathbf{r}_i(0)}^{\mathbf{r}_i(T)}$$

Then,

$$\int_{\mathbf{r}_1(0)}^{\mathbf{r}_1(T)} F_1 \; d\mathbf{r}_1 + \ldots + \int_{\mathbf{r}_N(0)}^{\mathbf{r}_N(T)} F_N \; d\mathbf{r}_N$$

$$= \frac{G}{2} m_1 \sum_{j=1, j\neq 1}^{N} \frac{m_j}{|\mathbf{r}_{1j}|} \Big|_{\mathbf{r}_1(0)}^{\mathbf{r}_1(T)} + \ldots + \frac{G}{2} m_N \sum_{j=1, j\neq N}^{N} \frac{m_j}{|\mathbf{r}_{Nj}|} \Big|_{\mathbf{r}_N(0)}^{\mathbf{r}_N(T)}$$

$$= \frac{G}{2} \sum_{i=1}^{N} m_i \sum_{j=1, j\neq i}^{N} \frac{m_j}{|\mathbf{r}_{ij}|} \Big|_{\mathbf{r}_i(0)}^{\mathbf{r}_i(T)}$$

Define the potential function $f : \mathbb{R}^{3N} \to \mathbb{R}^+$ by

$$f(\mathbf{r}) \quad = \quad \frac{G}{2} \sum_{i=1}^{N} m_i \sum_{j=1, j\neq i}^{N} \frac{m_j}{|\mathbf{r}_{ij}|}$$

This is the gravitational potential to the N-Body problem. (Computing the gradient confirms it). Putting all of this together gives the tenth constant of motion for the N-body problem. The integral is

$$\sum_{i=1}^{N} m_i \frac{|\mathbf{v}_i(0)|^2}{2} - \frac{G}{2} \sum_{i=1}^{N} m_i \sum_{j=1, j\neq i}^{N} \frac{m_j}{|\mathbf{r}_{ij}(0)|} = \sum_{i=1}^{N} m_i \frac{|\mathbf{v}_i(t)|^2}{2} - \frac{G}{2} \sum_{i=1}^{N} m_i \sum_{j=1, j\neq i}^{N} \frac{m_j}{|\mathbf{r}_{ij}(t)|}$$

for all $t \in \mathbb{R}$ along trajectories of the system.

4 Transformation Invariance of the Equations

There are certain coordinate changes that come up again and again in the study of N-body dynamics. For example, suppose that we are sitting in an inertial frame and know that N bodies have their initial velocities and positions in some plane, which may be affine from our point of view. Suppose that after computing their initial center of mass, and it's initial velocity we find that is is moving. Now we try to solve the equations of motion.

One might suspect that we have somehow chose coordinates poorly, and that the trajectories we find, say after numerically integrating the system, will seem more complicated than they would to an observer moving with the center of mass. For such an observer the plane of motion can be taken to be the xy plane. Of course the situation is often clarified by looking at the systems in just such coordinates.

The center of mass of the system moves in a constant direction with a constant. Then the frame of reference described above is an inertial frame. Since equations of motion were derived in an arbitrary inertial frame we expect that they are invariant under a coordinate change from one inertial frame to another. This is shown in the next section.

4.1 Linear Change of Variables

Theorem 3 (Invariance Under Affine Change of Coordinate) *The differential equation*

$$\ddot{\mathbf{r}}_i = G \sum_{j=1, j\neq i}^{N} \frac{m_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} \quad 1 \leq i \leq N \tag{5}$$

with initial conditions $\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{v}_1, \ldots, \mathbf{v}_N \in \mathbb{R}^3$ *is invariant (up to a rescaling of G) under the change of coordinates*

$$\mathbf{p}_i = A\mathbf{r}_i + \mathbf{b}$$

where $A \in GL(\mathbb{R}^3)$ *and* $\mathbf{b} \in \mathbb{R}^N$.

If we are given initial conditions in the $\mathbf{r}_i$ coordinates and asked to evolve them by 5 we are free to change variables to $\mathbf{p}_i$, solve the transformed equations, and apply the inverse transform to this solution, putting it back in terms of the original variables. What the theorem says is that the equation of motion for the new variables has the same form as 5, with the possibility that the constant G has changed.

Proof: A^{-1} exists by the assumption that $A \in GL(\mathbb{R}^3)$. Then

$$\mathbf{r}_i = A^{-1}(\mathbf{p}_i - \mathbf{b}) = A^{-1}\mathbf{p}_i + \mathbf{c}$$

where $\mathbf{c} = -A^{-1}\mathbf{b}$. Differentiating gives

$$\dot{\mathbf{r}}_i = A^{-1}\dot{\mathbf{p}}_i$$

and

$$\ddot{\mathbf{r}}_i = A^{-1}\ddot{\mathbf{p}}_i$$

Note that

$$\begin{aligned}
\mathbf{r}_{ij} &= \mathbf{r}_j - \mathbf{r}_i \\
&= (A^{-1}\mathbf{p}_j + \mathbf{c} - A^{-1}\mathbf{p}_i - \mathbf{c}) \\
&= A^{-1}(\mathbf{p}_j - \mathbf{p}_i) \\
&= A^{-1}\mathbf{p}_{ij}
\end{aligned}$$

so that

$$\begin{aligned} |\mathbf{r}_{ij}| &= |A^{-1}\mathbf{p}_{ij}| \\ &= |A^{-1}||\mathbf{p}_{ij}| \end{aligned}$$

where $|A^{-1}| = |\det(A^{-1})| \equiv \alpha$. Substituting these into 5 gives

$$\begin{aligned} \ddot{\mathbf{r}}_i &= A^{-1}\ddot{\mathbf{p}}_i \\ &= G \sum_{j=1, j\neq i}^{N} \frac{m_j}{\alpha^3 |\mathbf{p}_{ij}|^3} A^{-1}\mathbf{p}_{ij} \\ &= \tilde{G} A^{-1} \sum_{j=1, j\neq i}^{N} \frac{m_j}{|\mathbf{p}_{ij}|^3} \mathbf{p}_{ij} \end{aligned}$$

where $\tilde{G} = G/\alpha^3$. Now simply multiplying both sides by A, this is

$$\ddot{\mathbf{p}}_i = \tilde{G} \sum_{j=1, j\neq i}^{N} \frac{m_j}{|\mathbf{p}_{ij}|^3} \mathbf{p}_{ij}$$

But this is eqn (5) in terms of the variables $\mathbf{q}_i$ with only the constant scaled. Which gives the result.

$\square$

We can immediately note that if A has determinant plus or minus one, then $G = \tilde{G}$ and the equations are identical. In fact this is an if and only if statement.

4.2 Inertial Coordinate Change

Nothing in the previous argument is changed if the coordinate frame is translating at a constant velocity.

Explicitly we have that if

$$\mathbf{p}_i = A\mathbf{r}_i + (t\mathbf{d} + \mathbf{b}) \tag{6}$$

then

$$\mathbf{r}_i = A^{-1}\mathbf{p}_i + t\bar{\mathbf{d}} + \bar{\mathbf{c}}$$

where $\bar{\mathbf{d}} = -A^{-1}\mathbf{d}$ and $\bar{\mathbf{c}} = -A^{-1}\mathbf{c}$

Then

$$\ddot{\mathbf{r}}_i = A^{-1}\ddot{\mathbf{p}}_i$$

and

$$\mathbf{r}_{ij} = A^{-1}\mathbf{p}_{ij}$$

just as before. Since these are the two relations from which the rest of the proof follows, it goes through in this case as well. But eqn 6 is a completely general inertial change of variables. This establishes that

Theorem 4 (Inertial Frames) *The equations of motion for the N-body problem are invariant (up to change of gravitational constant) under inertial change of variable.*

4.3 Time Change

Occasionally it will be convenient for us to resale the time variable. This is a different kind of transformation from the ones considered above for several reasons. The first being that the vector field for the N-body problem is time invariant, so it's not clear that such a procedure is prudent. A second departure from the previous cases is that a time change is not a function from the phase space to itself whose effect on the equations of motion must be determined. With a time change we don't want to transform the initial variables at all, but instead to flow from the same initial conditions with time running at some modified rate. While this will not effect the vector field, it does effect the derivatives (the tangent space itself).

The right way to think about the time change is to think of this change of variables as a reparametrization of the flow. Then;

Theorem 5 (time change) *Suppose that $\phi : U \times \mathbb{R} \to \mathbb{R}^N$ is the flow generated by the vector field*

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

Then

$$\phi(\mathbf{x}, g(\mathbf{t}))$$

satisfies the differential equation

$$\frac{d}{dt}\mathbf{x} = g'(t)f(\mathbf{x}) \tag{7}$$

where $\mathbf{x} = \mathbf{x}(g(t))$.

Flows are defined and discussed in more detail in the second note set in the context of the variational equations. For now we take these things as standard. Good references are [HS], [MH], and [R].

Proof: The proof is a pedantic application of the definitions, but worth going through once. That ϕ is the flow generated by by the vector field $\dot{\mathbf{x}} = f(\mathbf{x})$ implies

$$\frac{d}{dt}\phi(\mathbf{x},t) = f(\phi(\mathbf{x},t))$$

for all fixed $\mathbf{x}$ and t. This is just the definition . Similarly, in order to show that $\phi(\mathbf{x}, g(t))$ solves the differential equation (7) it is sufficient to show that

$$\frac{d}{dt}\phi(\mathbf{x}, g(t)) = g'(t) f(\phi(\mathbf{x}, g(t)))$$

For this simply compute

$$\begin{aligned}
\frac{d}{dt}\phi(\mathbf{x}, g(t)) &= \frac{d}{d\tau}\phi(\mathbf{x},\tau)|_{\tau=g(t)}\frac{d}{dt}g(t) \\
&= g'(t) f(\phi(\mathbf{x},\tau))|_{\tau=g(t)} \\
&= g'(t) f(\phi(\mathbf{x}, g(t)))
\end{aligned}$$

□

A specific case worth singling out is the case of time reversal. Suppose we begin at an initial condition $\mathbf{x}_0$ and want to flow the point backward in time to $t = -T$. In the notation above, we want to compute $\phi(\mathbf{x}_0, -T)$. Then $g(t) = -t$ and the theorem tells that we can integrate the vector field

$$\dot{\mathbf{x}} = -f(\mathbf{x})$$

(where $\mathbf{x} = \mathbf{x}(-t)$) from $t = 0$ to $t = T$ and with $\mathbf{x}_0$ as initial condition.

5 N-Body Dynamics

In the previous sections we developed the equations of motion for systems of N-bodies interacting thought the law of mutual gravitational attraction, and explored many of the properties of these equations. However we have not yet discussed the motion of bodies in such a system at all. The rest of this note set is devoted to this.

First we consider two very simple (and similar) cases where the equations can be explicitly solved. However, in most situations it is not possible to find analytic solutions. Instead the system must typically be numerically integrated. We present several numerical experiments intended to illustrate the richness of the possible motions in the N-body problem, and hopefully to demonstrate some of the difficulties that arise when exploring such systems numerically.

5.1 The Kepler Problem; One Body Problem

The simplest case of N-body dynamics we can possible imagine is the case of one body. Here we imagine that in all of space there exists only one gravitating particle with mass $m_1 \equiv M$. Such a particle cannot self gravitate, so it experiences no forces and hence no acceleration, hence moves with a constant velocity.

We are free to choose coordinates centered at the position of this particle at time say $t_0 = 0$. Further we can allow the coordinates to move with the constant velocity of the particle as the equations of motion are unaltered by inertial change of coordinates. In these coordinates the particle is at rest at the origin.

Can anything happen in this situation? Imagine a second particle is introduced, but that this particle is so small that it does not effect the motion of the first mass. Such a particle is called a test particle. We can ask how a test particle will move in the field of the first.

This can be made more clear by noting that if M is the primary mass at the origin, and m_2 is the second body, then the equations of motion are

$$\ddot{\mathbf{r}}_1 = Gm_2 \frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} = 0 \qquad \mathbf{r}_1(0) = 0 \qquad \dot{\mathbf{r}}_1(0) = 0$$

(as $m_2 = 0$) and

$$\ddot{\mathbf{r}}_2 = GM \frac{\mathbf{r}_{21}}{|\mathbf{r}_{21}|^3} \qquad \mathbf{r}_2(0) = \mathbf{r}_0 \qquad \dot{\mathbf{r}}_2(0) = \mathbf{v}_0$$

where $\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2$. But the first mass is located at the origin, so $\mathbf{r}_1 = (0, 0)$. Integrating the first equation twice shows that the primary mass rests at the origin for all time as was desired. The equation of motion for the second body becomes

$$\ddot{\mathbf{r}}_2 = -GM \frac{\mathbf{r}_2}{|\mathbf{r}_2|^3} \qquad \mathbf{r}_2(0) = \mathbf{r}_0 \qquad \dot{\mathbf{r}}_2(0) = \mathbf{v}_0$$

Changing notation so that $\mathbf{r}_2 \equiv \mathbf{r}$ these are

$$\ddot{\mathbf{r}} = -GM \frac{\mathbf{r}}{|\mathbf{r}|^3} \qquad \mathbf{r}(0) = \mathbf{r}_0 \qquad \dot{\mathbf{r}}(0) = \mathbf{v}_0 \tag{8}$$

Determining the motions given by (8) is Kepler's Problem. The problem arise in it's own right in the study of central forces in mechanics. In celestial mechanics it has the interpretation described above. There, it is an accurate model for the interaction between a massive body like the sun, and a smaller body like an astroid, or perhaps Pluto or Mercury.

It turns out that the problem has as many constants of motion as it has unknowns. Then it can be 'integrated' in the following sense: Once initial conditions are given they determine the constants of motion. Fixing a constant reduces the possible motion to a certain fixed submanifold of the phase space.

There are enough constants of motion so that the intersection of these submanifold is a one-dimensional space, which must be the trajectory of the particle. We proceed following Arnold [Arn].

Firs we show that the Kepler Problem conserves angular momentum (the theorems above were proven for N-bodies with $N \geq 2$). Define the angular momentum of the infinitesimal body to be $h = \mathbf{r} \times \dot{\mathbf{r}}$. Then observe that

$$\frac{d}{dt}h = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0$$

as $\mathbf{u} \times \mathbf{u} = 0$ for all vectors $\mathbf{u}$, and

$$\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times -GM\frac{\mathbf{r}}{|\mathbf{r}|^3} = -\frac{GM}{|\mathbf{r}|^3}\mathbf{r} \times \mathbf{r} = 0$$

Then the constant

$$h_0 = \mathbf{r}_0 \times \mathbf{v}_0$$

is an integral of motion. It is an elementary fact of vector analysis that the equation $\mathbf{u} \times \mathbf{v} = c$ defines a plane in 3-space containing the origin. Then the submanifold defined by restricting to a constant value of the angular momentum is such a plane. This tells us that the motion of the test particle is planar, giving a reduction of dimension.

Assume for a moment that $h_0 = 0$. This occurs if and only if the initial velocity of the test particle is collinear with the initial position, which means the initial velocity is pointing either directly at, or directly away from the primary/origin. Further since h_0 is conserved the velocity points this way for all time.

In this case the particle's trajectory must lie on the line so defined. The dynamics restricted to this one-manifold are one degree of freedom dynamics. Now choose coordinates so that this line is the x-axis and orientation so that $\mathbf{r}_0 = x_0 > 0$. The equation of motion is

$$\ddot{x} = -GM\frac{x}{|x|^3} = -GM\frac{1}{x^2}$$

(as $x_0 > 0$ implies $|x| = x$. Note that the singularity prevents x from changing sign, as we will consider collisions irresolvable).

It is worthwhile do digress momentarily and develop some properties of one-freedom mechanical systems, i.e. systems given by the a second order equation $\ddot{x} = g(x)$, $x \in \mathbb{R}$. Such a system always has a potential functional defined by

$$f(x) = \int_{x_0}^{x} g(u)\, du$$

as the fundamental theorem of calculus then gives $d/dx f(x) = g(x)$. Adding a constant to the potential never changes the equations of motion, so we feel free to do so if it helps.

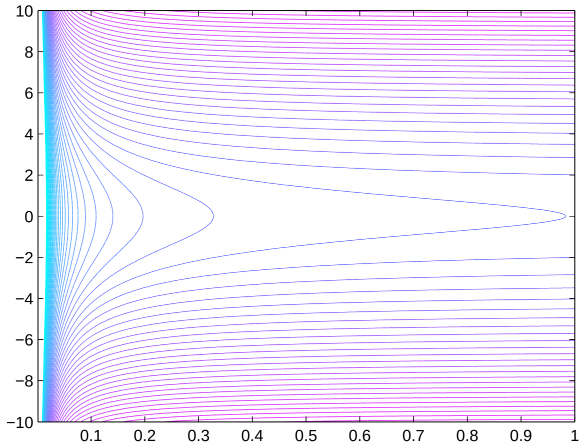

Figure 1: Implicit curves for many values of energy

Then a potential for the kepler problem when $h_0 = 0$ is

$$\tilde{f}(x) = \int_{x_0}^{x} -GM\frac{1}{u^2}\,du = GM\left(\frac{1}{x} - \frac{1}{x_0}\right)$$

Adding the constant $c = GM/x_0$ to this gives the convenient potential

$$f(x) = \frac{GM}{x}$$

and the problem conserves the total energy

$$E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - f(x)$$

In fact this holds for any one-freedom system. To see this simply compute

$$\frac{d}{dt}(T - f) = \dot{x}\ddot{x} - \dot{f}\dot{x} = \dot{x}(\ddot{x} - g(x)) = 0$$

Then solutions live on the curve in (two dimensional $(x, \dot{x})$) phase space defined by

$$E_0 = \frac{1}{2}\dot{x}^2 - \frac{GM}{x}.$$

This defines $\dot{x}$ implicitly as a function of x and gives the shape of the trajectory. A contour plot is shown in figure 1 for several values of energy. Each curve is a line of constant energy, and hence a solution curve for the problem. Picking an initial position and velocity determines an energy curve. The particle flows along the curve in the direction if decreasing potential.

Then there are two basic types of behavior. A phase point moves along the energy curve from higher velocity to lower velocity. As it does, it's position changes according to where it began on the curve. If the point begins with positive velocity then, as it moves along the curve it's velocity falls, and it's position increases. This can either continue indefinitely in which case the position goes to infinity, or at some point the velocity may go to zero. In this case the position begins to decrease and the point builds negative velocity.

In the case that the initial velocity is zero or negative, the fate of the particle is sealed. All such points loose position and pick up negative velocity. All of this conforms with the experience of throwing a ball straight up from the center of the earth. It may go up and come down. Given sufficient initial velocity it may escape. If it is thrown toward the center of the earth it cannot but collide.

The energy expression can readily be solved for $\dot{x}$ as a function of x giving

$$\dot{x} = \pm\sqrt{\frac{2GM}{x} + 2E_0} \tag{9}$$

where the sing chosen depends on the sign of the initial velocity, but this gives little more information than the contour plot. The resulting equation is separable and can in fact be integrated. However the antiderivative is to cumbersome to be of much value, and certainly cannot be solved for x. It seems that the energy approach is best in this situation.

It may seem like we have gone through lot of trouble simply to treat the degenerate case $h_0 = 0$, but the same ideas will yield the solution of the full Kepler problem. To see this assume now that $h_0 \neq 0$. Since this is a constant of motion, the angular velocity is non-zero for all time, in which case the velocity will never point directly toward or away from the origin. Since the Kepler Problem is planar this implies that the velocity vector must always be to one side of the position vector,pushing it always in either the clockwise or counterclockwise direction. However this means that the position vector cannot reverse directions.

This will be useful in a moment. First, we make some new definitions. Observe that the problem admits polar coordinates in a very natural way. Fix an arbitrary unit vector in the plane of motion of the test particle (we will specify a this vector precisely later). Any point in the plane can be described by specifying the angle it makes with the ray, and its magnitude.

Then if $\mathbf{r}$ is a vector in the plane we can write $\mathbf{r} = (r, \theta)$ where $r = |\mathbf{r}|$ and θ is as above. We define a rectangular frame by noting that $\mathbf{r} = (r\cos\theta, r\sin\theta) = r(\cos\theta, \sin\theta)$. Of course the vector $(\cos\theta, \sin\theta)$ has unit norm, so we define $\mathbf{e}_r = (\cos\theta, \sin\theta)$.

A second unit vector is given by $\mathbf{e}_\theta = (-\sin\theta, \cos\theta)$. Furthermore

$$\mathbf{e}_r \cdot \mathbf{e}_\theta = -\cos\theta\sin\theta + \cos\theta\sin\theta = 0$$

so these vectors constitute a non-inertial orthonormal basis for the plane of motion. Suppose we choose to orient the plane so that theta always increases in forward time. Then $\mathbf{e}_\theta$ always points in the direction of increasing theta.

Note that

$$\mathbf{r} = |\mathbf{r}|\frac{\mathbf{r}}{|\mathbf{r}|} = r\mathbf{e}_r$$

or

$$\mathbf{e}_r = \frac{\mathbf{r}}{|\mathbf{r}|}$$

so that $\mathbf{e}_r$ changes in time if $\mathbf{r}$ does. Since $\mathbf{e}_\theta$ is perpendicular to $\mathbf{e}_r$ it will vary in time as well.

The derivatives of these vectors are

$$\begin{aligned}
\frac{d}{dt}\mathbf{e}_r &= \frac{d}{dt}(\cos\theta, \sin\theta) \\
&= (-\sin\theta\ \dot{\theta}, \cos\theta\ \dot{\theta}) \\
&= \dot{\theta}(-\sin\theta, \cos\theta) \\
&= \dot{\theta}\ \mathbf{e}_\theta
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\mathbf{e}_\theta &= \frac{d}{dt}(-\sin\theta, \cos\theta) \\
&= (-\cos\theta\ \dot{\theta}, -\sin\theta\ \dot{\theta}) \\
&= -\dot{\theta}(\cos\theta, \sin\theta) \\
&= -\dot{\theta}\ \mathbf{e}_r
\end{aligned}$$

These allow us to compute the kinematic relation

$$\begin{aligned}
\dot{\mathbf{r}} &= (r\mathbf{e}_r)' \\
&= \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r \\
&= \dot{r}\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta
\end{aligned}$$

and similarly

$$\begin{aligned}
\ddot{\mathbf{r}} &= (\dot{r}\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta)' \\
&= (\dot{r}\mathbf{e}_r)' + (r\dot{\theta}\,\mathbf{e}_\theta)' \\
&= (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta
\end{aligned}$$

In these coordinates the differential equation for the Kepler problem is

$$\ddot{\mathbf{r}} = -\frac{GM}{r^2}\mathbf{e}_r$$

and in light of the kinematic expression for acceleration above this is

$$(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta = -\frac{GM}{r^2}\mathbf{e}_r$$

Equating the components gives the two scalar differential equations

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$

and

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

Consider the second equation first. This equation can be made exact by multiplying both sides by r. The right hand side will still be zero, however

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = \frac{d}{dt}(r^2\dot{\theta})$$

Since both terms vanish identically we have that

$$r^2\dot{\theta} = c$$

Note also that

$$\begin{aligned}
h_0 &= \mathbf{r} \times \dot{\mathbf{r}} \\
&= \mathbf{r} \times (\dot{r}\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta) \\
&= \mathbf{r} \times \dot{r}\mathbf{e}_r + \mathbf{r} \times r\dot{\theta}\,\mathbf{e}_\theta \\
&= 0 + r\dot{\theta}\mathbf{r} \times \mathbf{e}_\theta \\
&= r^2\dot{\theta}\;\mathbf{e}_r \times \mathbf{e}_\theta
\end{aligned}$$

where the vector part of this expression is the unit vector normal to the plane of motion (The cross product of the unit vectors points in the direction of the angular momentum because of the orientation convention).

Then the scalar part is the magnitude of the angular momentum expressed in these polar coordinates which shows that $c = |h_0|$. So the second scalar differential equation recovers the law of conservation of angular momentum.

This has an important geometric corollary. The quantity $r^2\dot{\theta}$ is twice the time rate of change of the area swept out by the vector $\mathbf{r}$. That this is constant implies that $\mathbf{r}$ sweeps equal areas in equal times. This is the content of Kepler's Second Law. In fact, all of the above depended on the fact that the force field was central, and holds in that generality.

All of this will aid in the analysis of the first equation. Square c and rearrange to give $\dot{\theta}^2 = c^2/r^4$. This allows the elimination of theta from the first of the scalar differential equations, and yields

$$\ddot{r} - r\frac{c^2}{r^4} = -\frac{GM}{r^2}$$

or

$$\ddot{r} = -\frac{GMr + c^2}{r^3} \equiv g(r).$$

This reduces the problem to a one-freedom dynamical system and the results developed above apply. A potential for the reduced problem is

$$\begin{aligned}
\tilde{f}(r) &= \int_{r_0}^{r} g(u)\, du \\
&= \int_{r_0}^{r} \left(-\frac{GMr + c^2}{r^3}\right) du \\
&= GM\left[\frac{1}{u}\right]_{r_0}^{r} - \frac{c^2}{2}\left[\frac{1}{u^2}\right]_{r_0}^{r} \\
&= \frac{GM}{r} - \frac{c^2}{2r^2} + \left(\frac{GM}{r_0} - \frac{c^2}{2r_0^2}\right)
\end{aligned}$$

Adding the constant $GM/r_0 - c^2/2r_0^2$ to this gives the "effective potential" for the decoupled problem

$$f(r) = \frac{GM}{r} - \frac{c^2}{2r^2}$$

where the fundamental theorem grantees that $d/dr f(r) = g(r)$ as discussed earlier in the section. Again a one-freedom system conserves it's energy functional. For the reduced problem this is

$$E(r, \dot{r}) = \frac{\dot{r}^2}{2} - f(r) = \frac{\dot{r}^2}{2} - \frac{GM}{r} + \frac{c^2}{2r^2}.$$

Solving for $\dot{r}$ in this expression gives

$$\dot{r} = \sqrt{\frac{2GM}{r} - \frac{c^2}{r^2} + 2E_0}$$

This can be separated to give the expression

$$\int_{t_0}^{t} dt = \int_{r_0}^{r} \frac{1}{\sqrt{\frac{2GM}{r} - \frac{c^2}{r^2} + 2E_0}}\, dr$$

which we will return to in a moment. First we develop Kepler's First Law of Planetary Motion, which is a statement about the geometry of the path of the test particle in configuration space. As is well known, it says that the paths are conic sections.

We have that $\dot{\theta} = c/r^2$, so that θ can be expressed as some unknown function of r. The chain rule gives

$$\frac{d}{dt}\theta = \frac{d}{dr}\theta \frac{d}{dt}r$$

which is equal to c/r^2. Then

$$\frac{d}{dr}\theta = \frac{c/r^2}{\dot{r}} = \frac{c}{r^2\sqrt{\frac{2GM}{r} - \frac{c^2}{r^2} + 2E_0}}$$

or

$$\theta(r) = \int_{r_0}^{r} \frac{c/\rho^2}{\sqrt{\frac{2GM}{\rho} - \frac{c^2}{\rho^2} + 2E_0}}\, d\rho$$

It is a tedious exercise, but one can show that integrand has an antiderivative. The computation is a series of 'u-substitutions'. First, let $u = 1/r$. Then we have

$$\theta(r) = \int \frac{c/\rho^2}{\sqrt{\frac{2GM}{\rho} - \frac{c^2}{\rho^2} + 2E_0}}\, d\rho = -\int \frac{1}{\sqrt{-u^2 + au + b}}\, du$$

where $a = GM/2c^2$ and $b = 2E_0/c^2$. The polynomial under the radical has roots

$$u_{1,2} = \frac{-a \pm \sqrt{a^2 - 4(-1)b}}{2(-1)} = \frac{a \mp \sqrt{a^2 + 4b}}{2} \equiv \alpha \mp \beta$$

and so factors as

$$-(u - \alpha + \beta)(u - \alpha - \beta) = -[(u - \alpha)^2 - \beta^2]$$

Then letting $u - \alpha = v$ gives

$$-\int \frac{1}{\sqrt{-u^2 + au + b}}\, du = -\int \frac{1}{\sqrt{-[(u - \alpha)^2 - \beta^2]}}\, du$$

$$= -\int \frac{1}{\sqrt{-[v^2 - \beta^2]}}\, dv$$

$$= -\int \frac{1}{\sqrt{\beta^2 - v^2}}\, dv$$

which is a standard integral, and gives the result (It could be argued that this is only valid if the discriminant is positive. But if it is not one completes the square and obtains an inverse trig integral anyway). At any rate, a little algebra gives

$$\theta = \cos^{-1}\left(\frac{\frac{c}{r} - \frac{GM}{c}}{\sqrt{2E_0 + \frac{(GM)^2}{c^2}}}\right).$$

This is solved for r so that

$$r = \frac{p}{1 + e\cos\theta}$$

where

$$p = \frac{c^2}{GM} \qquad e = \sqrt{1 + \frac{2E_0c^2}{(GM)^2}}.$$

Clearly this is the equation in polar coordinates for a conic section with eccentricity e and parameter p. The constant of integration is eliminated by choosing the vector from which we measure angles to point in such a way that at time zero the position vector and the velocity vector are perpendicular. That there must be such a time follows from the conservation of E and c and makes explicit use of the Kepler potential. (If this is not done the argument of the cosine term contains a phase shift, which corresponds to the test particle being located at an arbitrary point on the conic section at time zero).

In a moment this information will allow us to design orbits with prescribed properties. First however we comment that Kepler's Third Law of Planetary Motion is just a computation away. We can write

$$\dot{\theta} = \frac{c}{r^2} = \frac{c}{\left(\frac{p}{1+e\cos\theta}\right)^2} = \frac{c(1 + e\cos\theta)^2}{p^2}$$

which separates into

$$t_f - t_0 = \frac{p^2}{c}\int_{\theta_0}^{\theta_f} \frac{1}{(1 + e\cos\theta)^2}\, d\theta$$

Integrating from $\theta_0 = 0$ to ${}_f = 2\pi$ when $0 < e < 1$ will give $t_f - t_0 = T$, the period of the orbit about the ellipse. This integral can be treated as a contour integral in the complex plane by a well know complex analysis trick. The result is

$$T = 2\pi\frac{a^{3/2}}{\sqrt{GM}}$$

where

$$a = \frac{p}{1-e^2} = \frac{GM}{2|E_0|}$$

which is the result.

Now we address the issue of finding initial conditions that produce a conic orbit with some desired properties. For example, suppose we want to find an elliptic orbit with prescribed eccentricity e_0 and and that we want the maximum distance from the primary to the test particle to be d, for a system with known G and M. We will require the initial velocity to be perpendicular to the initial position vector ($\dot{r}_0 = 0$), and require the trajectory to orbit with counter clockwise rotation.

From analytic geometry it is know that the maximum distance is given by $d = p/(1-e)$ (similarly the minimum distance is $p/(1+e)$). Then

$$e^2 = 1 + \frac{2E_0c^2}{(GM)^2}$$

or

$$E_0c^2 = (GM)^2\frac{e^2-1}{2}$$

The quantities on the right hand side are all givens for the problem. Then we know the product on the left. From this we obtain c^2 by using

$$d = \frac{p}{1-e} = \frac{c^2/GM}{1-\sqrt{1+\frac{2E_0c^2}{(GM)^2}}}$$

Rearranging this gives

$$c^2 = dGM\left(1-\sqrt{1+\frac{2E_0c^2}{(GM)^2}}\right)$$

where the right hand side is know by virtue of the fact that we have already determined E_0c^2. This gives c^2. The maximum distance is d, but this is also the initial condition r_0 as we are taking time zero to be when the velocity and position are perpendicular. Then we can compute $\dot{\theta}_0 = c/r_0^2$. Here we choose the sign of c positive so as to give the desired rotation.

The kinematic relations derived previously give

$$\dot{\mathbf{r}}_0 = \dot{r}_0\mathbf{e}_r + r_0\dot{\theta}_0\mathbf{e}_\theta$$

But every term here is known, as $\dot{r}_0 = 0$, $r_0 = d$ and we just computed $\dot{\theta}$.

5.2 The Two body Problem

The Kepler Problem is useful as an approximation when one mass is very small, but it will turn out to be the key to solving the general two body problem. First note that the problem lives in a twelve dimensional phase space. Then the ten known constants of motion will be insufficient to provide a complete solution. However, as sometimes happens, a judicious choice of coordinates will collapse the problem to one we can solve.

Recall that the equations of motion are

$$\ddot{\mathbf{r}}_1 = \frac{Gm_2}{|\mathbf{r}_{12}|^3}\mathbf{r}_{12} \qquad \ddot{\mathbf{r}}_2 = \frac{Gm_1}{|\mathbf{r}_{21}|^3}\mathbf{r}_{21}$$

and consider the new variable $\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1$. We will try to derive new equations of motion in this variable. Differentiating twice gives

$$\begin{aligned}
\ddot{\mathbf{r}} &= \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 \\
&= \frac{Gm_1}{|\mathbf{r}_{21}|^3}\mathbf{r}_{21} - \frac{Gm_2}{|\mathbf{r}_{12}|^3}\mathbf{r}_{12} \\
&= \frac{Gm_1}{|\mathbf{r}_{21}|^3}(\mathbf{r}_1 - \mathbf{r}_2) - \frac{Gm_2}{|\mathbf{r}_{12}|^3}(\mathbf{r}_2 - \mathbf{r}_1) \\
&= \frac{Gm_1}{|\mathbf{r}|^3}(\mathbf{r}_1 - \mathbf{r}_2) + \frac{Gm_2}{|\mathbf{r}|^3}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= -\frac{G(m_1 + m_2)}{|\mathbf{r}|^3}\mathbf{r}
\end{aligned}$$

Letting $M \equiv m_1 + m_2$ we have the equation

$$\ddot{\mathbf{r}} = -\frac{GM}{|\mathbf{r}|^3}\mathbf{r}$$

which, we are pleased to find, is Kepler's Problem. Then our conclusions about Kepler's Problem carry over to the two body problem, and as we would have expected, Kepler's Laws apply to actual planets and satellites.

Note that the coordinate change reduced the dimension of the system from twelve to six (even before we appeal to conservation laws). That this worked is something of a spell of good luck. In general there is no reason to hope that the difference between two subsystems of a dynamical system is itself a dynamical system. The symmetry of the this particular problem carries the day again.

Nevertheless, it's interesting that the coordinate change is not inertial. This is obvious due to the fact that the equations of motion are not invariant under this change of variables. It's also clear if you think about the motions in the two frames. Observers on say the sun, who are watching the earth, see the center of mass of the two bodies orbiting the sun in an elliptic orbit.

However an observer who insists on sitting at the center of mass and observing the two bodies can certainly do so from an inertial reference frame. From there you see the origin of the first reference frame orbiting you in an ellipse. From the inertial frame at the center of mass the first frame is accelerating and as such cannot be inertial. The trick of changing to a convenient non-inertial frame comes up again in the study of the circular restricted three body problem.

A simple, but important problem that will come up often in the course of these notes is the problem of designing a two body orbit for bodies with known masses, whose distances from one another are know at either apogee or parage, and whose eccentricity in prescribed. We would like to use our understanding of the Kepler Problem to find appropriate initial conditions.

To solve the problem we have to decide on coordinates for the two body problem. Suppose we choose a frame whose origin is at the center of mass of the system at time zero and which moves with the constant velocity of the center of mass. In such a frame the origin is the center of mass for all time.

Again, we assume at time zero the initial velocity is perpendicular to the position vector, and we orient the frame so that the bodies lie initially on the x axis. This gives the conditions

$$\frac{1}{m_1+m_2}(m_1x_1(0)+m_2x_2(0))=0$$

and

$$\frac{1}{m_1+m_2}(m_1v_1(0)+m_2v_2(0))=0$$

From these we obtain the relations

$$x_1(0)=-\frac{m_2}{m_1}x_2(0)$$

and

$$v_1(0)=-\frac{m_2}{m_1}v_2(0)$$

Suppose we design an orbit with the required eccentricity for the Kepler problem and decide the initial velocity should be $\mathbf{v}_0=\dot{\theta}_0\mathbf{e}_\theta$. We can use the relations above to transform this back to the inertial frame.

Recall that $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$ so that we certainly have $\dot{\mathbf{r}}=\dot{\mathbf{r}}_2-\dot{\mathbf{r}}_1$.

Then

$$\begin{aligned} x_2(0)-x_1(0) &= x_2(1+\frac{m_2}{m_1}) \\ &= r_0 \end{aligned}$$

or

$$x_2(0) = \frac{r_0}{1 + \frac{m_2}{m_1}}$$

Using this with the above formulas gives determines x_1. Similarly one can show

$$\dot{x}_2(0) = \frac{r_0\dot{\theta}_0}{1 + \frac{m_2}{m_1}}$$

This last is due to the fact that

$$\dot{\mathbf{r}}_0 = \dot{r}_0\mathbf{e}_r + r_0\dot{\theta}_0\mathbf{e}_\theta$$

and here $\dot{r}_0 = 0$. $\dot{x}_2(0)$ in turn determines $\dot{x}_1(0)$.

Lets find a two body orbit with $d = 2.5$, $m_1 = 0.75$, $m_2 = 0.25$, $G = 1$, and eccentricity $e = 0.7$ (a fairly long ellipse). Then for the Kepler Problem $M = m_1 + m_2 = 1$. The formulas from the previous section give

$$\begin{aligned}
E_0c^2 &= -0.255\\
c^2 &= 0.75\\
c &= 0.8660254\\
\dot{\theta}_0 &= 0.138564\\
v_0 &= 0.34641\\
p &= 0.75\\
a &= 1.470588235\\
x_2(0) &= 1.875\\
x_1(0) &= -0.625\\
\dot{x}_2(0) &= 0.2598076211\\
\dot{x}_1(0) &= -0.086602504\\
T &= 11.205119674
\end{aligned}$$

Here v_0 is the initial condition in the Kepler problem. Integrating these initial conditions over this time interval gives the picture in figure 2.
The red orbit is mass one, and the blue orbit is mass two.

We can read the relevant data off the graph. At time zero the bodies are at apogee. They should be 2.5 distance units apart, and we see that they are. The orbits are clearly elliptic, and rotate about the origin as desired. Each orbit has a circle at its initial condition, and a star at it's final condition. We can see that these overlap, which shows that the period calculation is correct. The orbit has all the desired properties and the design method seems to be correct. This will be of use in later sets of notes.

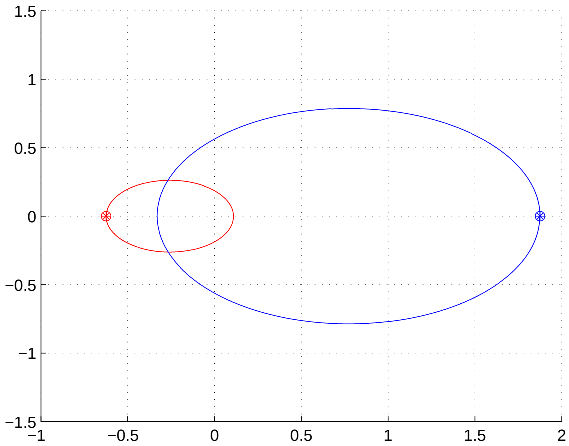

Figure 2: The designed orbits

5.3 N-Body 'Zoo'

When one considers the N-body problem for $N > 2$ there are less and less analytical results as N increases. Certainly there are specific configurations of large numbers of bodies about which much is known. But in general, when handed a specific initial configuration of a large number of bodies, there is little to do but study the system numerically.

In this section, in order to illustrate the variate of phenomena exhibited by the N-body problem and some of the difficulties involved in integrating it, we numerical integrate the N-body problem with $N = 5, G = 1$, $t_f = 10$ and require $|E(0) - E(t_i)| \leq 10^{-12}$ throughout the integration. The initial data for this experiment is

$$\begin{pmatrix} i & m_i & r_{i0} & \alpha_{i0} & \beta_{i0} & v_{i0} & \alpha_{vi0} & \beta_{vi0} \\ - & mu & du & deg & deg & du/tu & deg & deg \\ 1 & 1.0 & 1.0 & 0.0 & 0.0 & 0.6 & 90 & 0.0 \\ 2 & 1.0 & 1.0 & 72.0 & 0.0 & 0.6 & 90 & 0.0 \\ 3 & 1.0 & 1.0 & 144.0 & 0.0 & 0.6 & 90 & 0.0 \\ 5 & 1.0 & 1.0 & 216.6 & 0.0 & 0.6 & 90 & 0.0 \\ 5 & 1.0 & 1.0 & 288.0 & 0.0 & 0.6 & 90 & 0.0 \end{pmatrix}$$

Here, some explanation of the coordinates are needed.

When considering a large number of bodies, often some kind of symmetry is desired in their initial conditions. This is easier to attain with spherical coordinates than with standard rectangular coordinates. Then the positions are given as

$$\mathbf{r}_i(t_0) = \begin{pmatrix} x_{i0} \\ y_{i0} \\ z_{i0} \end{pmatrix} = \begin{pmatrix} r_{i0}\cos\alpha_{i0}\cos\beta_{i0} \\ r_{i0}\sin\alpha_{i0}\cos\beta_{i0} \\ r_{i0}\sin\beta_{i0} \end{pmatrix}$$

and a frame is defined at each m_i for specifying the i^{th} initial velocity. This is

$$\mathbf{r}_i = \frac{\mathbf{r}_i}{|\mathbf{r}_i|}$$

$$\mathbf{s}_i = \frac{\mathbf{k}_i \times \mathbf{r}_i}{|\mathbf{k}_i \times \mathbf{r}_i|}$$

and

$$\mathbf{t}_i = \mathbf{r}_i \times \mathbf{s}_i$$

The initial velocities are specified in spherical coordinates relative to this frame.

$$\mathbf{v}_i^{rst}(t_0) == \begin{pmatrix} v_{i0}\cos\alpha_{vi0}\cos\beta_{vi0} \\ v_{i0}\sin\alpha_{vi0}\cos\beta_{vi0} \\ v_{i0}\sin\beta_{vi0} \end{pmatrix}$$

Note that the initial conditions are only specified in these coordinates. The coordinates are integrated in cartesian coordinates. The necessary transformation is preformed in the program itself. All the user has to do is specify the configuration in the symmetric coordinates. Integrating the given initial conditions we obtain figure 3

The resulting trajectories are as symmetrical as the initial conditions. It looks like we have five elliptical orbits about the common center of mass. The configuration seems to be stable as it persists for several periods (much more will be said about stability in future sets of notes). It is natural to wonder how closely the numerical integration resembles the actual dynamics of the N-body problem.

A qualitative estimate of the error in the integration can be obtained by examining the drift in the energy as a function of time. This is shown in figure 4. The N-body problem has energy as a conserved quantity. Yet we see our numerical system does not. Nevertheless energy drifts very little during the integration. Our numerical system is a small perturbation of the actual problem and the near preservation of energy gives reason to hope that the qualitative and quantitative differences between the two systems are small at least on this time scale.

It's reasonable to think that when we have a stable configuration such as this, and a small drift in the error, then our numerics should be in good agreement with the mathematical model. Next we consider the same problem with the same initial conditions, but we change the mass of m_1 from 1 to 2. Doubling

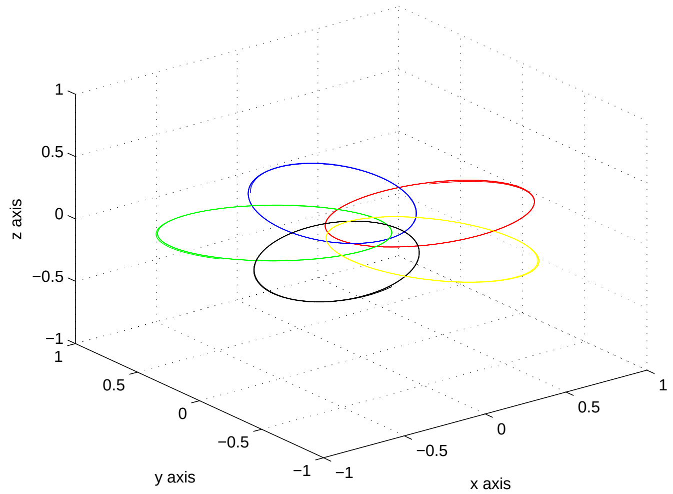

Figure 3: The trajectories of the particles

circles

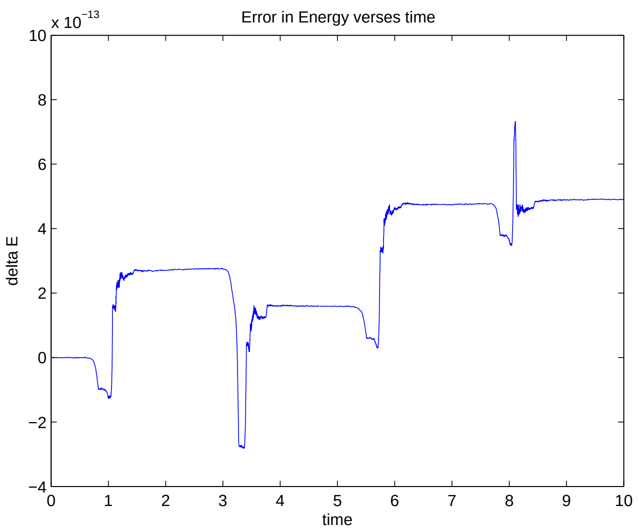

Figure 4: The energy error as a function of time

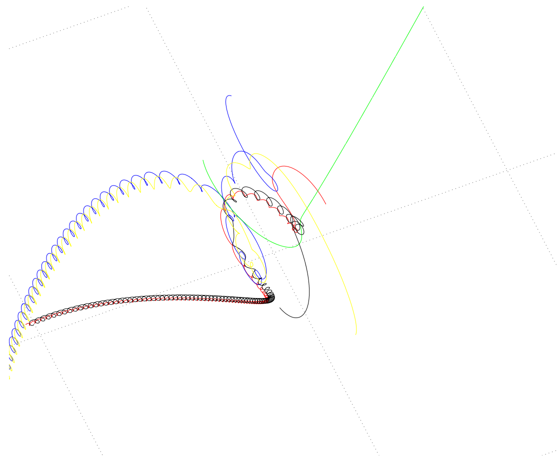

Figure 5: A close look at the new trajectory (birds nest)

the mass of m_1 doubles it's influence over the other particles and collisions result.

The increased mass breaks the symmetry of the system and the behavior is completely changed, as can be seen figure 5.

We can see from the distant view that one orbit at least seems to be unbounded (figure 6). Wether or not such diffusive orbits exist in the actual system is an interesting question. By experimenting with the numerics one finds that the energy tends to drift the most, or even jump, when the bodies pass near each other. In other words 'near misses' have a tendency to push energy into or out of the numerical system.

The drift in the energy which is shown in fir 7, is on the order of ten to the minus seven here, which is still small. However if we were suspicious of our numerics before, then so much more so here. It's not hard to imagine a scenario where several bodies keep having near collisions so energy keeps getting dumped into the numerical system. This extra energy could drive all the particles apart, but does it have to? Perhaps a situation could develop where some bodies experience bounded motion with recurrent near collisions and that the extra energy goes into driving some of the other particles out of bounds.

One could argue that the energy drift is only of the order of ten to the minus seven, and hence the numerics should be valid to roughly the seventh decimal place, in which case the numerical integration should give qualitatively the correct behavior of the N-body system. Indeed, when the configuration is stable this may be an accurate description of the state of affairs.

The flaw in this reasoning is that if the system is in an unstable regime, where chaotic behavior is possible, then it may be extremely sensitive to initial conditions, in which case even such small perturbations could be meaningful. Imagine that each jump in the energy is considered as a small perturbation of

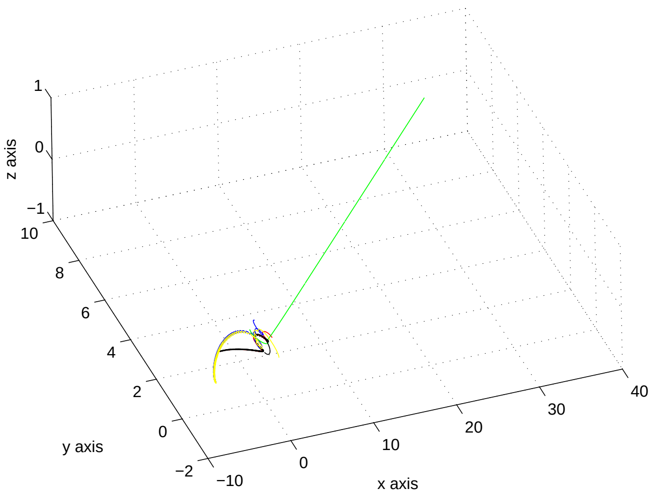

Figure 6: the birds nest from a far

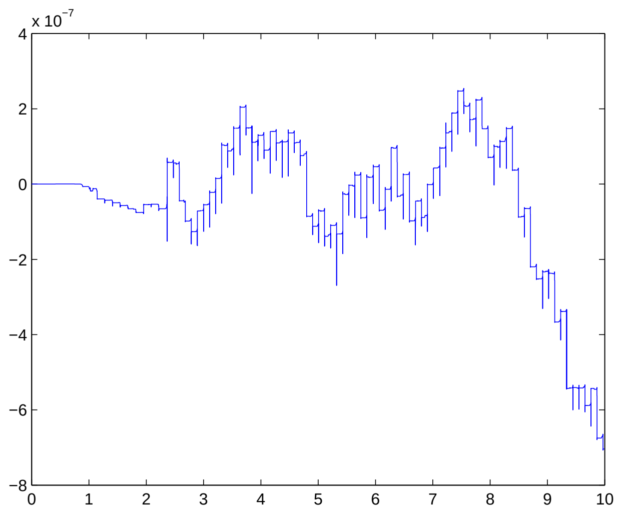

Figure 7: The energy error as a function of time for the birds nest

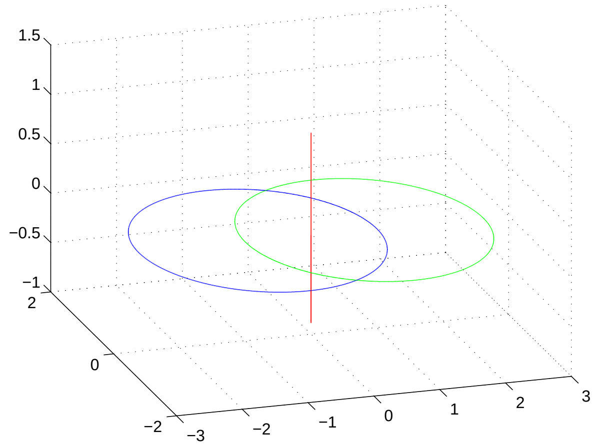

Figure 8: The configuration of the 3 bodies in the Sitnikov Problem

the initial conditions. Then even one such jump could change the long term qualitative behavior if the system, and many such jumps over an extended time could be disastrous.

The interplay between stable and unstable behavior is a recurrent theme throughout N-body dynamics, and will come up often in these notes. This example gives some of the flavor of the considerations that will have to be dealt with more seriously later.

5.4 Numerical Study: Sitnikov Problem

The two body problem admits analytic solution. All it's bounded trajectories are periodic, and all it's unbounded trajectories are go to infinity in both positive and negative time. In fact we know that all solutions are conic sections in the proper coordinate frame. The two body problem is one of the last problems about which so much can be said.

The addition of even one more body to the problem increases the possible complexity of the resulting dynamics without bound. If one desires a qualitative description of the 3-body dynamics then restricting to particular configurations of the 3 masses is often the only way to proceed. One popular configuration is known as Sitnikov's Problem.

In this problem two bodies of equal mass revolve about their center of mass in elliptic orbits in the xy-plane. A third, and much smaller body is placed on the z axis, with initial velocity parallel to this axis as well. The configuration is shown in fig 8. Note that the design method of the previous sections is used to achieve the desired elliptic orbits for the primaries.

The third body is small enough that the two body dynamics of the primaries is not destroyed (in fact on can take the third body to have zero mass, but we do not do this here). Then the motion of the third body will be restricted to the z-axis, but must by no means be regular. In fact this problem has been shown to exhibit behavior semi-conjugate to symbolic dynamics.

Historically the importance of the problem is theoretical rather than physical. The problem was studied by Sitnikov, and Moser (among others), and was one of the first Hamiltonian systems which could be shown to exhibit what is now called chaotic behavior. It gives rise to a twist mapping on the disk, and seems to have either motivated, or at least to have been a fundamental example in the early development of **KAM** theory (Kolmogorov, Arnold, and Moser). The classic reference on this problem is [Mo].

Here we will only study the problem numerically, as an illustration of the variety of behavior that arises in even simple three body configurations (The last several sets of notes will go into much greater detail on a version of the three body problem known as the circular restricted three body problem, which is worthy of study on both theoretical and practical grounds).

The dynamics of this system can be captured by a two dimensional mapping of a disk, or of the plane. In order to describe the state of the dynamical system it is sufficient to know the position of the primaries, and the position and velocity of the third body. Then the Sitnikov Problem is a three dimensional flow.

The position of the primaries is complectly determined by the angle between say the x-axis and one of the bodies. Then the phase space is composed of one angular, and two real variables. This can be reduced by one further dimension by taking an appropriate Poincare Section. (Poincare sections are discussed in more detail in the forth set of notes, but an excellent reference for dynamical systems theory in general is [R]).

Suppose we pick an initial condition for the system and mark the angular variable and the velocity of the third body every time it crosses the xy-plane. This defines a mapping from the cylinder to itself as follows; each point on the cylinder is an angle and a velocity at a time where the third body has position $z = 0$. The image of said point is the point on the cylinder corresponding to the angle of the primaries and the velocity of the third body at it's next return to the xy-plane (let's say that the image can be ∞ as well).

By studying this mapping we can gain an understanding of the behavior of the dynamics. Two numerical representations of this are shown in figures 9 and 10

For these computations we simulated one hundred initial conditions between $z = 0.1$ and $z = 1.5$. An initial condition is integrated as long as necessary to find three hundred intersections with the xy-plane. The angle of the primaries, and the magnitude of the velocity are plotted (only the magnitude need be considered due to the symmetry of the problem).

Figure 9 shows the effects. Another interesting close up is seen in figure 10. The mapping has a "banded" structure that is not evident from the first picture. Near the origin the (which is a fixed point) the crossings occur on invariant curves which are topological circles.

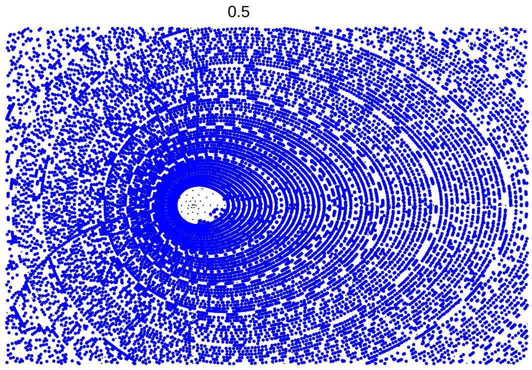

Figure 9: A Poincare Mapping of the Sitnikov Problem

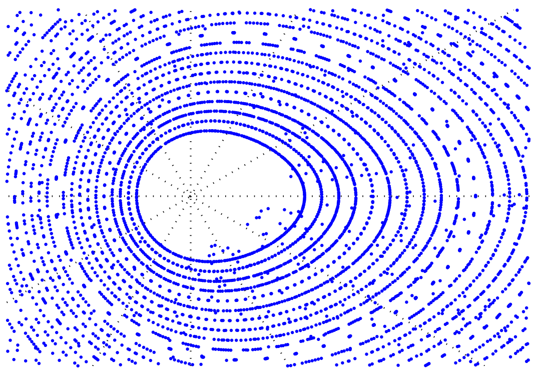

Figure 10: A closer look at the mapping

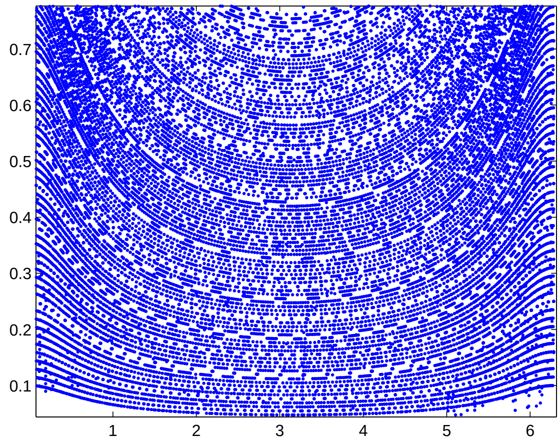

Figure 11: Lift of the cylinder mapping to an Euclidean domain

This indicates periodic motion in the trajectory of the third body. The circles which seem to be filled densely correspond to periodic motion which is not a rational multiple of the period if the primaries, while "spotted" filling of the circles is periodic motion whose frequency is a rational multiple of the primary period.

The invariant curves are not geometric circles but rather have an egg-like shape. The suggests a richer harmonic content than would simple circles. We don't expect the motion to be sinusoidal (except perhaps very close to the origin). Rather the trajectories should have some modulation.

The same information is shown in figures 11 and 12, but in another convenient form. These plots show the mapping lifted to a map in the plane. Now the invariant circles appear as stretched sinusoids, or periodic curves. The stretching indicates the presence of harmonic content. A densely filled curve corresponds to a densely filled circle and similarly for a spotted curve. The horizontal axis is the angular coordinate and the vertical axis is the velocity coordinate. Again the closeup shows clearly that the mapping is periodic for velocities closer to zero.

In terms of **KAM** theory the mapping is close to completely integrable at the origin. However in both the original plots and the plots of the lift it seems that as you move away from zero in the velocity coordinate, something seems to change. The closed curves seem to break up in both maps.

First, lets say that these maps are expensive to compute. Hundreds of initial conditions should be integrated, and each must be integrated for a long time. Then all the zeros must be found. Looking at 10, its clear that as the curves get larger, more and more points are necessary to see wether the circles are

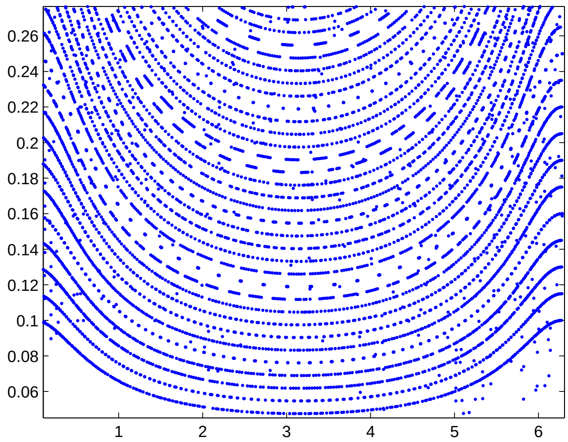

Figure 12: Close up of the lift for smaller amplitudes

filled densely or not. Then it's not completely clear wether the invariant curves are breaking, or wether we are just not seeing enough points (crossings) as the velocity coordinate increases.

It can be shown that the problem is integrable when the eccentricity of the primaries is zero; i.e. when the orbits of the primaries are circular. In this case all motion occurs on invariant circles. Moreover, as is shown in the two pictures, the oscillations of the third body are sinusoidal in character, though they may be out of phase with the primary frequency.

While it's nice to see invariant circles int the polar plots, we will stick with the lift for the moment because it's easy to read and see what's going on. The integrable system is shown in figures 13 and 14. Now the invariant circles are geometric circles. We see the same effect as before with some circles being filled densely and some behaving like rational rotation.

Lets continue this investigation, by increasing the eccentricity of the primaries and seing how this effects the third body. In the last run the eccentricity was zero. This time we increase it to $e = 0.25$.

The results are shown in figures 15 and 16. Already something has happened. It looks like for small enough velocities the mapping is still integrable, but the invariant curves are no longer geometric circles. Some harmonic distortion has developed. Also around $\dot{z} = 0.24$ is looks like there is some disturbance in the banded structure, although it's back shortly there after.

The wave trajectory in figure 16 shows that there is definitely more harmonic content in the wave than before. From just this information it's hard to say more. We may return to this later but for now lets move forward.

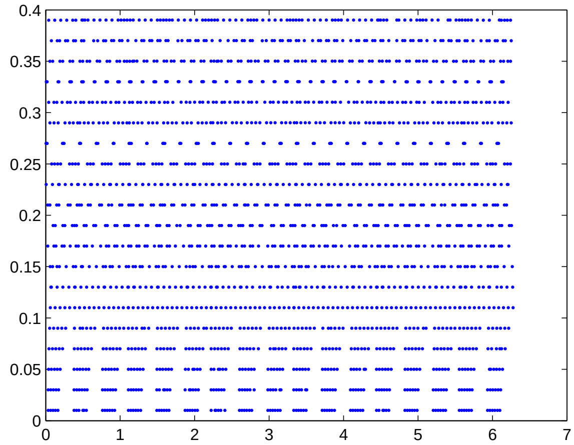

Figure 13: Poincare mapping when the system is integrable

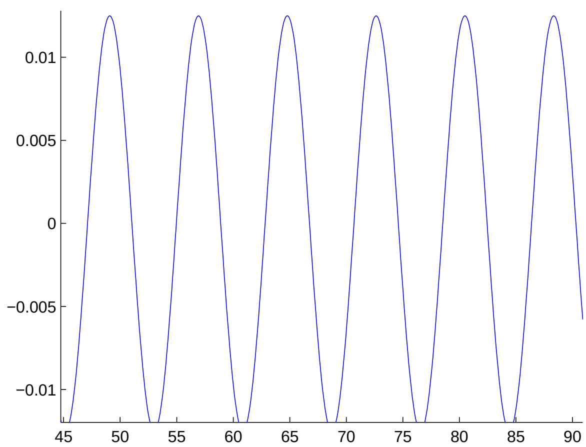

Figure 14: A generic trajectory in the integrable system

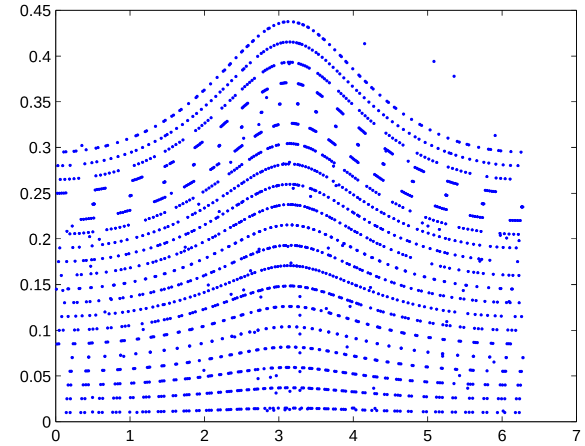

Figure 15: Map of the system when $e = 0.25$

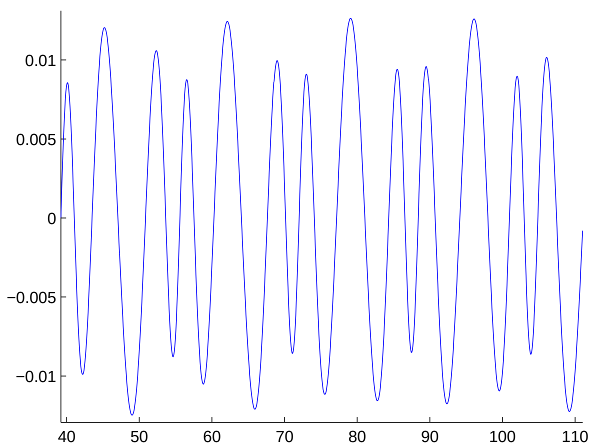

Figure 16: A trajectory in the $e = 0.25$ system

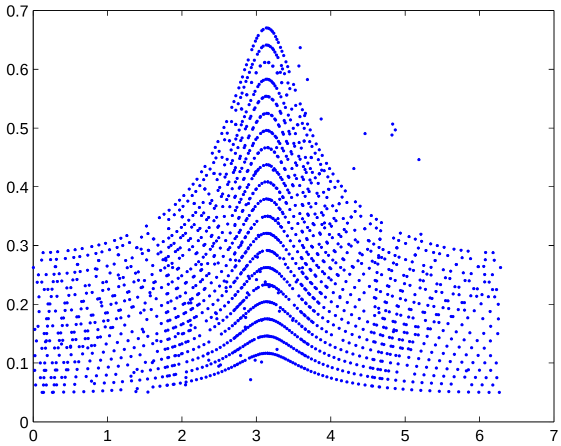

Figure 17: The $e = 0.5$ map system

In the next set $e = 0.5$. The results are in figures 17 and 18. It's interesting, but what seems to happen here is that, while there are more harmonics, the behavior seems more regular. The motion seems to be on invariant curves, and the trajectory, while very modulated, seems to be periodic. Certainly our map only captures a small range of $\dot{z}$ values, but it's interesting that the effects of the perturbation are not necessarily uniform.

We try another run, with an eccentricity $e = 0.7255738$, just to keep it interesting. For this map we have also used significantly more points and iterations. Now you can see a real change in the structure of the map. We'll give more pictures this time as there is more to look at. First, figures 19 and 20 show the kind of view we have been looking at. The first of these is the lift and it's clear that the invariant circle structure is broken. Without going into too much detail about what is in it's place, we simply point out that this is the kind of picture that **KAM** theory predicts, when invariant circles break in a twist map. The picture needs more resolution but the gaps in the picture are probably little copies of the phase space of the pendulum.

The times series (which is just a small sample of the possible behavior in the system) looks less regular as well. Figure 21 shows a longer view of the same signal. It seems that the wave form is 'noisier' than the ones above, and there are two amplitude ranges that seem to occur often.

Before we leave this example we present a view of the mapping as a disk map, in the original coordinates. Again, this illustrates that the invariant circle structure has been replaced by something more complex.

The difference in the dynamics that occurs for larger values of eccentricity makes sense if we pause to consider the possible configurations of the three

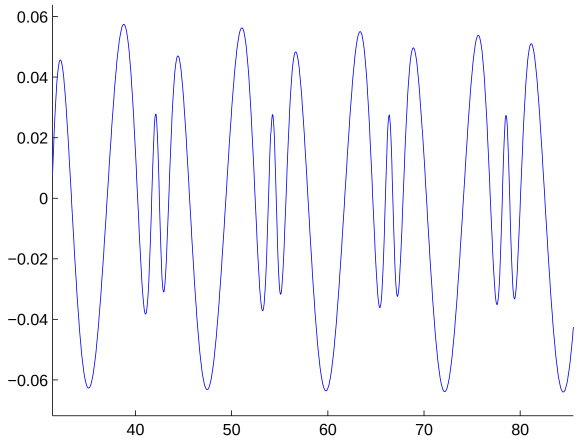

Figure 18: A trajectory in the $e = 0.5$ system

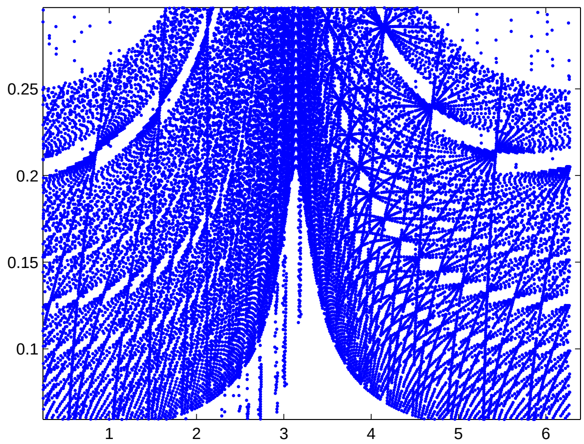

Figure 19: Map of the $e = 0.725$ system

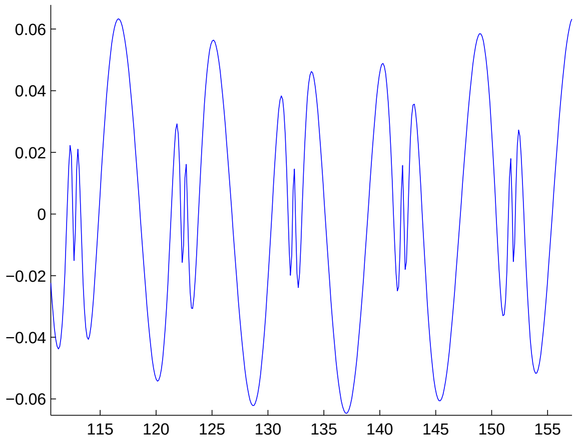

Figure 20: A trajectory in the $e = 0.725$ system

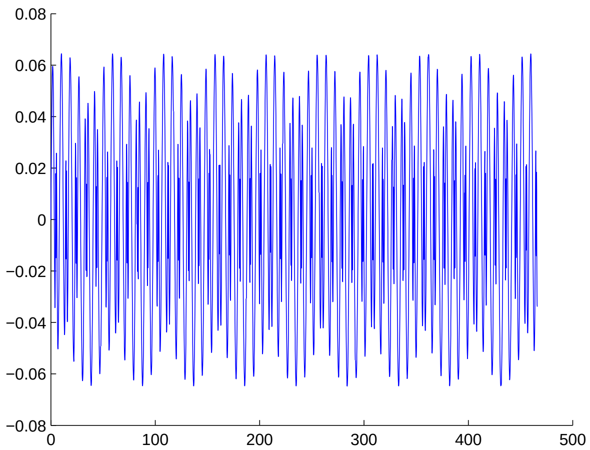

Figure 21: A long view of the same trajectory in the $e = 0.5$ system

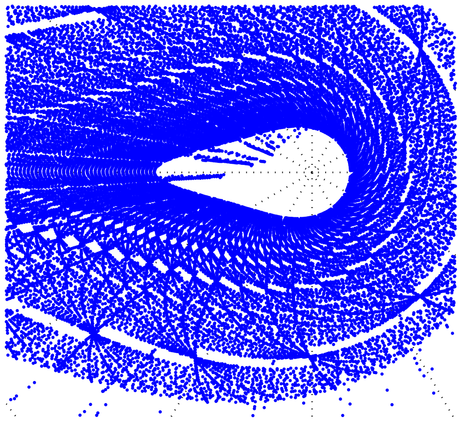

Figure 22: The mapping in angular coordinates

bodies. When the third body nears the xy-plane then the primaries may be near perigee in which case their influence on the third body is large. On the other hand, if they are in apogee then their influence will be less.

This can be thought of as giving the third body a 'kick' every time it passes the plane. Since it's possible even in the integrable case for the third body to oscillate with a frequency rationally independent of the primary frequency, it's possible for these kicks to interact with the smaller body in very complicated ways.

In [Mo] it is shown that this problem admits symbolic dynamics. Then we expect to find orbits that oscillate wildly with no discernable pattern. Even though we see many complicated looking orbits, it's hard to tell if we are looking at periodic orbits with long periods, or truly chaotic behavior.

On the other hand dynamical properties such as mixing, ergodicity, and chaos are bound up together. While it's not necessarily true that they imply one another, a strong indicator and indeed a necessary condition for this kind of complex phenomena is sensitivity to initial conditions. In such a system no amount of knowledge about the initial configuration of the system is even enough and predictability will eventually break down.

We can try to get a feel for the sensitivity to initial conditions in this system by examining a few orbits, and perturbing their initial data. We begin with an orbit from the invariant circle regime say when $e = 0.25$, where we expect at least a kind of Lyapunov stability. We integrate a reference orbit with $\dot{y}_0 = 0.2$ and second orbit where we perturb only the $\dot{y}_0$ initial condition, but by a fairly substantial amount ($\epsilon \approx 0.05$). The results are shown in figures 23 and 24.

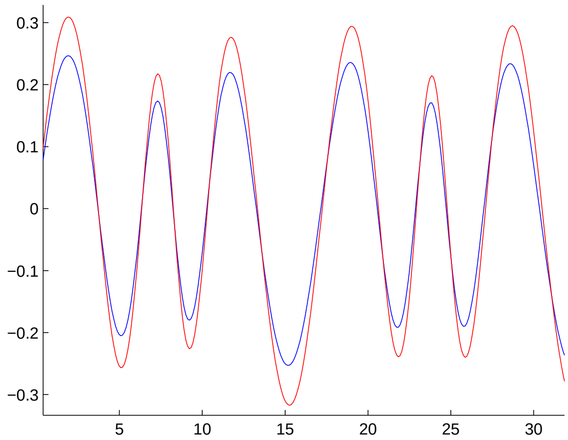

Figure 23: The mapping in angular coordinates

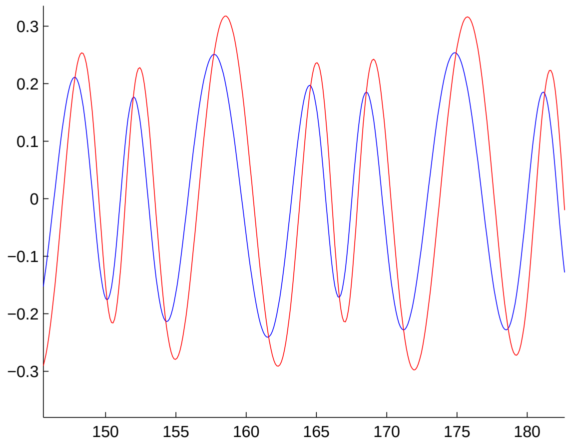

Figure 24: The mapping in angular coordinates

In the first figure the integration is from time zero to time thirty, and even for this substantial perturbation the two trajectories are almost identical. The next figure shows the same two conditions but later. From time one-fifty to time one-eighty. Now the matching is not as good. Never the less it's clear that all that's happening is that the blue trajectory (the reference orbit) is just falling behind the perturbation. Their qualitative form is still almost identical (even though the modulation of the trajectories is fairly complicated). It's safe to conjecture that this system is fairly stable (as the Poincare Map suggested).

The question of stability in the case of higher eccentricity, or further from integrability, is much more complicated and will be taken up in a later set of notes.

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